J. Appl. Prob. **44,** 366–378 (2007)
Printed in England

A DIFFUSION APPROXIMATION FOR MARKOV RENEWAL PROCESSES

STEVEN P. CLARK,* *University of North Carolina at Charlotte*
PETER C. KIESSLER,** *Clemson University*

Abstract

For a Markov renewal process where the time parameter is discrete, we present a novel method for calculating the asymptotic variance. Our approach is based on the key renewal theorem and is applicable even when the state space of the Markov chain is countably infinite.

Keywords: Markov renewal process; functional central limit theorem; asymptotic variance

2000 Mathematics Subject Classification: Primary 60G10
Secondary 60F17

1. Introduction

Let $\{(J_n, Y_n, T_n);\ n \in \mathbb{Z}_+ \cup \{-1\}\}$, where $\mathbb{Z}_+$ is the set of nonnegative integers, be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that

(i) J_n takes values in a countable set E,

(ii) Y_n is real valued,

(iii) $T_{-1} < 0 \le T_0 < T_1 < \cdots$,

(iv) T_n is integer valued.

Let $\mathcal{F}_n = \sigma((J_m, Y_m, T_m);\ m \le n)$ and assume that

$$\mathrm{P}\{J_{n+1} = k, Y_{n+1} \in B, T_{n+1} - T_n \le t \mid \mathcal{F}_n\} = \mu_k(B) Q(J_n, \{k\} \times [0, t]).$$

Here, for each $k \in E$, $\mu_k(\cdot)$ is a probability measure on $\mathcal{B}$, the Borel subsets of the real line, $B \in \mathcal{B}$, and $Q(k, \cdot)$ is a probability measure on $\mathcal{P}(E) \otimes \mathcal{B}$, where $\mathcal{P}(E)$ is the collection of all subsets of E and $t \in \mathbb{Z}_+$. We note that, given J_n, Y_n is independent of $\{(J_n, T_n);\ n \in \mathbb{Z}_+\}$ and that, for $n_1 \ne n_2$, Y_{n_1} and Y_{n_2} are conditionally independent given $\{(J_n, T_n);\ n \in \mathbb{Z}_+\}$. Moreover, the process $\{J_n;\ n \in \mathbb{Z}_+\}$ is a Markov chain which we assume is aperiodic, irreducible nonnull recurrent with stationary vector $\boldsymbol{\nu}$. In addition, we assume that, for each $j \in E$, the distribution function $Q(j, E \times (0, t])$ has period one. We specify the distribution of (J_0, Y_0, T_0) later. In this setting, $\{(J_n, T_n);\ n \in \mathbb{Z}_+\}$ is a Markov renewal process.

Received 4 January 2005; revision received 16 April 2007.
* Postal address: Department of Finance and Business Law, University of North Carolina at Charlotte, 9201 University City Boulevard, Charlotte, NC 28075, USA. Email address: spclark@email.uncc.edu
** Postal address: Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, USA. Email address: kiesslp@clemson.edu

We motivate the main result of this paper with the following example from queueing theory.

Example 1. Suppose that T_n is the time of the nth arrival to a single server queue, J_n is the customer's type, and Y_n the amount of work the customer brings. Let W_t be the amount of work that arrives by time t. To apply heavy traffic approximations for the virtual waiting time process it is necessary to have a functional central limit theorem (FCLT) for $W_.$. One approach is to first develop an FCLT for $\{Y_n\}$ and use a random change of time to obtain an FCLT for $W_.$. See [10, Chapter 13] for details on random time changes. When $\{Y_n\}$ is stationary, which is the case when $\{J_n\}$ is stationary, we may apply an FCLT for stationary sequences. The asymptotic variance term in the FCLT is given by

$$\sigma^2 = \operatorname{var}(Y_0) + 2\sum_{n=0}^{\infty} \operatorname{cov}(Y_0, Y_n)$$

(see, for example, [3], [7], or [5]). Using the Poisson equation (see [11, Chapter 2]), we can develop an expression for σ^2 using solutions, $\boldsymbol{x}$, to $\boldsymbol{x}(\boldsymbol{I} - \boldsymbol{P}) = y$ for the appropriate vector $\boldsymbol{y}$, where $\boldsymbol{I}$ is an identity matrix and $\boldsymbol{P}$ is the one-step transition matrix for $\{J_n\}$. In [4], σ^2 was calculated using spectral theory.

Our concern in this paper is to develop an FCLT for $W_.$ and a method for finding its asymptotic variance. The method presented here for calculating the asymptotic variance is based on the key renewal theorem and is applicable even when the state space of the Markov chain is countably infinite. The FCLT is stated as Theorem 1 and the expression for σ^2 is given in Theorem 5. Assumptions (iii) and (iv) in the statement of Theorem 1 are not given in terms of the Markov renewal process but rather in terms of a related Markov chain defined in Section 2. In Section 3, we provide conditions on the Markov renewal process for assumptions (iii) and (iv) to hold. In particular, we show that (iii) is implied by appropriate rates of convergence to the stationary distribution of the Markov chain $\{J_n;\ n \in \mathbb{Z}_+\}$ and bounds on the $T_{n+1} - T_n$s. This is accomplished by applying rate of convergence results for Markov chains given in [8] and [6].

