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Third Engel groups and the Macdonald-Neumann conjecture

S. Bachmuth and H.Y. Mochizuki

There exists a non-solvable group which is third Engel. More generally, the existence of a non-solvable group in which every n-generator subgroup is nilpotent of class at most $2n - 1$ is confirmed.

1. Introduction

G is an Engel group if for any elements x, y in G, the commutator $[x, y, y, \ldots, y] = 1$ where y is repeated $n(x, y)$ times and $n(x, y)$ may depend on x and y. If $n = n(x, y)$ is independent of x and y, then G is said to be a *bounded Engel group of degree* n or simply an n-th *Engel group*.

The first example of a non-solvable Engel group was a consequence of the work of Golod-Šafarevič [2]. (See the example in Herstein [7], p. 124.) The group constructed in Section 4 of [1] is the first example of a nonsolvable bounded Engel group, although the exact value of the degree n was not determined in [1]. This value has now been determined to be three, and reasons for its interest will now be advanced.

In the above notation, if $n = 1$, G is abelian, and if $n = 2$, G is metabelian. Also well-known is the fact that bounded Engel 2-groups need not be nilpotent. However, Gupta in [3] has shown that third Engel 2-groups are solvable. His work is based on results of Gupta and Weston [4] on groups of exponent 4 and results of Heineken [5] on third Engel groups, and he actually shows that a third Engel group is the extension of

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a solvable group by a group of exponent 5 . Thus, the problem of whether third Engel groups are solvable reduces to the corresponding question for 5-groups. The existence of such a nonsolvable group had already been conjectured by Macdonald and Neumann in [10], p. 557. We show

THEOREM 1. *There exists a third Engel group of exponent* 5 *which is not solvable.*

COROLLARY 1. *There exists a nonsolvable group all of whose two generator subgroups are nilpotent of class (at most) three.*

COROLLARY 2. *The variety of* 2*-metabelian groups (that is, each two generator subgroup is metabelian) is a nonsolvable variety.*

The Macdonald-Neumann conjecture is that there exist $[2 \to 3]$ 5-groups of nilpotency class m for each $m \geq 1$. An $[n \to k]$ *group* is a group such that every n-generator subgroup is nilpotent of class $\leq k$. Theorem 1 is equivalent to the Macdonald-Neumann conjecture as pointed out by Gupta in [3]. Direct verification of the conjecture readily follows by using Zusatz 2 of Heineken [6] which states that the group of Theorem 1 is automatically a $[2 \to 3]$ group. Hence the Corollary 1. In relation to the group of Theorem 1, we have in fact the more general

THEOREM 2. *A third Engel group of exponent* 5 *is an* $[n \to (2n-1)]$ *group. In particular the group of Theorem* 1 *is a nonsolvable* $[n \to (2n-1)]$ *group.*

Gupta has shown that groups of type $[n \to (2n-2)]$ are nilpotent by abelian and hence solvable. (In fact, according to Neumann [11], p. 98, Heineken has shown that groups of this type without elements of order 2 are nilpotent of class at most $3n - 3$.) Furthermore, Gupta has shown that $[n \to (2n-1)]$ groups are extensions of a solvable group by a group of exponent 5 . Thus Theorem 2 points out where the dichotomy occurs between solvability and nonsolvability for groups of type $[n \to k]$.

2. Some required notation and results

Let R be the free associative noncommutative ring of characteristic 5 with identity generated by indeterminates $x_1, x_2, x_3, \ldots$. Let L be the Lie ring in R generated by the x_i where addition in L is the same as in R and Lie multiplication in L is commutation $[x, y] = xy - yx$.

Let H denote the ideal of R generated by all elements

$$g(x, y) = x^2y - 3xyx + 3yx^2 ,$$

where x, y are any elements of L, or, alternately, by all elements

$$h(x, y, z) = xyz + zyx + 2yzx + 2yxz ,$$

where x, y, z are arbitrary elements of L. H is a Lie substitution ideal, which means that if a polynomial P in the x_i is in H, so is the polynomial P' obtained from P by substituting elements of L for the x_i. Let U be the ideal of R generated by all monomials in the x_i with a repeated indeterminate factor. We state the main result of [1].

THEOREM A. *In* $R/(H+U)$ *, let* G *be the group generated by the* $(1+x_i)$ mod $(H+U)$ *. Then* G *is a nonsolvable group of exponent* 5 *.*

In the next section, we will show that G is third Engel.

