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EXISTENCE OF FINITE GROUPS WITH CLASSICAL COMMUTATOR SUBGROUP

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Abstract

Given a group G, we may ask whether it is the commutator subgroup of some group $\mathscr{G}$. For example, every abelian group G is the commutator subgroup of a semi-direct product of $G \times G$ by a cyclic group of order 2. On the other hand, no symmetric group S_n $(n > 2)$ is the commutator subgroup of any group $\mathscr{G}$. In this paper we examine the classical linear groups over finite fields K of characteristic not equal to 2, and determine which can be commutator subgroups of other groups. In particular, we settle the question for all normal subgroups of the general linear groups $GL_n(K)$, the unitary groups $U_n(K)$ $(n \neq 4)$, and the orthogonal groups $O_n(K)$ $(n \geqslant 7)$.

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1. Preliminaries

If x and y are elements of a group G, the commutator of x and y, written $[x,y]$, is the element $x^{-1}y^{-1}xy$. The commutator subgroup of G is denoted by G'. We call G a *C-group* if it is the commutator subgroup of some group $\mathscr{G}$. We denote by $o(x)$ the order of x, by x^* the inner automorphism of G induced by x, and by $\langle x \rangle$ the subgroup generated by x.

We now give three theorems which are needed later.

THEOREM 1. *Let H be a characteristic subgroup of G, $x \in G$. Suppose that there is no element $\varphi \in (\operatorname{Aut} H)'$ such that $x^*|_H = \varphi$. Then G is not a C-group.*

PROOF. Suppose $\mathscr{G}' = G$. As H is characteristic in G, and G is characteristic in $\mathscr{G}$, H is characteristic in $\mathscr{G}$. Now x is a product of commutators in $\mathscr{G}$, each of which acts on H (via conjugation) as an element of $(\operatorname{Aut} H)'$. Hence $x^*|_H = \varphi$ for some $\varphi \in (\operatorname{Aut} H)'$, and the result follows.

THEOREM 2. *Suppose $\varphi \in \operatorname{Aut} G$ has order s. Extend G by the cyclic group $\langle \varphi \rangle$ of order s to obtain a group $\bar{G} = \langle G, \varphi \rangle$ with relations*

$$\text{those of } G, \quad \varphi^s = 1, \quad \varphi^{-1} g \varphi = g^{\varphi} \quad (g \in G).$$

Then $\bar{G}' = \langle G', g^{\varphi-1} \mid g \in G \rangle$.

The proof is straightforward and is omitted. Clearly $\bar{G}' \subseteq G$, and if equality holds, we have constructed a group of which G is the commutator. If $x \in G$, we define the *x-order* of φ, denoted by $o(\varphi, x)$, to be the order of the element φx in $\bar{G}$. It is easy to see that $o(\varphi, x)$ is a multiple of s.

THEOREM 3. *Let G be a group, $x \in G$, $\varphi, \psi \in \operatorname{Aut} G$ with $[\varphi, \psi] = x^*$. Then there exists a group $\mathcal{G}$ with $\mathcal{G}' \subseteq G \subseteq \mathcal{G}$ and $x \in \mathcal{G}'$.*

PROOF. We construct $\mathcal{G}$ by consecutive cyclic extensions of G. Suppose that $o(\varphi) = s$, $o(\psi) = t$, and let $o(\varphi, x) = n$. Extend G by the cyclic group $\langle \bar{\varphi} \rangle$ of order n to obtain a group $\bar{G}$ with relations

$$\text{those of } G, \quad \bar{\varphi}^n = 1, \quad \bar{\varphi}^{-1} g \bar{\varphi} = g^{\varphi} \quad (g \in G).$$

We now extend ψ to the generators of $\bar{G}$ by defining

$$\psi: \begin{array}{l} g \to g^{\psi} \quad (g \in G), \\ \bar{\varphi} \to \bar{\varphi} x. \end{array}$$