2. Preliminaries

For $t \in \mathbb{Z}_+ \cup \{-1\}$, define

$$(X_t, Z_t, L_t) = \begin{cases} (J_n, Y_n, 0) & \text{if } t = T_n, \\ (J_n, 0, t - T_n) & \text{if } T_n < t < T_{n+1}, \end{cases}$$

and

$$\mathcal{G}_t = \sigma(\chi_0, \dots, \chi_t),$$

where we have set $\chi_t = (X_t, Z_t, L_t)$. To see that the process $\{(X_t, Z_t, L_t);\ t \in \mathbb{Z}_+\}$ is a Markov chain, note that on $\{X_t = j, L_t = l\}$,

$$\{X_{t-l} = j, L_{t-l} = 0, \dots, X_t = j, L_t = l\}.$$

So, for some nonnegative integer n,

$$T_{n+1} - T_n > l \quad \text{and} \quad J_n = j.$$

It follows that

$$
\begin{aligned}
&\mathrm{P}\{X_{t+1} = k, Z_{t+1} \in B, L_{t+1} = u \mid \mathcal{G}_t\} \\
&= \begin{cases}
\dfrac{\mathrm{P}\{T_{n+1} - T_n > l+1 \mid X_n = j\}}{\mathrm{P}\{T_{n+1} - T_n > l \mid X_n = j\}} & \text{if } k = j,\ u = l+1,\ 0 \in B, \\[2ex]
\dfrac{\mathrm{P}\{J_{n+1} = k, Y_{n+1} \in B, T_{n+1} - T_n = l+1 \mid X_n = j\}}{\mathrm{P}\{T_{n=1} - T_n > l \mid X_n = j\}} & \text{if } u = 0, \\[2ex]
\dfrac{Q(j, \{k\} \times (l+1, \infty))}{Q(j, E \times [l+1, \infty))} & \text{if } k = j, u = l+1, 0 \in B, \\[2ex]
\dfrac{Q(j, \{k\} \times \{l+1\})\mu_k(B)}{Q(j, E \times [l+1, \infty))} & \text{if } u = 0.
\end{cases}
\end{aligned}
$$

In addition, $\{(X_t, L_t);\ t \in \mathbb{Z}_+\}$ is a Markov chain whose one-step transition matrix is given by the above equation with $B = \mathbb{R}$. We denote the $[(j,l),(k,u)]$th element of the tth power of this transition matrix by $p^t_{(j,l),(k,u)}$. The probability measure γ denotes the initial distribution of $\{\chi_n;\ n \in \mathbb{Z}_+\}$, with

$$\gamma(\{j\} \times B \times \{l\}) = \mathrm{P}\{X_0 = j, Z_0 \in B, L_0 = l\},$$

and

$$\gamma_{j,l} = \mathrm{P}\{X_0 = j, L_0 = l\}.$$

The notation $\gamma p^t_{(k,u)}$ denotes the probability that, at t, $X_t = k$ and $L_t = u$, given that the initial distribution is γ.

The assumptions on $\{(J_n, Y_n, T_n);\ n \in \mathbb{Z}_+\}$ imply that $\{(X_t, Z_t, L_t);\ t \in \mathbb{Z}_+\}$ is ergodic and, hence, has an invariant measure that is also a limiting distribution which is denoted by π. The measure π on $(\mathcal{P}(E) \otimes \mathcal{B} \otimes \mathcal{P}(\mathbb{Z}_+))$ is given by

$$
\pi(\{k\} \times B \times \{s\}) = \begin{cases}
\dfrac{\mu_k(B)\boldsymbol{\nu}_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}} & \text{if } s = 0, \\[2ex]
\dfrac{1_{\{0 \in B\}}\boldsymbol{\nu}_k Q(k, E \times (s, \infty))}{\boldsymbol{\nu} \cdot \boldsymbol{m}} & \text{if } s > 0,
\end{cases}
$$

where $1_{\{A\}}$ denotes the indicator function of an event A, $\boldsymbol{\nu}_k$ is the kth component of the vector $\boldsymbol{\nu}$, $E = \{1, \dots, k, \dots\}$, and $\boldsymbol{m} = (m_1, \dots, m_k, \dots)$ with

$$m_k = \mathrm{E}_k[T_1] < \infty.$$

For $k \in E$ and $s \in \mathbb{Z}_+$ set $\pi_{k,s} = \pi(\{k\} \times \mathbb{R} \times \{s\})$. The Markov chain $\{(X_t, Z_t, L_t);\ t \in \mathbb{Z}_+\}$ is stationary when the initial distribution is given by π. In this case, the process $\{Z_t;\ t \in \mathbb{Z}_+\}$ is also stationary.

To see that π is the stationary distribution, we observe that $\{(X_t, Z_t, L_t);\ t \in \mathbb{Z}_+\}$ is regenerative with regeneration times being the times at which $X = 1$ and $L = 0$. Thus, by Smith's theorem (see, for example, [9, p. 265]), we obtain

$$\pi(\{k\} \times B \times \{s\}) = \frac{\mathrm{E}[\text{amount of time spent in } \{k\} \times B \times \{s\} \text{ during a cycle}]}{\mathrm{E}[\text{cycle time}]}. \tag{1}$$

For some $c > 0$ independent of k, $c\boldsymbol{v}_k$ is the expected number of visits to k during a cycle and m_k is the expected time spent in k during each visit. Thus, the denominator of (1) is equal to $c\boldsymbol{v}\cdot\boldsymbol{m}$. The expected time spent in $\{k\}\times B\times\{s\}$ during a cycle is the expected number of visits to k, multiplied by the probability that a visit lasts at least s, multiplied by either the probability that the mark is in B if $s = 0$, or 1 if $0 \in B$ (if $s > 0$), or 0 if $0 \in B^c$ (if $s > 0$), where B^c denotes $c \notin B$. That is,

$$\begin{aligned}&\mathrm{E}[\text{amount of time spent in } \{k\}\times B\times\{s\} \text{ during a cycle}]\\ &\quad = \begin{cases} c\mu_k(B)\boldsymbol{v}_k & \text{if } s = 0,\\ 1_{\{0\in B\}}c\boldsymbol{v}_k Q(k, E\times(s,\infty)) & \text{if } s > 0.\end{cases}\end{aligned}$$

