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THE COMPUTATION OF THE HODGKIN–SNAITH OPERATION IN $K_*(Z\times BU, Z/p)$

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Introduction

This paper builds on the work of V. P. Snaith [5] (in particular Section 6) and gives a more explicit determination of the result.

Recently J. E. McLure has given a satisfactory account of Dyer–Lashof operations in K-theory. For any E_∞-space Y and for any $r\geqq 2$ there is a map

$$Q: K_\alpha(Y, Z/p^r)\to K_\alpha(Y, Z/p^{r-1})$$

with specific properties (see [3], Theorem 1). Earlier, L. Hodgkin [2] and V. P. Snaith [5] constructed an operation

$$\bar{Q}: K_\alpha(Y, Z/p)\to K_\alpha(Y, Z/p)/\mathrm{Ind}\,(Y),$$

where $\mathrm{Ind}(Y)=\{x^p\,|\,x\in K_\alpha(Y, Z/p)\}$ is the indeterminancy of $\bar{Q}$. (Throughout the paper p denotes an odd prime.) The construction of $\bar{Q}$ given in [2] and [5] fails to go through if the Bockstein homomorphism β is nonzero. But as far as this paper is concerned [5] is correct (this is explained more fully in [1]). The relation between Q and $\bar{Q}$ is as follows:

Let $x\in\operatorname{Ker}\beta\subset K_*(Y, Z/p)$ and let $y\in K_*(Y, Z/p^2)$ be a lifting of x, then $\bar{Q}(x)=Q(y)$. The lifting y of x is not unique. It follows from the properties of Q that $\bar{Q}$ is well defined $\mathrm{mod}\,\mathrm{Ind}(Y)$. In this paper we study $\bar{Q}$ and refer to it as the Hodgkin–Snaith operation.

To describe the main result we use the notation of [5].

Theorem 1. *Let $D\subset K_0(Z\times BU, Z/p)=Z/p[u_0^{-1}, u_0, u_1,\ldots]$ denote the translates (under $u_0\in K_0(1\times BU, Z/p)$) of the decomposables in the algebra $K_0(0\times BU, Z/p)$.*

Then, mod D,

$$p_0^{-p+1}\bar{Q}(u_k)=\sum_{m=0}^{p-1}(-1)^m u_{kp+m}\quad \textit{if}\quad k\geqq 1$$

and

$$u_0^{-p+1}\bar{Q}(u_0)=\sum_{m=1}^{p-1}(-1)^m u_m.$$

The elements $u_0, u_1, u_2,\ldots$ are mod p reductions of integral classes. If $x_0, x_1, x_2,\ldots$ denote their mod p^2 reductions then from the properties of Q follows immediately

Theorem 2. *The operation $Q: K_*(Z\times BU, Z/p^2)\to K_*(Z\times BU, Z/p)$ is given by*

$$u_0^{-p+1}Q(x_k)=\sum_{m=0}^{p-1}(-1)^m u_{kp+m} \bmod D \quad \textit{if} \quad k\geqq 1$$

and

$$u_0^{-p+1}Q(x_0)=\sum_{m=1}^{p-1}(-1)^m u_m \bmod D.$$

The iterated operations $\bar{Q}^t$ can be computed (using the formulas for $\bar{Q}(x\cdot y)$ and $\bar{Q}(x+y)$). In particular we obtain the following

Corollary 1. *Let t be a positive integer. Then*

$$u_0^{-p^t+1}\bar{Q}^t(u_0)=\sum_{m=p^{t-1}}^{p^t-1}(-1)^m u_m \bmod D.$$

The corollary can be applied to study the homomorphism $K_*(Q(S^0), Z/p)\to K_*(Z\times BU, Z/p)$ (induced from the canonical infinite loop map $Q(S^0)\to Z\times BU$). From [2] we know that $K_*(Q(S^0), Z/p)=Z/p[\theta_1^{-1},\theta_1,\theta_p,\theta_{p^2},\ldots]$, where θ_{p^t} represents $\bar{Q}^t(\theta_1)$. Therefore $K_*(Q_0(S^0), Z/p)=Z/p[\tilde{\theta}_1,\tilde{\theta}_2,\ldots]$, where $\tilde{\theta}_t=\theta_1^{-p^t}\theta_{p^t}$. By naturality $\tilde{\theta}_t$ is mapped to $u_0^{-p^t}\bar{Q}(u_0)$.

Corollary 2. *Under the canonical homoporphism $K_*(Q_0(S^0), Z/p)\to K_0(0\times BU, Z/p)$ the element $\tilde{\theta}_t$ is mapped to*

$$u_0^{-1}\sum_{m=p^{t-1}}^{p^t-1}(-1)^m u_m$$

modulo decomposable elements.

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1. Preliminaries

We have to recall the K-homology of $CP^\infty=BS^1$ and of $Z\times BU$. Let η be the canonical line bundle over CP^∞, and let $c=\eta-1$. Since S^1 is an abelian group the multiplication induces a map $M: BS^1\times BS^1\to BS^1$. With this map $K^*(BS^1)$ and $K_*(BS^1)$ are well-known Hopf algebras. A description of $K_*(BS^1)$ will be given in Section 2.

Let $u_n\in K_0(BS^1)$ be defined by $\langle u_n, c^m\rangle=\delta_{n,m}$. We consider now c as a map $BS^1\to BU$.

