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ARE CHAOTIC FUNCTIONS REALLY CHAOTIC

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We give a class C of continuous functions from $[0, 1]$ onto itself which are chaotic in the sense of Li and Yorke, but with the property that almost all (in the sense of Lebesgue) points of $[0, 1]$ are eventually fixed. For some continuous functions from $[0, 1]$ onto itself which are not in C, we also show that their non-wandering sets are all equal to the interval $[0, 1]$.

1. Introduction

Let $f(x)$ be a continuous function from the interval $[0, 1]$ into itself. For any x_0 in $[0, 1]$ and any natural number m, let $f^m(x_0)$ denote the mth iterate of x_0 under f. x_0 is called a fixed point of $f(x)$ if $f(x_0) = x_0$. x_0 is called a periodic point of $f(x)$ with minimal period n if $f^n(x_0) = x_0$ and $f^k(x_0) \neq x_0$, $k = 1, \ldots, n-1$. x_0 is called an eventually periodic point of $f(x)$ if, for some natural number m, $f^m(x_0)$ is a periodic point of $f(x)$. x_0 is called an eventually fixed point of $f(x)$ if, for some natural number m, $f^m(x_0)$ is a fixed point of $f(x)$. x_0 is called an asymptotically periodic point of $f(x)$ if there is a periodic point y of $f(x)$ such that $\left|f^k(y)-f^k(x_0)\right|$ tends to zero as k tends to infinity.

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In [10], it is shown that if $f(x)$ is a continuous function from the interval $[0, 1]$ into itself and if $f(x)$ has a periodic point with minimal period 3, then there is an uncountable set S (a scrambled set) such that

(1) $f(x)$ has infinitely many periodic points of different periods, and

(2) every x in S is not asymptotically periodic and for each x, y in S, $x \neq y$, we have

$$0 = \liminf_{k\to\infty} |f^k(x)-f^k(y)| < \limsup_{k\to\infty} |f^k(x)-f^k(y)| .$$

This result has been extended to include those continuous functions from $[0, 1]$ into itself with a periodic point whose period is not an integral power of 2 ([3], [8], [11]). Such functions have been called chaotic by several people ([3], [8], [9], [10], [11]). It is shown in [3] (see [9] also) that the set of all chaotic functions from $[0, 1]$ into itself contains an open dense subset of the space of all continuous functions from $[0, 1]$ into itself with the max norm.

But as indicated in [4, pp. 21-22] and [12], such scrambled sets S for any chaotic function may have (Lebesgue) measure zero. Therefore, the chaotic behavior may be essentially unobservable. In [4, p. 18], the function $f(x) = 1 - 1.401155 \ldots x^2$ is given. For this function $f(x)$, it is said that almost all (in the sense of Lebesgue) initial points are attracted to the same stable, but nonperiodic orbit. However, the dynamical behavior on this stable, nonperiodic orbit appears to be chaotic even though this orbit has (Lebesgue) measure zero.

In this paper we give a class C of continuous functions from $[0, 1]$ onto itself which are chaotic in the sense of Li and Yorke, but are very well behaved from a physical point of view. To be specific, all functions in C are piecewise linear and have "flat bottoms". For all these functions, we show that almost all points of $[0, 1]$ are mapped onto the same, but unstable fixed point $x = 1$. That is, almost all points of $[0, 1]$ are eventually fixed. Therefore, from a physical point of view, every function in C is not chaotic after all. This seems to suggest that chaotic functions should be further classified (see [9] also).