The initial distribution for (J_0, Y_0, T_0) is given as the forward recurrence time when $L_t = 0$ given that the initial distribution on (X_0, Z_0, L_0) is γ. Thus,

$$\mathrm{P}\{J_0 = j, Y_0\in B, T_0 = t\} = \begin{cases}\gamma(\{j\}\times\{B\}\times\{0\}) & \text{if } t = 0,\\ \displaystyle\sum_{i\in E}\sum_{l\geq 1}\gamma_{i,l}\frac{Q(i,\{j\}\times\{l+t\})}{Q(i, E\times(l,\infty))}\mu_j(B) & \text{if } t > 0.\end{cases}$$

Notice that if the initial distribution for (X_0, Z_0, L_0) is π then

$$\mathrm{P}\{J_0 = j, Y_0\in B, T_0 = t\} = \sum_{i\in E}\frac{\boldsymbol{v}_i}{\mu\boldsymbol{m}}Q(i,\{j\}\times(t,\infty))\mu_j(B),$$

which agrees, realizing that we are working with a discrete time process, with stationary forward recurrence time given in [2, Example 6.18 of Chapter 10].

In what follows, $\mathrm{E}[\cdot]$ will denote expectation under the assumption that the initial distribution is γ, $\mathrm{E}_\pi[\cdot]$ will denote expectation when the process is stationary, and $\mathrm{E}_j[\cdot]$ will denote conditional expectation given that $J_0 = j$ and $T_0 = 0$.

For $n\in\mathbb{Z}_+$ and a positive integer N, define

$$\zeta_{N,n} = \frac{1}{\sqrt{N}}(Z_n - \mathrm{E}_\pi[Z_0])$$

and, for all real numbers $\tau \geq 0$, define

$$S_\tau^{(N)} = \sum_{n\leq N\tau}\zeta_{N,n}.$$

The process $S^{(N)}$ has sample paths in the space D_∞ of real-valued functions on $[0,\infty)$ that are right-continuous with left-hand limits. We let d_∞^o be the metric on D_∞ given in [1, p. 168]. We use the notation '$\xrightarrow{\mathrm{D}}$' to denote convergence in distribution of stochastic processes with sample paths in D_∞.

Theorem 1. *Suppose that the following conditions hold:*

(i) $K_1 \equiv \sup_{j\in E}\int_{\mathbb{R}}|x|\mu_j(\mathrm{d}x) < \infty$,

(ii) $K_2 \equiv \sup_{j\in E}\int_{\mathbb{R}}x^2\mu_j(\mathrm{d}x) < \infty$,

(iii) $\sum_{(j,l)\in E\times\mathbb{Z}_+}\pi_{j,l}\|p^t_{(j,l),\cdot} - \pi_\cdot\|^2_{tv} = o(t^{-2-\delta})$, *for some* $\delta > 0$,

https://doi.org/10.1239/jap/1183667407 Published online by Cambridge University Press

(iv) *for a fixed* $i^* \in E$ *and with* $\tau_{i^*} = \inf\{t = 1, 2, \dots : (X_t, L_t) = (i^*, 0)\}$, *we have*

$$\mathrm{E}_{(i^*,0)}[\tau_{i^*}] < \infty.$$

Then, provided that $\sigma_Z > 0$, *we have*

$$S_{\cdot}^{(N)} \xrightarrow{\mathrm{D}} \sigma_Z B_{\cdot},$$

where $B_{\cdot}$ *is a standard Brownian motion and*

$$\sigma_Z^2 = \operatorname{var}_\pi(Z_0) + 2\sum_{t=1}^{\infty} \operatorname{cov}_\pi(Z_0, Z_t). \tag{2}$$

The proof of Theorem 1 is carried out in two steps. The result is first proved in the special case where the process $\{(X_t, Z_t, L_t);\ t = 0, 1, \dots\}$ is stationary. The result stated in Theorem 1 will follow as a special case using a coupling argument. As stated, Theorem 1 may appear unappealing in that conditions (iii) and (iv) are not given in terms of the semi-Markov kernel of the Markov renewal process $\{(J_n, Y_n, T_n);\ n = -1, 0, 1, \dots\}$. However, in Section 3, we develop conditions for the semi-Markov kernel that imply conditions (iii) and (iv) of Theorem 1.

Proof of Theorem 1. First assume that the initial distribution is π. In this case the process $\{(X_t, Z_t, L_t);\ t = 0, 1, \dots\}$ is stationary. By [3, Theorem 7.6, Chapter 7], it suffices to show that

$$\sum_{n=1}^{\infty} [\mathrm{E}_\pi[\mathrm{E}_\pi[Z_t - c \mid X_0, L_0]^2]]^{1/2} < \infty,$$

where

$$c = \sum_{j' \in E} \int_{\mathbb{R}} \pi_{j',0}\, x \mu_{j'}(\mathrm{d}x),$$

or, equivalently, it suffices to show that

$$\mathrm{E}_\pi[\mathrm{E}_\pi[Z_t - c \mid X_0, L_0]^2] = o(t^{-2-\delta}).$$

This follows since the left-hand side of the above equation is equal to

$$\begin{aligned}
&\sum_{(j,l) \in E \times \mathbb{Z}_+} \Bigg[\sum_{(j',l') \in E \times \mathbb{Z}_+} \int_{\mathbb{R}} (p^t_{(j,l),(j',l')} - \pi_{j',l'} 1_{\{0\}}(j') x) \mu_{j'}(\mathrm{d}x) \Bigg]^2 \\
&\quad \le K_1^2 \sum_{(j,l) \in E \times \mathbb{Z}_+} \pi_{j,l} \| p^t_{(j,l),\cdot} - \pi_{\cdot} \|^2_{tv} \\
&\quad = o(t^{-2-\delta}),
\end{aligned}$$

by assumptions (i) and (iii).

