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CONDITIONS ON NEAR-RINGS WHICH IMPLY THAT NIL N-SUBGROUPS ARE NILPOTENT

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1. Introduction

We assume the reader to be familiar with the basic definitions of near-rings, N-subgroups etc. as presented, for example, in (**4**). Throughout, N will denote a left near-ring (i.e. $a, b, c \in N$ imply $a(b+c) = ab + ac$) in which $0n = 0$ for each $n \in N$. We say that N is *strictly semiprime* if $A^2 = (0)$ implies $A = (0)$ where A is an N-subgroup of N. An N-subgroup A is *nilpotent* if $A^n = (0)$ for some positive integer n and an element $a \in N$ is *nil* if $a^n = 0$ for some n. An element $a \in N$ is *regular* if $ax = 0$ or $xa = 0$ implies $x = 0$.

Theorem 1. *N is strictly semiprime if and only if N has no nonzero nilpotent N-subgroups.*

Proof. We need only suppose that N is strictly semiprime and that A is a nonzero nilpotent N-subgroup. Then for some $k \geqslant 3$, $A^k = (0)$ whilst $A^{k-1} \neq (0)$. Hence we can choose $a_1, a_2, \ldots, a_{k-1} \in A$ with $a_1a_2 \ldots a_{k-1} \neq 0$. But then $(a_1a_2 \ldots a_{k-2}A)^2 \subseteq A^{2k-2} \subseteq A^k = (0)$ from which we see that $a_1a_2 \ldots a_{k-2}A = (0)$ contrary to assumption.

Theorem 2. *If the near-ring N is distributively generated by S, or has a regular element, then N is strictly semiprime if and only if $xNx = (0)$ implies $x = 0$.*

Proof. Suppose $xNx = (0)$ implies $x = 0$ and A is an N-subgroup of N with $A^2 = (0)$. If $a \in A$ then $aNa = (0)$ and so $a = 0$ and $A = (0)$. Conversely, if N is strictly semiprime it has no nilpotent N-subgroups by Theorem 1. If N has a regular element then $xNx = (0)$ implies $(xN)^2 = (0)$ and thus $xN = (0)$ from which $x = 0$. Alternatively, if N is distributively generated the result follows by (**5**; Lemma 14).

We wish to establish some results concerning nil and nilpotent N-subgroups analogous to those in ring theory.

2. Nilpotent and nil N-subgroups

Theorem 3. *If N is a strictly semiprime near-ring with the maximum condition on right annihilators and if N is either distributively generated or has a regular element then a nil N-subgroup of N is zero.*

Proof. If A is a nil N-subgroup and $a \in A$ with $a \neq 0$ then $Na \neq (0)$. Choose $ta \in Na$ with $r(ta)$ maximal among all $r(za)$ with $za \neq 0$. If $x \in N$ then $xta \in Na$ and either $xta = 0$ or, for some $k > 1$, $(xta)^k = 0 \neq (xta)^{k-1}$. Since $r(ta) \subseteq r((xta)^{k-1})$, if $xta \neq 0$ we have $r(ta) = r((xta)^{k-1})$ and in either case $(ta)x(ta) = 0$. Thus $taNta = (0)$ yielding $ta = 0$ which is false. It follows that $A = (0)$ as required.

If N is a near-ring with the maximum condition on right ideals the family of nilpotent ideals of N will contain a maximal element.

Ramakotaiah (**6**; 3.3) proved that the sum of two nilpotent ideals is nilpotent and hence there will be a unique maximal nilpotent ideal which we denote by W.

Theorem 4. *If N has an identity or is distributively generated, if W contains all the nilpotent N-subgroups of N and if N has the maximum condition on right ideals then a nil N-subgroup is nilpotent.*

Proof. Let $\bar{N} = N/W$. If N has an identity so also has $\bar{N}$ and if N is distributively generated then so is $\bar{N}$. A right annihilator in $\bar{N}$ is the image of a right ideal in N under the canonical homomorphism of N to $\bar{N}$ and thus $\bar{N}$ has the maximum condition on right annihilators. By Theorem 1, $\bar{N}$ is strictly semiprime and then by Theorem 3 each nil N-subgroup is zero. But if I is a nil N-subgroup of N its image $\bar{I}$ is a nil $\bar{N}$-subgroup of $\bar{N}$ and so $I \subseteq W$. Since W is nilpotent, I is nilpotent.