Now let $f(x)$ be a continuous function from $[0, 1]$ into itself and let x_0 be an element of $[0, 1]$. We shall call x_0 recurrent if, for every $\delta > 0$, there exists a natural number m such that $\left|f^m(x_0)-x_0\right| < \delta$. We shall call x_0 non-wandering if, for every $\delta > 0$, there exists y in $[0, 1]$ with $|x_0-y| < \delta$ and $\left|f^m(y)-x_0\right| < \delta$ for some natural number m. Let P_f be the set of all periodic points of $f(x)$, R_f the set of all recurrent points of $f(x)$, and Ω_f the set of all non-wandering points of $f(x)$. Then it is obvious that $\overline{P}_f \subseteq \overline{R}_f \subseteq \Omega_f$. In [6], it is shown that $\overline{P}_f = \overline{R}_f$. But in general, $\overline{R}_f \neq \Omega_f$. In this note, we also show that, for every $f(x)$ in a class C' containing the above-mentioned class C (see Section 2), $\overline{P}_f = \overline{R}_f = \Omega_f = \mathrm{Cl}([0, 1]\backslash K)$, where Cl denotes the closure and K is the set of all x in $[0, 1]$ such that $f^m(x) = 1$ for some natural number m. In particular, if $f(x) : [0, 1] \to [0, 1]$ is defined by $f(x) = 2x$ if $0 \le x \le \frac{1}{2}$, and $f(x) = 2 - 2x$ if $\frac{1}{2} \le x \le 1$; then we show that the non-wandering set of this $f(x)$ is the whole interval $[0, 1]$.

In [1, 2], one-parameter families of "flat top" functions are studied and discussed. In particular, they proved the "supergeometric" convergence of the period doubling sequence which are peculiar to the trapezoid functions. The results are in contrast to those found by Feigenbaum for mappings which are quadratic in a neighborhood of the critical point. Since all our functions in the above-mentioned class C are "bottom flat", they are conjugate to "flat top" functions through the function $g(x) = 1 - x$. Therefore, all results obtained in [1, 2] for top flat functions also hold for all functions in the class C.

In Section 2, we state our main results (Theorems 1 and 2). In Section 3 we give a proof of Theorem 1. The proof of Theorem 2 is similar and shall be omitted. In the last section we pose an open question.

2. Statement of main results

Given any two real numbers $m_1 < 0$ and $m_2 > 0$. For positive

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integers m and i with $1 \le i \le 2^m$, we define $n_{m,i}$ recursively as follows:

$$n_{1,1} = m_1 \ , \quad n_{1,2} = m_2 \ ,$$

and

$$n_{m+1,i} = \begin{cases} [n_{m,(i+1)/2}] \cdot m_2 \ , & \text{if } i \text{ is odd and } n_{m,(i+1)/2} < 0 \ , \\ [n_{m,(i+1)/2}] \cdot m_1 \ , & \text{if } i \text{ is odd and } n_{m,(i+1)/2} > 0 \ , \\ [n_{m,i/2}] \cdot m_1 \ , & \text{if } i \text{ is even and } n_{m,i/2} < 0 \ , \\ [n_{m,i/2}] \cdot m_2 \ , & \text{if } i \text{ is even and } n_{m,i/2} > 0 \ . \end{cases}$$

These numbers $n_{m,i}$ can also be obtained by way of binary trees as follows (see Figure 1).

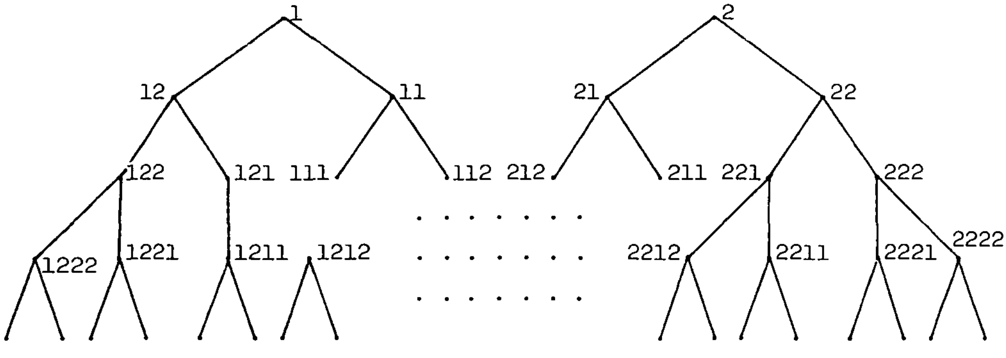

FIGURE 1

In Figure 1, on each level m, there are 2^m nodes. Each node is represented by a string of 1's or 2's. To get the explicit representation of these nodes, we start with 1 and 2 on the first level. Then from each node on the mth level, we attach 1 or 2 to obtain 2 nodes on the $(m+1)$th level. The rule of attachment, from left to right, is as follows:

(1) from the first node on the mth level, we attach 2 first and then 1 to get the first and second nodes on the $(m+1)$th level;

(2) from the second node on the mth level, we attach 1 first and then 2 to get the third and fourth nodes on the

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$(m+1)$th level;

(3) repeat the above procedures; but note that we attach, from left to right on the mth level, the numbers 2, 1, 1, 2 repeatedly to obtain representations of nodes on the $(m+1)$th level as shown in Figure 1.