Let $\{(X_t, Z_t, L_t);\ t \in \mathbb{Z}_+\}$ have initial distribution γ and let $\{(\tilde{X}_t, \tilde{Z}_t, \tilde{L}_t);\ t \in \mathbb{Z}_+\}$ be an independent copy of $\{(X_t, Z_t, L_t);\ t \in \mathbb{Z}_+\}$, but having initial distribution π. Let

$$\tau = \inf\{t \in \mathbb{Z}_+ : (X_t, L_t) = (\tilde{X}_t, \tilde{L}_t) = (i^*, 0)\}.$$

From [6, Theorem 3.1, Chapter II], assumption (iv) implies that $\tau < \infty$ with probability one. Define $\{(\hat{X}_t, \hat{Z}_t, \hat{L}_t);\ t \in \mathbb{Z}_+\}$ by

$$(\hat{X}_t, \hat{Z}_t, \hat{L}_t) = \begin{cases} (X_t, Z_t, L_t) & \text{if } t \le \tau, \\ (\tilde{X}_t, \tilde{Z}_t, \tilde{L}_t) & \text{if } t \ge \tau. \end{cases}$$

We show, with the obvious notation, that $d^o_\infty(\hat{S}^N_\cdot, \tilde{S}^N_\cdot) \to 0$ in probability for which it suffices to show that $d^o_m(\hat{S}^N_\cdot, \tilde{S}^N_\cdot) \to 0$ for each positive integer m. To this end, note that, for each $t = 0, 1, \dots,$

$$\begin{aligned} |\tilde{S}^N_t - \hat{S}^N_t| &\le \frac{1}{\sqrt{N}} \sum_{n=1}^{Nt} |\tilde{Z}_n - \hat{Z}_n| \\ &= \sum_{n=1}^{\min(Nt,\tau)} |\tilde{Z}_n - \hat{Z}_n| \\ &\le \sum_{n=1}^{\tau} |\tilde{Z}_n - \hat{Z}_n|. \end{aligned}$$

Since the right-hand side of the above equation does not depend on t and

$$d^o_m(\hat{S}^N_\cdot, \tilde{S}^N_\cdot) \le \max_{0 \le t \le m} |\hat{S}^N_t - \tilde{S}^N_t|,$$

we have, for $\varepsilon > 0$,

$$\begin{aligned} \mathrm{P}\{d^o_m(\hat{S}^N_\cdot, \tilde{S}^N_\cdot) > \varepsilon\} &\le \mathrm{P}\bigg\{\frac{1}{\sqrt{N}} \sum_{n=1}^{\tau} |\tilde{Z}_n - \hat{Z}_n| > \varepsilon\bigg\} \\ &\to 0 \end{aligned}$$

since $\sum_{n=1}^{\tau} |\tilde{Z}_n - \hat{Z}_n|$ is a finite valued random variable.

By the first part of the proof, $\tilde{S}^N_\cdot \Longrightarrow \sigma B_\cdot$. By [1, Theorem 3.1], $\hat{S}^N_\cdot \Longrightarrow \sigma B_\cdot$. Since $S^N_\cdot$ and $\hat{S}^N_\cdot$ have the same distribution, we have $S^N_\cdot \Longrightarrow \sigma B_\cdot$, and the proof is complete.

In Section 3, we give sufficient conditions for Theorem 1(iii) and (iv) to hold in terms of the Markov renewal process $\{(J_n, Y_n, T_n);\ n \in \mathbb{Z}_+\}$.

To motivate the calculation of σ^2_Z, consider the special case where E consists of a single point and $Y_n = 1$ for all n. In this case, $W_\cdot$ is the counting process for a stationary renewal process. Then (see, for example, [1, p. 154] or [3, p. 107]), we obtain

$$W^m_\cdot \xrightarrow{\mathrm{D}} \sigma_Z B_\cdot,$$

where $\sigma^2_Z = \sigma^2/m^3$, σ^2 is the variance of the inter-renewal distribution and m is the expected value. We observe that the expression for σ^2_Z in Theorem 1 reduces to σ^2/m^3 in the renewal case. Our method, applied to a renewal counting process, for determining σ^2_Z is to first observe that

$$\sum_{t=1}^{\infty} \mathrm{cov}_\pi(Z_t, Z_0) = \lim_{t\to\infty} \frac{1}{m} \mathrm{E}_0\bigg[W_t - \frac{t}{m}\bigg],$$

where $\mathrm{E}_0[(\cdot)]$ is the expectation for the ordinary renewal process. We then develop a renewal equation for $\mathrm{E}_0[W_t - t/m]$ and apply the key renewal theorem to obtain the limit. For a nonarithmetic renewal process these calculations are well known; see [2, pp. 297–298]. Once we have the infinite sum, it is an easy matter to compute σ_Z^2.