Using the fact that $[\varphi, \psi] = x^*$, it is easily checked that this indeed defines an automorphism of $\bar{G}$. We can now extend $\bar{G}$ by the cyclic group $\langle \bar{\psi} \rangle$ of order $\bar{t}$, where $\bar{t}$ is the order of ψ in $\operatorname{Aut} \bar{G}$. We obtain a group $\mathcal{G}$ with relations

$$\text{those of } G, \qquad \bar{\varphi}^n = 1, \quad \bar{\varphi}^{-1} g \bar{\varphi} = g^{\varphi}, \quad \bar{\psi}^{\bar{t}} = 1,$$
$$\bar{\psi}^{-1} g \bar{\psi} = g^{\psi}, \quad \bar{\psi}^{-1} \bar{\varphi} \bar{\psi} = \bar{\varphi} x.$$

A simple calculation shows that

$$\mathcal{G}' = \langle G', x, g^{\varphi - 1}, g^{\psi - 1} \mid g \in G \rangle.$$

Hence $\mathcal{G}$ has the desired properties.

2. General linear groups

Let $GL = GL_n(K)$ be the group of non-singular $n \times n$ matrices ($n > 1$) over the finite field $K = \mathbf{F}_q$ of $q = p^k$ elements ($p > 2$), and let

$$SL = SL_n(K) = \{X \in GL \mid \det X = 1\}.$$

It is known (Dieudonné, 1951) that $\operatorname{Aut} SL$ is generated by automorphisms of the following types:

(i) $A \to X^{-1} A X$, where $X \in GL$.

(ii) $A \to A^{\sigma}$, where $\sigma \in \operatorname{Aut} K$.

(iii) $A \to (A^{-1})^t$.

We denote automorphisms of these three types by φ, χ, ψ respectively.

We wish to determine $(\operatorname{Aut} SL)'$. A simple calculation shows that $(\operatorname{Aut} SL)'$ is generated by the elements $[w_1, w_2]$, where w_1, w_2 run through the three types φ, χ, ψ. As $GL' = SL$, any commutator $[\varphi_1, \varphi_2]$ is clearly an inner automorphism of SL. Since $\operatorname{Aut} K$ is abelian, we have $[\chi_1, \chi_2] = 1$.

Suppose $\varphi\colon A\to X^{-1}AX$ and $\chi\colon A\to A^{\sigma}$. Then $[\varphi,\chi]\colon A\to X^{-\sigma}XAX^{-1}X^{\sigma}$. Now $\det(X^{-1+\sigma})=(\det X)^{p^r-1}$, where $\sigma\colon K\to K$ is given by $y^{\sigma}=y^{p^r}$ for all $y\in K$. As p^r-1 is even, $\det(X^{-1+\sigma})$ is a square in K. Thus $[\varphi,\chi]$ is an automorphism of type (i), induced by an element of GL with square determinant.

Now suppose $\varphi\colon A\to X^{-1}AX$, and $\psi\colon A\to(\mathrm{A}^{-1})^t$. Then

$$[\varphi,\psi]\colon A\to X^tXAX^{-1}(X^{-1})^t.$$

Since $\det(X^tX)$ is a square, $[\varphi,\psi]$ is of type (i), induced by an element of GL with square determinant. Finally, $[\chi,\psi]=1$. We conclude that

$$(\mathrm{Aut}\,SL)'\subseteq\{X^*\mid X\in GL,\det X\text{ is a square}\}.$$

Except for $GL_2(\mathbf{F}_3)$, every non-central normal subgroup of GL contains SL. So let S be such a subgroup, $SL\subseteq S\subseteq GL$. Then $S'=SL$ and so SL is characteristic in S. Furthermore, $C_S(SL)=Z(S)$ and so by Theorem 1, a necessary condition for S to be a C-group is that $S/Z(S)\subseteq(\mathrm{Aut}\,SL)'$.

Let α be a generator of K^*, and let $[GL\colon S]=r$, so that $S=\{X\in GL\mid\det X\text{ is an }r\text{th power}\}$. If $Q=\operatorname{diag}(\alpha,1,1,\dots,1)$, then $S=\langle SL,Q^r\rangle$. Assume $\mathscr{G}'=S$. Since $Q^r\in\mathscr{G}'$, the above analysis implies that $(Q^r)^*|_{SL}=A^*|_{SL}$, where $A\in GL$, and $\det A$ is a square. As $C_{GL}(SL)$ consists of the scalar matrices, there is a $\lambda\in K$ such that $\det(\lambda Q^r)=\alpha^r\lambda^n$ is a square. If n is even and r is odd, we clearly have a contradiction. Hence in such cases, S is not a C-group.