Now let the ith node on the mth level be represented by $a_1 a_2 \ldots a_m$, where $a_k = 1$ or 2, $k = 1, \ldots, m$. Then it can be easily shown that $n_{m,i} = m_{a_1} m_{a_2} \ldots m_{a_m}$. Note that

$$\sum_{i=1}^{2^m} 1/|n_{m,i}| = \left(-(1/m_1)+(1/m_2)\right)^m$$

for all natural numbers m (recall that $m_1 < 0$ and $m_2 > 0$). This fact will be used later.

Given any two real numbers $0 < a \le b < 1$. With the above definition of $n_{m,i}$, we now define $a_{m,k}$ and $b_{m,k}$ recursively for all natural numbers m and k with $1 \le k \le 2^m-1$ as follows:

$$a_{1,1} = a\ ,\quad b_{1,1} = b\ ,\quad a_{m+1,1} = (b-1)/n_{m,1}\ ,\quad b_{m+1,1} = (a-1)/n_{m,1}\ ,$$

$$a_{m+1,k} = a_{m,k/2} \text{ and } b_{m+1,k} = a_{m,k/2} \text{ if } 1 < k < 2^{m+1}-1 \text{ and } k \text{ even,}$$

$$a_{m+1,k} = b_{m,(k-1)/2} + a/n_{m,(k+1)/2} \text{ and}$$
$$b_{m+1,k} = b_{m,(k-1)/2} + b/n_{m,(k+1)/2} \text{ if } 1 < k \le 2^{m+1}-1\ ,\ k \text{ odd,}$$
$$\text{and } (k+1)/2 \text{ even,}$$

$$a_{m+1,k} = a_{m,(k+1)/2} + b/n_{m,(k+1)/2} \text{ and}$$
$$b_{m+1,k} = a_{m,(k+1)/2} + a/n_{m,(k+1)/2} \text{ if } 1 < k \le 2^{m+1}-1\ ,\ k \text{ odd,}$$
$$\text{and } (k+1)/2 \text{ odd.}$$

Note that $b_{m+1,k} - a_{m+1,k} = (b-a)/|n_{m,(k+1)/2}|$ for all $1 \le k \le 2^{m+1}-1$ and k odd. This fact will be used later.

Let $g(x)$ be a continuous function from $[0, 1]$ onto itself and let

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$0 < a \le b < 1$. If $g(x)$ is linear on $[a, b]$ and $g([a, b]) = [0, 1]$, then we say $g(x)$ is fully linear on $[a, b]$. If $g(x)$ is fully linear on $[a, b]$ and $g(a) = 1$, $g(b) = 0$, then we say $g(x)$ is n.f. linear (fully linear with negative slope) on $[a, b]$. If $g(x)$ is fully linear on $[a, b]$ and $g(a) = 0$, $g(b) = 1$, then we say $g(x)$ is p.f. linear (fully linear with positive slope) on $[a, b]$.

Now we can state our main theorems.

THEOREM 1. *Let* $0 < a \le b < 1$ *and let* $f(x)$ *be a continuous function from* $[0, 1]$ *onto itself with the following three properties:*

(i) $f(x) = 0$ *for all* $a \le x \le b$ *;*

(ii) $f(0) = f(1) = 1$ *;*

(iii) f *is n.f. linear on* $[0, a]$ *and p.f. linear on* $[b, 1]$.

Assume that the slopes of $f(x)$ on $[0, a]$ and on $[b, 1]$ are m_1 and m_2 respectively. For these values of m_1, m_2, a , and b , let $n_{m,i}$, $a_{m,k}$, and $b_{m,k}$ be defined as above. Also let P, R , and Ω be defined as in the introduction with respect to the above $f(x)$ (we suppress the subscript f). The the following hold.