In the case when N is distributive so that $a, b, c \in N$ imply $(a+b)c = ac + bc$ it is easy to see that every nilpotent N-subgroup is contained in a nilpotent ideal and hence in W. Trivially, a distributive near-ring is distributively generated and so

Corollary. *If N is distributive with the maximum condition on right ideals then every nil N-subgroup is nilpotent.*

For a near-ring N we define the *distributor ideal* to be the ideal generated by all elements of N of the form $(a+b)c - bc - ac$ and denote it by $D(N)$. N is distributive if $D(N) = (0)$. Inductively we now define a *distributor series* $\{D^k(N)\}$ by $D^1(N) = D(N)$, $D^k(N)$ is the ideal generated by the elements $(u+v)c - vc - uc$ where $u, v \in D^{k-1}(N)$ and $c \in N$. Clearly $D^1(N) \supseteq D^2(N) \supseteq \cdots \supseteq D^k(N) \supseteq \cdots$. Then N is *weakly distributive* if $D^n(N) = 0$ for some n.

Theorem 5. *For each positive integer k, $D(N)^k \subseteq D^k(N)$.*

Proof. The result is evident for $k = 1$. Suppose $D(N)^{k-1} \subseteq D^{k-1}(N)$. If $X = \{(a+b)c - bc - ac \colon a, b, c \in N\}$ and $u \in D(N)^{k-1}$ we see that, working modulo $D^k(N)$, $uX = (0)$. Since $D(N)^{k-1}N \subseteq D(N)^{k-1}$ we have $X \subseteq r(D(N)^{k-1})$ where $r(D(N)^{k-1})$ is an ideal of N. Thus $D(N) \subseteq r(D(N)^{k-1})$ and thus $D(N)^k \subseteq D^k(N)$.

Corollary 1. *If N is weakly distributive then $D(N)$ is nilpotent.*

Combining this with the corollary to Theorem 4 and the observation that $N/D(N)$ is a distributive near-ring we have

Corollary 2. *If N is weakly distributive with the maximum condition on right ideals every nil N-subgroup is nilpotent.*

Corollary 3. *If N is a distributively generated near-ring whose additive group is soluble and if N has the maximum condition on right ideals then every nil N-subgroup of N is nilpotent.*

Proof. Since the additive group of N is soluble it follows from (**1**; 4.4.5) that N is weakly distributive. The result now follows from Corollary 2.

3. Nil subnear-rings

For rings, Herstein and Small (**2**) proved that the maximum condition on both right and left annihilators was sufficient to ensure that nil subrings were nilpotent. In (**3**) there is a proof of this result which can be modified to establish the corresponding result for near-rings. Because of a corollary that we wish to establish as a consequence of this result, we have made more extensive modifications than are necessary.

Theorem 6. *If N has the maximum condition on both left and right annihilators, a nil subnear-ring of N is nilpotent.*