Our derivation of σ_Z^2, in the setting described at the beginning of this section, is accomplished by steps similar to those suggested above. Theorem 3 develops an appropriate Markov renewal theorem and Theorem 4 determines the limiting solution. Using Theorem 4, we determine in Theorem 5 the sum of the covariances and, thus, σ_Z^2.

3. Assumptions (iii) and (iv) of Theorem 1

We elaborate on conditions (iii) and (iv) in Theorem 1 by exploiting coupling arguments for rates of convergence. Following [6], for the process $\{(X_t, L_t);\ t \in \mathbb{Z}_+\}$ consider a version $\{(X_t^{(i,l)}, L_t^{(i,l)});\ t \in \mathbb{Z}_+\}$ having initial distribution $\delta_{(i,l)}$ for each pair (i,l), and a version $\{(X_t^{\pi}, L_t^{\pi});\ t \in \mathbb{Z}_+\}$ having initial distribution π. Fix a state i^* and let

$$T_{(i,l),\pi} = \inf\{t = 0, 1, \dots;\ (X_t^{(i,l)}, L_t^{(i,l)}) = (X_t^{\pi}, L_t^{\pi}) = (i^*, 0)\}.$$

Using [6, Equation (8.3), p. 35], for $\alpha > 1$, we have

$$t^{-\alpha} \| p_{i,l;\cdot}^t - \pi_\cdot \|_{tv}^2 \le 4t^{-\alpha}\, \mathrm{P}\{T_{(i,l),\pi} > t\}$$

and, hence,

$$\sum_{(i,l)\in E\times\mathbb{Z}_+} \pi_{i,l} t^{-\alpha} \| p_{i,l;\cdot}^t - \pi_\cdot \|_{tv}^2 \le 4t^{-\alpha}\, \mathrm{P}\{T_{\pi,\pi} > t\}. \tag{3}$$

Suppose that

$$\mathrm{E}_{\pi}[\tau_0^{\alpha}] < \infty, \qquad \mathrm{E}_{(i^*,0)}[\tau_1^{\alpha}] < \infty,$$

where

$$\tau_0 = \inf\{t = 0, 1, \dots;\ (X_t, L_t) = (i^*, 0)\},$$
$$\tau_1 = \inf\{t = 1, 2, \dots;\ (X_t, L_t) = (i^*, 0)\},$$

then, by [6, proof of Theorem 8.6, Chapter I], the right-hand side of (3) goes to zero as $t \to \infty$. That is, condition (iii) is satisfied. Since

$$\mathrm{E}_{(i^*,0)}[\tau_1] \le \mathrm{E}_{(i^*,0)}[\tau_1^{\alpha}],$$

condition (iv) will also be satisfied.

By construction $\tau_0 = T_{R_0}$ and $\tau_1 = T_{R_1}$, where

$$R_0 = \inf\{n = 0, 1, \dots;\ J_n = i^*\}, \qquad R_1 = \inf\{n = 1, 2, \dots;\ J_n = i^*\}.$$

Theorem 2. *If conditions (iii) and (iv) in Theorem 1 are replaced by the conditions*

(iii′) *with* $\alpha = 2(1+\delta)$

$$m(\alpha) = \sup_{(i,j)} \left\{ \sum_{t\in\mathbb{Z}_+} t^{\alpha+1} \frac{Q(i, \{j\} \times \{t\})}{Q(i, \{j\} \times (0, \infty))} \right\} < \infty,$$

(iii″) $\mathrm{E}_{\boldsymbol{\nu}}[R_0^{\alpha}] < \infty$ *and* $\mathrm{E}_{i^*}[R_1^{\alpha}] < \infty$ *hold,*

and conditions (i) and (ii) of Theorem 1 hold, then so does the conclusion of Theorem 1.

https://doi.org/10.1239/jap/1183667407 Published online by Cambridge University Press

Proof. We need to show that (iii′) and (i″) imply (iii) and (iv) and in light of the above remarks it suffices to show that (iii′) and (i″) imply both $\mathrm{E}_\pi[\tau_0^\alpha]$ and $\mathrm{E}_{(i^*,0)}[\tau^\alpha]$ are finite.

Recalling that $\tau_0 = T_{R_0}$ and using the convexity of t^α, we have

$$\begin{aligned}
&\mathrm{E}_\pi[T_{R_0}^\alpha] \\
&\quad = \sum_{n=0}^{\infty} \sum_{i_0,\dots,i_{n-1} \neq i^*} \mathrm{E}_\pi[T_n^\alpha 1(J_0 = i_0, \dots, J_{n-1} = i_{n-1}, J_n = i^*)] \\
&\quad \le \sum_{n=0}^{\infty} \sum_{i_0,\dots,i_{n-1} \neq i^*} (n+1)^{\alpha-1} \sum_{k=0}^{n} \mathrm{E}_\pi[(T_k - T_{k-1})^\alpha 1(J_0 = i_0, \dots, J_{n-1} = i_{n-1}, J_n = i^*)],
\end{aligned} \tag{4}$$

where $T_{-1} \equiv 0$. Using the initial distribution for J_0, T_0, and (iii′), we obtain