(1) For each natural number m ,

(a) the graph of $f^m(x)$ contains exactly 2^m line segments whose slopes are not zero. These nonzero slopes, from left to right, are $n_{m,i}$, $i = 1, 2, \ldots, 2^m$, respectively.

(b) $$f^m(x) = \begin{cases} 0 \text{ on } [a_{m,k}, b_{m,k}] \text{ for all } 1 \le k \le 2^m-1 \text{ and } k \text{ odd,} \\ 1 \text{ on } [a_{m,k}, b_{m,k}] \text{ for all } 1 < k < 2^m-1 \text{ and } k \text{ even.} \end{cases}$$

(c) the equation $f^m(x) = x$ has exactly 2^m distinct solutions. These solutions are $(a_{m,1})/(1+a_{m,1})$, $(a_{m,k})/(1+a_{m,k}-b_{m,k-1})$, $k = 3, 5, \ldots, 2^m-1$,

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$(b_{m,k})/(1+b_{m,k}-a_{m,k+1})$, $k = 1, 3, 5, \ldots, 2^m-3$, and 1 . In particular,

$$x = 1/\left(1-m_1m_2^{m-1}\right) = [a(1-b)^{m-1}]/[a(1-b)^{m-1}+1]$$

is a periodic point of $f(x)$ with minimal period m .

(2) If $a < b$, then the following hold.

(a) For almost all (in the sense of Lebesgue) points x in $[0, 1]$, $f^m(x) = 1$ for some natural number m . (Note that $x = 1$ is a fixed point of $f(x)$. So almost all points of $[0, 1]$ are eventually fixed.)

(b) $\overline{P} = \overline{R} = \Omega = [0, 1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{2^m-1} (a_{m,n}, b_{m,n}) = \mathrm{Cl}([0, 1]\setminus K)$, where K is the set of all x in $[0, 1]$ such that $f^m(x) = 1$ for some natural number m . (So Ω is a Cantor-like set with Lebesgue measure zero.)

(3) If $a = b$, then $\overline{P} = \overline{R} = \Omega = [0, 1] = \mathrm{Cl}(K)$, where K is the set of all x in $[0, 1]$ such that $f^m(x) = 1$ for some natural number m .

REMARK. If $a = 1/3$ and $b = 2/3$, then the non-wandering set of this $f(x)$ is the usual Cantor ternary set. See [7] for details and other related results.

The following theorem is a generalization of Theorem 1. Since the proof is similar to that of Theorem 1, we shall omit it.

THEOREM 2. *Let* $n > 1$ *be any integer and let*

$$0 < a_1 \le b_1 < c_1 \le d_1 < a_2 \le b_2 < c_2 \le d_2 < \ldots < a_{n-1} \le b_{n-1} < c_{n-1} \le d_{n-1} < a_n \le b_n < 1 .$$

Let $f(x)$ *be a continuous function from* $[0, 1]$ *onto itself with the following four properties:*

(i) $f(x) = 0$ *for all* $a_i \le x \le b_i$, $i = 1, 2, \ldots, n$;

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(ii) $f(x) = 1$ *for all* $c_i \le x \le d_i$, $i = 1, 2, \ldots, n-1$;

(iii) $f(0) = f(1) = 1$;

(iv) $f(x)$ *is fully linear on each of the following intervals:* $[0, a_1]$, $[d_i, a_{i+1}]$, $[b_i, c_i]$, $i = 1, 2, \ldots, n-1$, *and* $[b_n, 1]$.

Then the following hold.

(1) *For every natural number* m ,

(a) *the graph of* $f^m(x)$ *contains exactly* $(2n)^m$ *line segments whose slopes are not zero. These nonzero slopes can be obtained by a recursive formula similar to that in Theorem* 1. (It would be easier to use the k-ary tree method. Start from the first level with n nodes labelled $1, 2, \ldots, n$, the rule of attachment now, from left to right, is

$n, n-1, \ldots, 2, 1, 1, 2, \ldots, n-1, n$.)

(b) *the equation* $f^m(x) = x$ *has exactly* $(2n)^m$ *distinct solutions. These solutions can be easily computed by a recursive formula similar to that in Theorem* 1. (*Therefore, the coordinates of all periodic points of* $f(x)$ *can be explicitly expressed.*) *In particular,* $x = a_1\left[(1-b_n)^{m-1}\right]/\left[a_1(1-b_n)^{m-1}+1\right]$ *is a periodic point of* $f(x)$ *with minimal period* m .