Let $y_1, y_2, \ldots, y_n$ be distinct indeterminates, $n \geq 3$, and let $T_3(y_1, y_2, \ldots, y_n)$ be the multilinear (degree 1 in each y_i) part of

$$[(1+y_1)(1+y_2) \ldots (1+y_n)-1]^3 .$$

It readily follows that

$$(1) \quad T_3(\ldots, y_i, y_{i+1}, \ldots) = T_3(\ldots, y_{i+1}, y_i, \ldots) + T_3(\ldots, [y_i, y_{i+1}], \ldots) .$$

If T is the ideal of R generated by the $T_3(z_1, \ldots, z_n)$, $n \geq 3$, where the z_i are in L, then $T \subseteq H$ (Lemma 4 of [1]). As a result (see Section 4 of [1]), we have $(g-1)^3 = 0$ for all $g \in G$.

We will also need the following three theorems.

THEOREM B. *In* R

(i) *a monomial of the form* $M_1x_iM_2x_iM_3$ *is congruent to* αPx_i^2 mod H *, for all indeterminates* x_i *, where* P *is a polynomial in the* x_j *and* α *is an integer modulo* 5 *,*

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(ii) modulo H, x_i^2 *commutes with all elements of* R *, and*

(iii) if a monomial M *in the* x_i *has an indeterminate factor repeated three or more times, then* $M \equiv 0 \mod H$.

THEOREM C (Heineken, Hauptsatz 3 of [5]). *In a group without elements of even order, the set of all elements* g *satisfying* $[g^{\pm 1}, y, y, y] = 1$ *for all* y *in the group forms a subgroup.*

THEOREM D. *The associated Lie ring of a third Engel group of exponent* 5 *is a third Engel Lie ring of characteristic* 5 .

All parts of Theorem B are trivial consequences of the results or methods in Section 3 of [1]. A proof of Theorem D readily follows from the same arguments used in Theorem 4 of Higman [9]. Theorem C is proved by Heineken using right-normed notation. But as shown by Lemma 1 of [10], the third Engel conditions in the right-normed and the left-normed notations are equivalent, that is,

$\left[y, [y, [y, x]]\right] = 1$ if and only if $[x, y, y, y] = 1$ for elements x and y in a group G .

3. Proof of Theorem 1

Let G be the group of Theorem A in Section 2. To prove Theorem 1 we need to show that G is a third Engel group. By Theorem C, we need to prove that $[g, y, y, y] = 1$ for all $y \in G$ where g is a generator of G . Put $g = 1 + x$ and $y = 1 + P$ where x is an indeterminate (generator of R) and P is a polynomial in the x_i .

LEMMA 1. $[g, y, y, y] \equiv 1 + PxP^2 + P^2xP \mod (H+U)$.

Proof. $g^{-1} = 1 - x$ and $y^{-1} = 1 - P + P^2$ modulo $(H+U)$, since $x^2 \equiv P^3 \equiv 0 \mod (H+U)$. Thus, all terms omitted from $[g, y, y, y]$ either have two occurrences of x or have a factor of P^3 .

LEMMA 2. $PxP^2 \equiv P^2xP \equiv 0 \mod (H+U)$.

Proof. Modulo U , PxP^2 is the sum of polynomials $T_3^*\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ where the x's are indeterminates and

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$T_3^*(x_1, x_2, \ldots, x_k)$ is the sum of monomials $M_1xM_2M_3$, M_i coming from the i-th factor of P and $M_1M_2M_3$ having degree one in x_i , $1 \le i \le k$. Note that the sum of the monomials $M_1M_2M_3$ is $T_3(x_1, x_2, \ldots, x_k)$. We need only prove that $T_3^*(x_1, x_2, \ldots, x_k) \equiv 0 \mod H$ for $k \ge 3$. We proceed by induction on k . For $k = 3$, we have

$$T_3^*(x_1, x_2, x_3) = x_1xx_2x_3 + x_1xx_3x_2 + x_2xx_1x_3 + x_2xx_3x_1 + x_3xx_1x_2 + x_3xx_2x_1 .$$

But the sum of these six monomials is in H because it is just the linearization of $x_1xx_1x_1$ which is certainly in H from part *(ii)* or *(iii)* of Theorem B. Therefore, assume inductively that $T_3^*(x_1, x_2, \ldots, x_k) \equiv 0 \mod H$. Since H is a substitution ideal, $T_3^*(z_1, z_2, \ldots, z_k) \equiv 0 \mod H$ with $z_i \in L$. We need the equation

$$(2)\quad T_3^*(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{k+1}) - T_3^*(x_1, \ldots, x_{i+1}, x_i, \ldots, x_{k+1}) = T_3^*(x_1, \ldots, x_ix_{i+1}-x_{i+1}x_i, \ldots, x_{k+1}) .$$

(The proof of (2) is essentially the same as that for (1).)