$$\begin{aligned}
&\mathrm{E}_\pi[T_0^\alpha 1(J_0 = i_0, \dots, J_{n-1} = i_{n-1}, J_n = i^*)] \\
&\quad = \sum_{t=1}^{\infty} t^\alpha \frac{\boldsymbol{v}_{i_0}}{\boldsymbol{v} \cdot \boldsymbol{m}} Q(i_0, \{i_1\} \times (t, \infty)) Q(i_1, \{i_2\} \times (0, \infty)) \cdots Q(i_{n-1}, \{i^*\} \times (0, \infty)) \\
&\quad \le \frac{m(\alpha)}{\boldsymbol{v} \cdot \boldsymbol{m}} \boldsymbol{v}_{i_0} Q(i_0, \{i_1\} \times (0, \infty)) Q(i_1, \{i_2\} \times (0, \infty)) \cdots Q(i_{n-1}, \{i^*\} \times (0, \infty)).
\end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}
&\mathrm{E}_\pi[T_0^\alpha 1(J_0 = i_0, \dots, J_{n-1} = i_{n-1}, J_n = i^*)] \\
&\quad \le \frac{m(\alpha)}{\boldsymbol{v} \cdot \boldsymbol{m}} \boldsymbol{v}_{i_0} Q(i_0, \{i_1\} \times (0, \infty)) Q(i_1, \{i_2\} \times (0, \infty)) \cdots Q(i_{n-1}, \{i^*\} \times (0, \infty)).
\end{aligned} \tag{6}$$

Inserting (5) and (6) into (4) and applying (i″) gives

$$\begin{aligned}
\mathrm{E}_\pi[T_{R_0}^\alpha] &\le \frac{m(\alpha)}{\boldsymbol{v} \cdot \boldsymbol{m}} \sum_{n=0}^{\infty} \sum_{i_0,\dots,i_{n-1} \neq i^*} (n+1)^\alpha \boldsymbol{v}_i Q(i_0, \{i_1\} \times (0, \infty)) Q(i_1, \{i_2\} \times (0, \infty)) \cdots \\
&\qquad\qquad \times Q(i_{n-1}, \{i^*\} \times (0, \infty)) \\
&= \frac{m(\alpha)}{\boldsymbol{v} \cdot \boldsymbol{m}} \mathrm{E}_{\boldsymbol{v}}[(R_0 + 1)^\alpha] \\
&\le 2^{\alpha-1} \frac{m(\alpha)}{\boldsymbol{v} \cdot \boldsymbol{m}} \mathrm{E}_{\boldsymbol{v}}[R_0^\alpha] \le \infty.
\end{aligned}$$

Showing $\mathrm{E}_{(i^*,0)}[T_{R_1}^\alpha] < \infty$ is similar, and the proof is complete.

4. Calculating the asymptotic variance

In this section we calculate the variance term given in (1) as an application of the key renewal theorem.

For $k \in E$ and $t = 0, 1, \dots$, let

$$N_t^k = \sum_{n=0}^{\infty} 1_{\{J_n = k\}} 1_{\{0 \le T_n \le t\}}.$$

https://doi.org/10.1239/jap/1183667407 Published online by Cambridge University Press

The process $\{N_t^k;\ t = 0, 1, \dots\}$ counts the number of times that the Markov renewal process $\{(J_n, T_n);\ n = 0, 1, 2, \dots\}$ visits the state k. Let

$$m_k = \mathrm{E}_k[T_1], \qquad \sigma_k^2 = \mathrm{E}_k[(T_1 - m_k)^2].$$

Now define

$$\delta_{i,k} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Finally, for each k, define the function $h : E \times \{0, 1, \dots\} \to \mathbb{R}$ by

$$h_k(i, t) = \mathrm{E}_i\left[N^k(t) - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t\right].$$

Theorem 3. *The function h_k satisfies the Markov renewal equation*

$$h_k(i, t) = \delta_{i,k} - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} \sum_{s=0}^{t-1} Q(i, E \times (s, \infty)) + \sum_{s=1}^{t} \sum_{j \in E} h_k(j, t - s) Q(i, \{j\} \times \{s\}).$$

Proof. For $t = 0, 1, \dots$, we have

$$\mathrm{E}_i\left[N_t^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t\right] = \mathrm{E}_i\left[\left(N_t^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t\right) 1_{\{T_1 > t\}}\right] + \mathrm{E}_i\left[\left(N_t^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t\right) 1_{\{T_1 \leq t\}}\right]. \tag{7}$$

When $T_1 > t$, we have

$$N_t^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t = \delta_{i,k} - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t,$$

so that the first term on the right-hand side of (7) is equal to

$$\left(\delta_{i,k} - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t\right) \mathrm{P}_i\{T_1 > t\} = \left(\delta_{i,k} - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} t\right) Q(i, E \times (t, \infty)). \tag{8}$$

We rewrite the second term on the right-hand side of (7) as

$$\begin{aligned} &\mathrm{E}_i\left[\left(N_{T_1-}^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} T_1 + N_t^k - N_{T_1-}^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} (t - T_1)\right) 1_{\{T_1 \leq t\}}\right] \\ &\quad = \sum_{s=1}^{t} \sum_{j \in E} \left\{\left(\delta_{i,k} - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot m} s\right) + \mathrm{E}_j\left[N_{t-s}^k - \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} (t - s)\right]\right\} Q(i, \{j\} \times \{s\}) \\ &\quad = \delta_{i,k} Q(i, E \times [0, t]) - \sum_{s=1}^{t} \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} s Q(i, E \times \{s\}) \\ &\qquad + \sum_{s=1}^{t} \sum_{j \in E} h_k(j, t - s) Q(i, \{j\} \times \{s\}). \end{aligned} \tag{9}$$

Inserting (8) and (9) into (7) gives

$$\begin{aligned} h_k(i, t) = \delta_{i,k} &- \frac{\boldsymbol{v}_k}{\boldsymbol{v} \cdot \boldsymbol{m}} \left(t Q\left(i, E \times (t, \infty)\right) + \sum_{s=1}^{t} s Q(i, E \times \{s\})\right) \\ &+ \sum_{s=1}^{t} \sum_{j \in E} h_k(j, t - s) Q(i, \{j\} \times \{s\}). \end{aligned} \tag{10}$$