Lemma 1.1 (see [5], pp. 198 and 199). *Let the image of u_n under the map*

$$K_0(BS^1)\xrightarrow{c_*}K_0(BU)\xrightarrow{u_0}K_0(1\times BU)\subset K_0(Z\times BU)$$

be also denoted by u_n. *Then*

$$K_*(Z \times BU) = Z[u_0^{-1}, u_0, u_1, \ldots].$$

(*Note that* $u_0, u_1, u_2, \ldots \in K_0(1 \times BU)$, $u_0 \in K_0(1 \times BU) \cong K_0(BU)$ *corresponds to* 1.)

We will also use the symbol u_n for the mod p reduction of u_n. Our goal is the computation of the indecomposable part of $\bar{Q}(u_k)$.

This computation is based on the following

Theorem 1.2. *Let* $k \geqq 1$. *The indecomposable part of* $u_0^{-p}\bar{Q}(u_k)$ *is equal to the image of*

$$u_{kp} - \sum_{j=1}^{p-1} M_*(\psi_*^j u_{p-1} \otimes u_{kp}) \tag{*}$$

under $c_*: K_0(BS^1, Z/p) \to K_0(0 \times BU, Z/p)$, *where* ψ_*^j *is the dual of the Adams operation* ψ^j. *The indecomposable part of* $u_0^{-p}\bar{Q}(u_0)$ *is equal to*

$$-\sum_{j=1}^{p-1} \psi_*^j u_{p-1} \quad \textit{under } c_*.$$

This theorem appears in the proof of Corollary 6.3.6 of [5]. It is not difficult to prove that the factor μ appearing in [5] is -1. The extra summand u_{kp} in (*) comes from

$$-\sum_{\mathbf{t}} \frac{1}{|G_{\mathbf{t}}|} i_*(u_{t_1} \otimes \cdots \otimes u_{t_p}) \text{ in Lemma 6.3.2 of [5],}$$

where the sum is taken over all ordered partitions $\mathbf{t} = (t_1, \ldots, t_p)$ of kp with $t_1 \neq t_p$. This sum is therefore empty if $k = 0$. A proof of 1.2 is also given in [1].

2. Proof of the Theorem

By Theorem 1.2 we have to calculate the element

$$\sum_{j=1}^{p-1} M_*(\psi_*^j u_{p-1} \otimes u_{kp}) \in K_0(BS^1, Z/p),$$

i.e. their coefficients with respect to the basis $u_1, u_2, \ldots$.

Theorem 1 follows immediately from

Proposition 2.1. *For any* $k \geqq 0$ *we have*

$$\sum_{j=1}^{p-1} M_*(\psi_*^j u_{p-1} \otimes u_{kp}) = -\sum_{j=1}^{p-1} (-1)^j u_{kp+j} \textit{ in } K_0(BS^1, Z/p).$$

In order to prove Proposition 2.1 we identify $K_0(BS^1)$ with a subring of $Q[X]$ (see [4]). More precisely, there are the following standard facts:

(1) $K_0(BS^1)$ can be identified with the subring of $Q[X]$ generated by the elements

$$\binom{X}{n} = X(X-1)\dots(X-n+1)/n!,\ n=0,1,\dots.$$

The element u_n corresponds to $\binom{X}{n}$.

(2) The operation ψ_*^j is given by $\psi_*^j(X)=jX$, or more generally $(\psi_*^j f)(X)=f(jX)$ for any polynomial f.

Proposition 2.1 is therefore equivalent to

Proposition 2.2. Mod p *we have, for any* $k \geqq 0$,

$$\sum_{j=1}^{p-1} \binom{jX}{p-1}\binom{X}{kp} \equiv -\sum_{j=1}^{p-1} (-1)^j \binom{X}{kp+j}.$$

Proof. We use the identity $\binom{kp+j}{j}\binom{X}{kp+j} = \binom{X}{kp}\binom{X-kp}{j}$. If we reduce it mod p for $0<j<p$ we obtain

$$\binom{X}{kp+j} \equiv \binom{X}{kp}\binom{X}{j}.$$

So it remains to prove

$$\sum_{j=1}^{p-1} \binom{jX}{p-1} \equiv -\sum_{j=1}^{p-1} (-1)^j \binom{X}{j} \bmod p.$$

Hence we have to check that the two polynomials over $\mathbb{F}_p$ are the same. But over $\mathbb{F}_p$ they are both equal to X^{p-1}, since their values at $X=a$ are 0 for $a=0\in\mathbb{F}_p$ and 1 for $a\in\mathbb{F}_p^*$.

Remark. In [4] it is shown that the elements $u_0, u_1, \dots, u_{p-1}$ and $P_{l,t} = -\sum_{m=lp^t}^{(l+1)p^t} (-1)^m u_m$, $1 \leqq l \leqq p-1$, $t \geqq 1$ generate the subspace of $K_*(BS^1, Z/p)$ invariant under ψ_*^q (under the hypothesis $p \mid q-1$ and $p^2 \nmid q-1$). The indecomposable part of $\bar{Q}^t(u_k)$ can be expressed by their images in $K_*(Z \times BU, Z/p)$. For example,

$$u_0^{-p^t+1}\bar{Q}(u_0) = -\sum_{l=1}^{p-1} P_{lt-1} \bmod D.$$

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