(2) *If* $\sum_{i=1}^{n} (b_i-a_i)^2 + \sum_{i=1}^{n-1} (d_i-c_i)^2 \ne 0$, *then the following hold:*

(a) *for almost all* x *in* $[0, 1]$, $f^m(x) = 1$ *for some natural number* m (*so almost all points of* $[0, 1]$ *are eventually fixed*);

(b) $\overline{P} = \overline{R} = \Omega = \mathrm{Cl}([0, 1]\backslash K)$, *where* K *is the set of all* x *in* $[0, 1]$ *such that* $f^m(x) = 1$ *for some natural*

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number m .

(3) *If* $\sum_{i=1}^{n} (b_i-a_i)^2 + \sum_{i=1}^{n-1} (d_i-c_i)^2 = 0$, *then* $\overline{P} = \overline{R} = \Omega = [0, 1] = \mathrm{Cl}(K)$, *where* K *is the set of all* x *in* $[0, 1]$ *such that* $f^m(x) = 1$ *for some natural number* m .

REMARK. The classes C' and C we mentioned in the introduction are defined as follows: C' is the set of all continuous functions from $[0, 1]$ onto itself satisfying the hypotheses of Theorem 2. C is the set of all continuous functions from $[0, 1]$ onto itself satisfying the hypotheses of Theorem 2 and the additional condition that

$$\sum_{i=1}^{n} (b_i-a_i)^2 + \sum_{i=1}^{n-1} (d_i-c_i)^2 \neq 0 .$$

Obviously, C is a proper subset of C' .

3. Proof of Theorem 1

Assume that $f^m(x)$ is n.f. (p.f. respectively) linear on $[c, d]$ with slope s . Since the composition of two linear functions is also linear, $f^{m+1}(x)$ divides the interval $[c, d]$ into three subintervals $[c, \alpha]$, $[\alpha, \beta]$, and $[\beta, d]$ such that

(a) $f^{m+1}(x)$ is n.f. linear on $[c, \alpha]$ with slope sm_2 $(sm_1$ respectively$)$,

(b) $f^{m+1}(x)$ is identically zero on $[\alpha, \beta]$, and

(c) $f^{m+1}(x)$ is p.f. linear on $[\beta, d]$ with slope sm_1 $(sm_2$ respectively$)$.

Also, $\alpha = d + b/s$ $(c + a/s$ respectively), $\beta = d + a/s$ $(c + b/s$ respectively), and $\beta - \alpha = (b-a)/|s|$. Therefore (1) (a) follows easily by induction on m .

To show (1) (b), we note that $x = 1$ is a fixed point of $f(x)$. So if $f^m(x_0) = 1$ for some natural number m , then $f^n(x_0) = 1$ for all integers $n \geq m$. Since $f^m(x) = 0$ or 1 on $[a_{m,k}, b_{m,k}]$, $k = 1, 2, \ldots, 2^m-1$, $f^{m+1}(x) = 1$ on $[a_{m,k}, b_{m,k}]$, $k = 1, 2, \ldots, 2^m-1$. Hence $f^{m+1}(x) = 1$, by definition, on $[a_{m+1,2k}, b_{m+1,2k}]$, $k = 1, 2, \ldots, 2^m-1$. That is, $f^{m+1}(x) = 1$ on $[a_{m+1,k}, b_{m+1,k}]$, $1 < k < 2^{m+1}-1$ and k even. Now let $c = b_{m,k}$, $d = a_{m,k+1}$, where $1 \leq k \leq 2^m-1$ and k odd (when $k = 2^m - 1$, we let $a_{m,k+1} = 1$). Then $f^m(x)$ is p.f. linear on $[c, d]$ with slope $n_{m,k+1}$. So, from above,

$$\alpha = b_{m,k} + a/n_{m,k+1} = a_{m+1,2k+1}$$

and

$$\beta = b_{m,k} + b/n_{m,k+1} = b_{m+1,2k+1} .$$

This proves most of (1) (b). The rest of (1) (b) can be proved similarly.