Suppose now that r is even. Let $B=\operatorname{diag}(\alpha^{r/2},1,1,\dots,1)$, and consider the following two automorphisms of S:

$$\varphi\colon A\to B^{-1}AB,$$
$$\psi\colon A\to(A^{-1})^t.$$

We find that $[\varphi^{-1},\psi^{-1}]=(BB^t)^*$. But $BB^t=Q^r$. Define $\mathscr{G}=\langle S,\varphi,\psi\rangle$, with relations as defined in Theorem 3. Then $\mathscr{G}'=S$ and so S is a C-group.

Finally, assume that both n and r are odd. Let $C=\operatorname{diag}(\alpha^{(n+r)/2},1,1,\dots,1)$, and consider the following two automorphisms of S:

$$\varphi\colon A\to C^{-1}AC,$$
$$\psi\colon A\to(A^{-1})^t.$$

We have $[\varphi^{-1},\psi^{-1}]=(CC^t)^*=(CC^tZ)^*$, where $Z=\alpha^{-1}I$. But

$$CC^tZ=\operatorname{diag}(\alpha^{n+r-1},\alpha^{-1},\alpha^{-1},\dots,\alpha^{-1})$$

has determinant α^r and so is in S. In fact, $S=\langle SL,CC^tZ\rangle$ since α is a generator of K^*. If we define $\mathscr{G}=\langle S,\varphi,\psi\rangle$ with relations as in Theorem 3, then $\mathscr{G}'=S$.

We may summarize the above results as follows:

THEOREM 4. *Let S be a subgroup of $GL_n(K)$, char $K\neq 2$, with $SL_n(K)\subseteq S\subseteq GL_n(K)$, and $[GL_n(K)\colon S]=r$. Then S is a C-group except when n is even and r is odd.*

It is easily checked that every proper normal subgroup of $GL_2(\mathbf{F}_3)$ is a C-group and so the theorem is true for any normal subgroup S of $GL_n(K)$.

3. Orthogonal and unitary groups

Let K be the finite field of p^h elements ($p>2$), and suppose that f is a non-degenerate symmetric bilinear form on a K-vector space V with index $\nu(f)\geqslant 1$. Denote by $O_n(K,f)$ the corresponding orthogonal group. If $\{e_i\}$, $i=1,2,\ldots,n$, is an orthogonal basis for V, and R is the (diagonal) matrix of f with respect to this basis, then $O_n(K,f)$ is realized as the set of all $A\in GL_n(K)$ with $ARA^t=R$. Let $\Omega_n(K,f)$ denote the commutator subgroup $O_n(K,f)'$ and set

$$O_n^+(K,f)=\{A\in O_n(K,f)\mid \det A=1\}.$$

Suppose now that h is even, so that K has a unique non-trivial involution σ, where $y^\sigma=y^{p^{h/2}}$ for all $y\in K$. Let g be a reflexive σ-linear form on V, and denote by $U_n(K,g)$ the corresponding unitary group. With respect to a suitable basis, $U_n(K,g)$ is realized as the set of all $A\in GL_n(K)$ with $A\tilde{A}=I$, where $\tilde{A}=(A^\sigma)^t$. Finally, set $U_n^+(K,g)=\{A\in U_n(K,g)\mid \det A=1\}$.

Using arguments similar to those used in the general linear case, we obtain the following:

THEOREM 5. *Suppose $n\geqslant 7$. If n is odd, the only non-central normal subgroups of $O_n(K,f)$ which are C-groups are $\Omega_n(K,f)$ and $\langle\Omega_n(K,f),-I\rangle$. If n is even, the only such C-groups are $\Omega_n(K,f)$ and $O_n^+(K,f)$.*

THEOREM 6. *Let S be a subgroup of $U_n(K,g)$, char $K\neq 2$, $n\neq 4$, with*

$$U_n^+(K,g)\subseteq S\subseteq U_n(K,g)$$

and $[U_n(K,g): S]=r$. Then S is a C-group except when n is even and r is odd.

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