Applying the summation by parts formula, we obtain

$$\sum_{s=1}^{t} sQ(i, E \times \{s\}) = tQ(i, E \times [0, t]) - \sum_{s=1}^{t} Q(i, E \times [0, s-1]),$$

which gives

$$\begin{aligned} tQ(i, E \times (t, \infty)) &+ \sum_{s=1}^{t} sQ(i, E \times \{s\}) \\ &= t(1 - Q(i, E \times [0, t])) + tQ(i, E \times [0, t]) - \sum_{s=1}^{t} Q(i, E \times [0, s-1]) \\ &= t - \sum_{s=1}^{t} Q(i, E \times [0, s-1]) \\ &= \sum_{s=0}^{t-1} (1 - Q(i, E \times [0, s])) \\ &= \sum_{s=0}^{t-1} Q(i, E \times (s, \infty)). \end{aligned} \tag{11}$$

Inserting (11) into the second term on the right-hand side of (10) yields

$$h_k(i, t) = \delta_{i,k} - \frac{\boldsymbol{\nu}_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \sum_{s=0}^{t-1} Q(i, E \times (s, \infty)) + \sum_{s=1}^{t} \sum_{j \in E} h_k(j, t-s) Q(i, \{j\} \times \{s\}),$$

which completes the proof.

Theorem 4. *For $k \in E$, we have*

$$\lim_{t \to \infty} h_k(i, t) = \frac{\boldsymbol{\nu}_k}{(\boldsymbol{\nu} \cdot \boldsymbol{m})^2} \sum_{j \in E} \boldsymbol{\nu}_j \frac{\sigma_j^2 + m_j^2 + m_j}{2}.$$

Proof. Set

$$g_k(i, t) = \delta_{i,k} - \frac{\boldsymbol{\nu}_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \sum_{s=0}^{t-1} Q(i, E \times (s, \infty)).$$

Then the key renewal theorem [2, Theorem 9.2.8] implies that

$$\begin{aligned} \lim_{t \to \infty} h_k(i, t) &= \sum_{t=0}^{\infty} \sum_{j \in E} \frac{\boldsymbol{\nu}_j}{\boldsymbol{\nu} \cdot \boldsymbol{m}} g_k(j, t) \\ &= \sum_{t=0}^{\infty} \sum_{j \in E} \frac{\boldsymbol{\nu}_j}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \left(\delta_{j,k} - \frac{\boldsymbol{\nu}_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \sum_{s=0}^{t-1} Q(j, E \times (s, \infty)) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{\infty} \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}}\Bigl(1 - \sum_{j\in E} \frac{\boldsymbol{v}_j}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{s=0}^{t-1} Q(j, E\times(s,\infty))\Bigr) \\
&= \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{t=0}^{\infty}\sum_{j\in E} \frac{\boldsymbol{v}_j m_j}{\boldsymbol{v}\cdot\boldsymbol{m}}\Bigl(1 - \frac{1}{m_j}\sum_{s=0}^{t-1} Q(j, E\times(s,\infty))\Bigr) \\
&= \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{j\in E} \frac{\boldsymbol{v}_j m_j}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{t=0}^{\infty} \frac{1}{m_j} \sum_{s=t}^{\infty} Q(j, E\times(s,\infty)) \\
&= \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{j\in E} \frac{\boldsymbol{v}_j}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{s=0}^{\infty}\sum_{t=0}^{s} Q(j, E\times(s,\infty)) \\
&= \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{j\in E} \frac{\boldsymbol{v}_j}{\boldsymbol{v}\cdot\boldsymbol{m}} \sum_{s=0}^{\infty} (s+1) Q(j, E\times(s,\infty)) \\
&= \frac{\boldsymbol{v}_k}{(\boldsymbol{v}\cdot\boldsymbol{m})^2} \sum_{j\in E} \boldsymbol{v}_j \frac{\sigma_j^2 + m_j^2 + m_j}{2},
\end{aligned}
$$

which completes the proof.

Set

$$
\xi_k = \mathrm{E}[Y_0 \mid J_0 = k, T_0 = 0] = \int_{\mathbb{R}} x\,\mu_k(\mathrm{d}x),
$$

$$
\xi_{k,2} = \mathrm{E}[Y_0^2 \mid J_0 = k, T_0 = 0] = \int_{\mathbb{R}} x^2\,\mu_k(\mathrm{d}x),
$$

and

$$
\xi = \sum_{k=1}^{K} \xi_k \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}}.
$$

Since the process $\{Z_t;\ t = 0, 1, \dots\}$ is assumed to be stationary, we have

$$
\xi = \mathrm{E}[Z_0].
$$

Theorem 5. *If*

$$
\sum_{i\in E}\sum_{k\in E} \sup_{t\geq 0} \boldsymbol{v}_i |\xi_i \xi_k h_k(i,t)| < \infty
$$

then

$$
\begin{aligned}
\sigma_Z^2 = &\sum_{k\in E} \frac{\boldsymbol{v}_k}{\boldsymbol{v}\cdot\boldsymbol{m}} \xi_{k,2} - \sum_{j\in E}\sum_{k\in E} \frac{\boldsymbol{v}_j \boldsymbol{v}_k}{(\boldsymbol{v}\cdot\boldsymbol{m})^2} \xi_j \xi_k \\
&+ \sum_{i\in E}\sum_{j\in E}\sum_{k\in E} \frac{\boldsymbol{v}_i \boldsymbol{v}_j \boldsymbol{v}_k}{(\boldsymbol{v}\cdot\boldsymbol{m})^3} \xi_i \xi_k (\sigma_j^2 + m_j^2 + m_j) - 2\sum_{i\in E} \frac{\boldsymbol{v}_i}{\boldsymbol{v}\cdot\boldsymbol{m}} \xi_i^2.
\end{aligned}
$$