To prove (1) (c), we note that $f^m(x)$ is n.f. linear on $[b_{m,k-1}, a_{m,k}]$, $k = 3, 5, \ldots, 2^m-1$. Their slopes are $-1/(a_{m,k}-b_{m,k-1})$, $k = 3, 5, \ldots, 2^m-1$, respectively. So their equations are $y = -(x-a_{m,k})/(a_{m,k}-b_{m,k-1})$, $k = 3, 5, \ldots, 2^m-1$, respectively. By letting $y = x$, we obtain that $x = a_{m,k}/(1+a_{m,k}-b_{m,k-1})$, $k = 3, 5, \ldots, 2^m-1$. This gives part of (1) (c). The other part of (1) (c) follows similarly.

For the proof of (2) (a) it suffices to show that

$$\lim_{m\to\infty} \sum_{k=1}^{2^m-1} (b_{m,k}-a_{m,k}) = 1 .$$

By definition of $a_{m,k}$ and $b_{m,k}$, we have, for every natural number m,

$$b_{m+1,k} - a_{m+1,k} = (b-a)/|n_{m,(k+1)/2}| , \quad k = 1, 3, 5, \ldots, 2^{m+1}-1 .$$

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So

$$\sum_{k=1}^{2^{m+1}-1} \left(b_{m+1,k}-a_{m+1,k}\right)$$

$$= (b-a)\left(1 + \sum_{k=1}^{m} \sum_{i=1}^{2^k} 1/|n_{k,i}|\right)$$

$$= (b-a) \sum_{k=0}^{m} \left(-(1/m_1)+(1/m_2)\right)^k$$

$$= (b-a)\left[1-\left(-(1/m_1)+(1/m_2)\right)^{m+1}\right]/\left(1+(1/m_1)-(1/m_2)\right)$$

$$= (b-a)\left[1-(a+1-b)^{m+1}\right]/[1-a-(1-b)] \quad \left(\text{since } m_1 = -1/a \text{ and } m_2 = 1/(1-b)\right)$$

$$= 1 - (a+1-b)^{m+1}$$

$\to 1$ as m tends to infinity.

This proves (2) (a).

To show (2) (b), let K_m, for each natural number m, be the set of all x in $[0, 1]$ such that $f^m(x) = 0$ or 1. Then $K_1 \subset K_2 \subset K_3 \subset \ldots$ and $\lim_{m\to\infty} \mu(K_m) = 1$, where μ is the Lebesgue measure. Let

$$W = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{2^m-1} \left(a_{m,k}, b_{m,k}\right) .$$

Then it is clear that $\Omega \subseteq [0, 1]\backslash W$. Now let x_0 be any element of $[0, 1]\backslash W$ with $x_0 \neq a_{m,k}$ or $b_{m,k}$ for all $m \geq 1$ and $k = 1, 2, \ldots, 2^m-1$. So $x_0 \notin K_m$ for all $m \geq 1$. For any $\delta > 0$, there exist, since $\lim_{n\to\infty} \mu(K_n) = 1$, a natural number m and two elements y, z in K_m such that $y < x_0 < z$ and $|y-z| < \delta$. Hence there exists k with $1 < k < 2^m-1$ such that $f^m(x)$ is fully linear on $\left[b_{m,k}, a_{m,k+1}\right]$ and $\left[b_{m,k}, a_{m,k+1}\right] \subseteq [y, z]$. Since the intersection points of the curves $y = f^m(x)$ and $y = x$ are period m (need not be minimal) points of

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$f(x)$, this shows that x_0 is in $\overline{P}$. Therefore $\overline{P} = \overline{R} = \Omega = [0, 1]\backslash W$. This completes the proof of (2) (b).

To prove (3), let K_m , for every natural number m , be the set of all x in $[0, 1]$ such that $f^m(x) = 0$ or 1 . Then K_m consists of exactly $2^m + 1$ distinct points and $K_1 \subset K_2 \subset K_3 \subset \ldots$. Also $f^m(x)$ is fully linear on any interval whose endpoints are two consecutive elements in K_m (that is, no other elements of K_m lie strictly in between). Let $\delta(K_m) = \max\{|c-d| \mid c, d \text{ are any two consecutive elements in } K_m\}$. Then it is clear that $\delta(K_{m+1}) < \max\{a, 1-a\}\delta(K_m)$. Therefore $\lim_{m\to\infty} \delta(K_m) = 0$ and the set $K = \bigcup_{m=1} K_m$ is dense in $[0, 1]$. The rest is easy.