Notice that the expression on the right-hand side of (2) has fewer arguments and so by our induction hypothesis is in H . Using this, we have

$$\begin{aligned}T_3^*(x_1, x_2, x_3, \ldots, x_{k+1}) &= 6T_3^*(x_1, x_2, x_3, \ldots, x_{k+1}) \\ &\equiv T_3^*(x_1, x_2, x_3, \ldots, x_{k+1}) + T_3^*(x_1, x_3, x_2, \ldots, x_{k+1}) \\ &\quad + T_3^*(x_2, x_1, x_3, \ldots, x_{k+1}) + T_3^*(x_2, x_3, x_1, \ldots, x_{k+1}) \\ &\quad + T_3^*(x_3, x_1, x_2, \ldots, x_{k+1}) + T_3^*(x_3, x_2, x_1, \ldots, x_{k+1}) \mod H .\end{aligned}$$

But again the six terms on the right-hand side are $\equiv 0 \mod H$, since they are the linearization of

$$T_3^*(x, x, x, x_4, \ldots, x_{k+1}) \equiv 0 \mod H .$$

This completes the induction. By symmetry $P^2xP \equiv 0 \mod H$. This proves

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Lemma 2 and hence Theorem 1.

4. Proof of Theorem 2

Let G be the free third Engel n-generator group of exponent 5. Theorem 1 and the results of Heineken and Gupta show that G is nilpotent of class $\geq 2n-1$. To show that G has class $\leq 2n-1$, it is sufficient to prove that the associated Lie ring of G is nilpotent of class $\leq 2n-1$. As a result of Theorem D we need to prove

LEMMA 3. *The free third Engel n-generator Lie ring L of characteristic 5 is nilpotent of class $\leq (2n-1)$.*

Proof. Let $a_1, a_2, \ldots, a_n$ generate L, and let R be the associative subring of the endomorphism ring of $(L, +)$ generated by the endomorphisms

$$A_i = \mathrm{ad}a_i : x \to [x, a_i]$$

together with the identity map. According to Higgins [8], R is a homomorphic image of the ring R/H of Section 2.

To prove that L is nilpotent of class $\leq 2n-1$, it is sufficient to prove that every left-normed product

$$c = \left[a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_{2n}}\right] = \left[\ldots\left[\left[a_{i_1}, a_{i_2}\right], a_{i_3}\right], \ldots, a_{i_{2n}}\right] = 0 .$$

We consider three cases:

(i) Some a_{i_j}, $j \geq 2$, appears at least three times in c. Then

$$c = a_{i_1}A_{i_2} \ldots A_{i_{2n}}$$
$$= 0 \text{ by } (iii) \text{ of Theorem B.}$$

(ii) a_{i_1} appears at least three times in c. Then

$$c = \alpha a_{i_1}A_{i_1}^2 P \text{ by } (i) \text{ and } (ii) \text{ of Theorem B}$$
$$= 0 .$$

(iii) Each a_i appears twice in c. Then

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$$c = \alpha a_{i_1} A_{i_1} M$$
$$= 0$$

where M is the product of the squares of the other indeterminates (by *(i)* and *(ii)* of Theorem B).

This exhausts all cases and completes the proof of Lemma 3.

References

[1] S. Bachmuth, H.Y. Mochizuki, D.W. Walkup, "Construction of a nonsolvable group of exponent 5", *Word problems* (edited by W.W. Boone, R.C. Lyndon, F.B. Cannonito. North-Holland, Amsterdam; in press).

[2] E.S. Golod, "On nil-algebras and residually finite p-groups" (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* 28 (1964), 273-276.

[3] N.D. Gupta, "Third Engel 2-groups are soluble", *Canad. Math. Bull.* (to appear).

[4] Narain D. Gupta and Kenneth W. Weston, "On groups of exponent four", *J. Algebra* 17 (1971), 59-66.

[5] Hermann Heineken, "Engelsche Elemente der Länge drei", *Illinois J. Math.* 5 (1961), 681-707.

[6] Hermann Heineken, "Über ein Levisches Nilpotenzkriterium", *Arch. Math.* 12 (1961), 176-178.

[7] I.N. Herstein, *Topics in ring theory* (Lecture notes, Univ. Chicago, Chicago, 1965).

[8] P.J. Higgins, "Lie rings satisfying the Engel condition", *Proc. Cambridge Philos. Soc.* 50 (1954), 8-15.

[9] Graham Higman, "On finite groups of exponent five", *Proc. Cambridge Philos. Soc.* 52 (1956), 381-390.

[10] I.D. Macdonald and B.H. Neumann, "A third-Engel 5-group", *J. Austral. Math. Soc.* 7 (1967), 555-569.

[11] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

Department of Mathematics,
University of California,
Santa Barbara,
California, USA.

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