Proof. Recall that

$$
\sigma_Z^2 = \mathrm{E}[(Z_0 - \xi)^2] + 2\sum_{t=1}^{\infty} \mathrm{E}[(Z_0 - \xi)(Z_t - \xi)]. \tag{12}
$$

To calculate the first term on the right-hand side of (12) we first note that

$$\mathrm{E}[Z_0^2] = \sum_{k \in E} \frac{\nu_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \xi_{k,2}$$

and

$$\mathrm{E}[Z_0]^2 = \xi^2 = \bigg(\sum_{k \in E} \xi_k \frac{\nu_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}}\bigg)^2.$$

It then follows that

$$\mathrm{E}[(Z_0 - \xi)^2] = \sum_{k \in E} \frac{\nu_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \xi_{k,2} - \sum_{j \in E} \sum_{k \in E} \frac{\nu_j \nu_k}{(\boldsymbol{\nu} \cdot \boldsymbol{m})^2} \xi_j \xi_k. \tag{13}$$

To calculate the infinite sum in (12), we note that

$$\begin{aligned}
\mathrm{E}[(Z_0 - \xi)(Z_t - \xi)] &= \mathrm{E}[Z_0(Z_t - \xi)] \\
&= \sum_{i \in E} \xi_i \frac{\nu_i}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \, \mathrm{E}[Z_t - \xi \mid J_0 = i, T_0 = 0] \\
&= \sum_{i \in E} \xi_i \frac{\nu_i}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \, \mathrm{E}_i[Z_t - \xi].
\end{aligned} \tag{14}$$

Since

$$\begin{aligned}
\mathrm{E}_i[Z_t] &= \sum_{k \in E} \mathrm{E}_i[Y_t(N_t^k - N_{t-1}^k)] \\
&= \sum_{k \in E} \xi_k \, \mathrm{E}_i[N_t^k - N_{t-1}^k],
\end{aligned}$$

the right-hand side of (14) is equal to

$$\sum_{i \in E} \xi_i \frac{\nu_i}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \sum_{k \in E} \xi_k \, \mathrm{E}_i\bigg[N_t^k - N_{t-1}^k - \frac{\nu_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}}\bigg].$$

Using (14) and Theorem 4, the second term in (12) is equal to

$$\begin{aligned}
&2 \lim_{n \to \infty} \sum_{t=1}^{n} \sum_{i \in E} \frac{\nu_i}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \xi_i \sum_{k \in E} \xi_k \, \mathrm{E}_i\bigg[N_t^k - N_{t-1}^k - \frac{u_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}}\bigg] \\
&\quad = 2 \sum_{i \in E} \sum_{k \in E} \frac{\nu_i}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \xi_i \xi_k \bigg(\lim_{t \to \infty} \mathrm{E}_i\bigg[N_t^k - \frac{t \nu_k}{\boldsymbol{\nu} \cdot \boldsymbol{m}}\bigg] - \delta_{i,k}\bigg) \\
&\quad = \sum_{i \in E} \sum_{j \in E} \sum_{k \in E} \frac{\nu_i \nu_j \nu_k}{(\boldsymbol{\nu} \cdot \boldsymbol{m})^3} \xi_i \xi_k (\sigma_j^2 + m_j^2 + m_j) - 2 \sum_{i \in E} \frac{\nu_i}{\boldsymbol{\nu} \cdot \boldsymbol{m}} \xi_i^2,
\end{aligned} \tag{15}$$

where in the first equality we applied the dominated convergence theorem which applies by assumption. Inserting (13) and (15) into (12) gives the desired expression for σ_Z^2.

https://doi.org/10.1239/jap/1183667407 Published online by Cambridge University Press

Acknowledgement

We are grateful to an anonymous referee for carefully reading the paper and for making insightful comments and suggestions that led to a better paper.

References

[1] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*. John Wiley, New York.
[2] ÇINLAR, E. (1975). *Introduction to Stochastic Processes*. Prentice Hall, Englewood Cliffs, NJ.
[3] DURRETT, R. (1991). *Probability: Theory and Examples*. Wadsworth and Brooks, Pacific Grove, CA.
[4] KEILSON, J. AND WISHART, D. M. G. (1964). A central limit theorem for processes defined on a Markov chain. *Proc. Camb. Philos. Soc.* **60,** 547–567.
[5] LIMNIOS, N. AND OPRIŞAN, G. (2001). *Semi-Markov Processes and Reliability*. Birkhäuser, Boston, MA.
[6] LINDVALL, T. (1992). *Lectures on the Coupling Method*. John Wiley, New York.
[7] MEYN, S. P. AND TWEEDIE, R. L. (1993). *Markov Chains and Stochastic Stability*. Springer, New York.
[8] PITMAN, J. W. (1974). Uniform rates of convergence for Markov chain transition probabilities. *Z. Wahrscheinlichkeitsth.* **29,** 193–227.
[9] RESNICK, S. I. (1992). *Adventures in Stochastic Processes*. Birkhäuser, Boston, MA.
[10] WHITT, W. (2002). *Stochastic-Process Limits*. Springer, New York.
[11] WHITT, W. (2002). *Stochastic-Process Limits*. Internet Supplement.