4. An open question

Let $h(x)$ be a continuous function from $[0, 1]$ into itself. For every x in $[0, 1]$, let

$$L_h(x) = \bigcap_{n=1}^{\infty} \mathrm{Cl}\{h^m(x) \mid m \text{ is any integer greater than or equal to } n\} .$$

It is obvious that $L_h(x)$ is compact and nonempty for every x in $[0, 1]$.

Fix any two real numbers a and b with $0 < a < b < 1$. Let $f : [0, 1] \to [0, 1]$ be defined by $f(x) = -(x-a)/a$ if $0 \le x < a$, $f(x) = 0$ if $a \le x \le b$, and $f(x) = (x-b)/(1-b)$ if $b < x \le 1$. For $0 \le \alpha \le 1$, let $f_\alpha(x) = \alpha f(x)$ and consider $f_\alpha(x)$ as a one-parameter family of continuous functions from $[0, 1]$ into itself with α as the parameter. To simplify notations, we denote, from now on, $L_{f_\alpha}(x)$ as $L_\alpha(x)$.

When $0 \le \alpha < a$, it follows from [5] that the unique fixed point $x = \alpha a/(\alpha+a)$ of $f_\alpha(x)$ is globally stable. So for all x in $[0, 1]$, $L_\alpha(x) = L_\alpha(\alpha a/(\alpha+a)) = \{\alpha a/(\alpha+a)\}$. In particular, $L_\alpha(x) = L_\alpha(\alpha)$ for all

x in $[0, 1]$.

When $\alpha = a$, every x with $0 < x < a/2$ or $a/2 < x \le a$ is a periodic point of $f_\alpha(x)$ with minimal period 2. But none is stable. In this case $L_\alpha\big((1+b)/2\big) = L_\alpha(a/2) = \{a/2\}$ and all other points are eventually periodic (with minimal period 2).

When $a < \alpha \le b$, $x = \alpha$ is a periodic point of $f_\alpha(x)$ with minimal period 2. In this case $L_\alpha\big((a+\alpha b)/(\alpha+a)\big) = L_\alpha\big(\alpha a/(\alpha+a)\big) = \{\alpha a/(\alpha+a)\}$ and $L_\alpha(x) = L_\alpha(\alpha) = \{0, \alpha\}$ for all $x \ne (a+\alpha b)/(\alpha+a)$, $\alpha a/(\alpha+a)$.

So far, we have shown that, for almost all α (in fact, with only one exception at $\alpha = a$) in $[0, b]$, $L_\alpha(x) = L_\alpha(\alpha)$ for almost all x (in fact, with only two exceptions at $x = (a+\alpha b)/(\alpha+a)$ and $x = \alpha a/(\alpha+a)$ when $a < \alpha \le b$) in $[0, 1]$.

Now let $\alpha_1 = \left[b+\sqrt{b^2+4a(1-b)}\right]/2$ and $\alpha_2 = \left[b+\sqrt{b^2+4b(1-b)}\right]/2$. Then period 3 points bifurcate spontaneously at $\alpha = \alpha_1$ and exist for all $\alpha_1 \le \alpha \le 1$. For $\alpha_1 \le \alpha \le \alpha_2$, $x = \alpha$ is a period 3 point of $f_\alpha(x)$ and it seems that $L_\alpha(x) = L_\alpha(\alpha)$ for almost all x in $[0, 1]$.

When $\alpha = 1$, Theorem 1 implies that $L_\alpha(x) = L_\alpha(\alpha) = L_1(1) = \{1\}$ for almost all x in $[0, 1]$.

Based on the above observation, we make the following conjecture. Note that $L_\alpha(0) = L_\alpha(1) = L_\alpha(\alpha)$ for all α in $[0, 1]$.

CONJECTURE. Let $f(x)$, $f_\alpha(x)$, and $L_\alpha(x)$ be defined as above. Then for almost all α in the parameter space $[0, 1]$, $L_\alpha(x) = L_\alpha(\alpha)$ for almost all x in $[0, 1]$.

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