Proof. Let P be a nil subnear-ring of N which is not nilpotent and $K = r(P^t) = r(P^{t+j})$ for each $j \geq 0$. Define $P_1 = \{x \in P : x \notin K\}$. Certainly P_1 is non-empty. If $z \in P_1$ implies $zP \subseteq K$ then $z \in P$ implies $P^t zP = (0)$ and $P^{t+2} = (0)$ which is false. Thus $\mathcal{P}_1 = \{r(x) : x \in P_1,\ xP \nsubseteq K\}$ is non-empty so we can choose $x_1 \in P_1$ with $r(x_1)$ maximal in $\mathcal{P}_1$. Notice that $Px_1 \nsubseteq K$. Now suppose we have defined $P_1, P_2, \ldots, P_k$; $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ and $x_1, x_2, \ldots, x_k$ with $x_i \in P_i$, $r(x_i)$ maximal in $\mathcal{P}_i$ and $P_i = \{x \in P:\ xx_{i-1} \ldots x_1 \notin K\} \neq \emptyset$. $\mathcal{P}_i = \{r(x):\ x \in P_i,\ xx_{i-1} \ldots x_1 P \nsubseteq K\} \neq \emptyset$. Then $Px_k x_{k-1} \ldots x_1 \subseteq K$ implies $P^{t+1} x_k x_{k-1} \ldots x_1 = (0)$ and $x_k x_{k-1} \ldots x_1 \in K$ which is false. Hence $P_{k+1} = \{x \in P:\ xx_k \ldots x_1 \notin K\} \neq \emptyset$. If $z \in P_{k+1}$ implies $zx_k x_{k-1} \ldots x_1 P \subseteq K$ then for $z \in P$ either $z \in P_{k+1}$ in which case $zx_k \ldots x_1 P \subseteq K$ or $z \notin P_{k+1}$ in which case $zx_k \ldots x_1 \in K$. Hence $P^t zx_k \ldots x_1 P = (0)$ for each $z \in P$ so $x_k \ldots x_1 P \subseteq K$ which is not true. Thus $\mathcal{P}_{k+1} = \{r(x):\ x \in P_{k+1},\ xx_k \ldots x_1 P \nsubseteq K\} \neq \emptyset$ so we can choose $x_{k+1} \in P_{k+1}$ with $r(x_{k+1})$ maximal in $\mathcal{P}_{k+1}$. Now define $a_k = x_k x_{k-1} \ldots x_1$. Clearly $r(a_1) = r(x_1) \subseteq r(a_k)$. We next observe that, for $j, m > 0$, $r(x_j) \subseteq r(x_{j+m} \ldots x_j)$ and $a_{j+m} \notin K$, $a_{j+m} P \nsubseteq K$ together imply $r(x_{j+m} \ldots x_j) \in \mathcal{P}_j$ from which $r(x_j) = r(x_{j+m} \ldots x_j)$. Using these results we now establish the existence of an infinite chain of left annihilators. From the construction of a_1 we observe that $a_1u \neq 0$ for some $u \in P$. Since $r(a_n) = r(a_1)$ it then follows that $a_nu \neq 0$ for each n. Now consider $x_na_{n+j}u$ for $j > 0$. If $x_na_{n+j}u \notin K$ then $x_na_{n+j} \notin K$. Putting $y = x_n$, $z = x_{n+j} \ldots x_{n+1}$ we get $x_na_{n+j}u = yzya_{n-1}u$. If $yzya_{n-1}P \not\subseteq K$ then, since $zy \in P$, for some integer $m > 1$, $(zy)^m = 0 \neq (zy)^{m-1}$. Now $y(zy)^{m-1} \neq 0$ implies that $(zy)^{m-1} \in r(yzy)$ but $(zy)^{m-1} \notin r(y)$ whilst $y(zy)^{m-1} = 0$ implies $y(zy)^{m-2} \neq 0$ and $(zy)^{m-2} \in r(yzy)$ but $(zy)^{m-2} \notin r(y)$. In either case, $r(y) \subsetneq r(yzy)$ which, since $r(yzy) \in \mathscr{P}_n$, contradicts the definition of $y = x_n$. Thus $yzya_{n-1}P \subseteq K$. Then $P^tyzya_{n-1}P = (0)$ implies $x_{n+t} \ldots x_{n+1}x_na_{n+j}P = (0)$ and thus $x_na_{n+j}P = (0)$ contradicting $x_na_{n+j}u \notin K$. It follows that $x_na_{n+j}u \notin K$. But then $P^tx_na_{n+j}u = (0)$ yields $x_na_{n+j}u = 0$. Thus for $j > 0$, $x_na_{n+j}u = 0$. Next suppose $j = 0$. Then $x_na_nu = x_n^2a_{n-1}u$. If $x_na_nu \notin K$ then $x_na_n \notin K$ and thus $x_n^2 \in P_n$ so that if $x_n^2a_{n-1}P \not\subseteq K$ we have $r(x_n^2) \in \mathscr{P}_n$ and $r(x_n^2) = r(x_n)$. For some $m > 1$, $x_n^m = 0$, $x_n^{m-1} \neq 0$. But $x_n^2x_n^{m-2} = 0 = x_nx_n^{m-2} = x_n^{m-1}$. Thus we have $x_n^2a_{n-1}P \subseteq K$ which implies that $a_nP \in r(x_{n+t} \ldots x_n) = r(x_n)$ as before and hence $x_na_nu = 0$. We now see that $x_n \in l(\{a_ku : k \geqq n\})$ whereas $x_na_{n-1}u \neq 0$ so $x_n \notin l(\{a_ku : k \geqq n-1\})$. This leads to a properly ascending chain of left annihilators which contradicts the maximum condition on left annihilators. Hence nil subnear-rings are nilpotent.

We say that an N-subgroup of N is *module essential* (*essential*) if it has non-zero intersection with all non-zero right ideals (N-subgroups) of N. Furthermore, N has *finite rank* provided each chain $A_1 \subset A_2 \subset \cdots$ of right ideals of N in which for each $i \geqslant 2$ there is a non-zero N subgroup $B_i \subset A_i$ with $A_{i-1} \cap B_i = (0)$ terminates finitely. Notice that finite rank is a maximum condition on right ideals and that when N is a ring this reduces to an equivalent definition to the usual one involving no infinite direct sums.

Lemma 1. *If N has finite rank and every module essential N-subgroup of N is essential then every chain $X_1 \supset X_2 \supset \cdots$ of N-subgroups in which, for each $i \geqslant 1$, there is a non-zero N-subgroup $Y_i \subset X_i$ with $Y_i \cap X_{i+1} = (0)$ terminates finitely.*

Proof. Construct an ascending chain of right ideals $A_1 \subset A_2 \subset \cdots$ as follows. Choose A_1 to be a right ideal of N maximal subject to $A_1 \cap X_1 = (0)$. Suppose A_k has been chosen. Choose A_{k+1} to be a right ideal of N maximal subject to $A_k \subset A_{k+1}$ and $A_{k+1} \cap X_{k+1} = (0)$. For each k, $A_k + X_k$ is module essential and hence essential in N. Now let $y \in Y_{k-1} \cap (A_k + X_k)$ with $y \neq 0$. Then $y \in X_{k-1} \cap (A_k + X_k) = (X_{k-1} \cap A_k) + X_k$. Since $Y_{k-1} \cap X_k = (0)$ we must have $B_k = X_{k-1} \cap A_k \neq (0)$. Further $B_k \subset A_k$ and $B_k \cap A_{k-1} = A_k \cap X_{k-1} \cap A_{k-1} = (0)$. Since N has finite rank, the chain $A_1 \subset A_2 \subset \cdots$ must terminate finitely and hence so also must $X_1 \supset X_2 \supset \cdots$.

This leads to the following

Theorem 7. *If every module essential N-subgroup of N is essential and*

N has the maximum condition on right annihilators and finite rank then each nil N-subgroup of N is nilpotent.

Proof. In the proof of Theorem 6 we constructed a chain of left annihilators $l\{a_k u: k \geq n\}$, in which the inclusions were strict at each stage, on the assumption that P was a non-nilpotent nil subnear-ring of N. Now put $A_n = rl\{a_k u: k \geq n\}$ to obtain a descending chain $A_1 \supset A_2 \supset \cdots$ of right ideals of N. Now $a_n u \in A_n$ so $a_n N \cap A_n \neq (0)$. If $a_n t \in A_{n+1}$ then $x_{n+1} \in l(A_{n+1})$ yields $x_{n+1} a_n t = a_{n+1} t = (0)$. Hence with $B_n = a_n N \cap A_n \neq (0)$ we have $B_n \cap A_{n+1} = (0)$. Applying Lemma 1 the chain $A_1 \supset A_2 \supset \cdots$ terminates finitely and hence N has maximum condition on left annihilators of the form $l(A_n)$. But these are just $l\{a_k u: k \geq n\}$.

Since finite rank is a maximum condition we have

Corollary 1. *If every module essential N-subgroup of N is essential and N has the maximum condition on right ideals then nil subnear-rings of N are nilpotent.*

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