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RESEARCH ARTICLE

A categorical action of the shifted 0-affine algebra

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Abstract
We introduce a new algebra $\mathcal{U} = \dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_n)$ called the shifted 0-affine algebra, which emerges naturally from studying coherent sheaves on n-step partial flag varieties through natural correspondences. This algebra $\mathcal{U}$ has a similar presentation to the shifted quantum affine algebra defined by Finkelberg-Tsymbaliuk. Then, we construct a categorical $\mathcal{U}$-action on a certain 2-category arising from derived categories of coherent sheaves on n-step partial flag varieties. As an application, we construct a categorical action of the affine 0-Hecke algebra on the bounded derived category of coherent sheaves on the full flag variety.

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1. Introduction

The derived category of coherent sheaves on an algebraic variety plays a crucial role in modern algebraic geometry and its related fields. In this article, we focus on exploring the categorical actions of a specific algebra—the shifted 0-affine algebra—on the derived categories of coherent sheaves on Grassmannians and n-step partial flag varieties.

1.1. *Categorical* $\mathfrak{sl}_2$*-Action*

Roughly speaking, a categorical action of a Kac-Moody Lie algebra $\mathfrak{g}$ consists of a collection of functors and categories that recover actions of Chevalley generators at the level of Grothendieck groups. Let $\{e, h, f\}$ be elements in $\mathfrak{g}$ that form an $\mathfrak{sl}_2$-triple.

A finite-dimensional representation V of $\mathfrak{sl}_2$ consists of a direct sum decomposition $V = \oplus_\lambda V_\lambda$ into weight spaces and linear maps $e : V_\lambda \to V_{\lambda+2}$ and $f : V_\lambda \to V_{\lambda-2}$, satisfying the relation $(ef - fe)|_{V_\lambda} = \lambda I_{V_\lambda}$. Such data can be depicted in the following diagram

$$\cdots \underset{f}{\overset{e}{\rightleftarrows}} V_{\lambda-2} \underset{f}{\overset{e}{\rightleftarrows}} V_\lambda \underset{f}{\overset{e}{\rightleftarrows}} V_{\lambda+2} \underset{f}{\overset{e}{\rightleftarrows}} \cdots . \tag{1.1}$$

This characterization leads to the following notion of $\mathfrak{sl}_2$ acting on categories.

Definition 1.1 [28, Definition 1.2]. A naive categorical $\mathfrak{sl}_2$-action consists of a sequence of additive categories $\mathcal{C}(\lambda)$ together with additive functors $\mathsf{E} : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda+2)$ and $\mathsf{F} : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda-2)$ for each $\lambda \in \mathbb{Z}$ such that there exist isomorphisms of functors[1]

$$\begin{aligned} \mathsf{EF}|_{\mathcal{C}(\lambda)} &\cong \mathsf{FE}|_{\mathcal{C}(\lambda)} \bigoplus Id_{\mathcal{C}(\lambda)}^{\oplus \lambda} \text{ if } \lambda \geq 0, \\ \mathsf{FE}|_{\mathcal{C}(\lambda)} &\cong \mathsf{EF}|_{\mathcal{C}(\lambda)} \bigoplus Id_{\mathcal{C}(\lambda)}^{\oplus -\lambda} \text{ if } \lambda \leq 0, \end{aligned} \tag{1.2}$$

where $Id_{\mathcal{C}(\lambda)}$ is the identity functor for $\mathcal{C}(\lambda)$.

The work of Beilinson-Lusztig-MacPherson [8] offers a geometric framework for a categorical $\mathfrak{sl}_2$-action, where the weight categories are defined as $\mathcal{C}(\lambda) = \mathrm{D}^b\mathrm{Con}(\mathrm{Gr}(k, N))$. These categories correspond to the derived categories of constructible sheaves on Grassmannians $\mathrm{Gr}(k, N)$, with $\lambda = N - 2k$. The functor $\mathsf{E} : \mathrm{D}^b\mathrm{Con}(\mathrm{Gr}(k, N)) \to \mathrm{D}^b\mathrm{Con}(\mathrm{Gr}(k-1, N))$ is given by pull-push along the following correspondence

[1] Note that we do not specify the data of these isomorphisms between functors in this definition.

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$$
\begin{tikzcd}
& \mathrm{Fl}(k-1,k) = \{0 \overset{k-1}{\subset} V' \overset{1}{\subset} V \overset{N-k}{\subset} \mathbb{C}^N\} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathrm{Gr}(k,N) & & \mathrm{Gr}(k-1,N)
\end{tikzcd}
\tag{1.3}
$$

where $\mathrm{Fl}(k-1,k)$ is the 3-step partial flag variety, and the numbers above the inclusions indicate the increase in dimensions. The functor F is given by the opposite pull-push.

Then, building on the work by Beilinson-Lusztig-MacPherson [8] and Chuang-Rouquier [20], the categories $\mathcal{C}(\lambda)$ and the functors E and F introduced above give a naive categorical $\mathfrak{sl}_2$-action. More precisely, the functors E and F satisfy (1.2) in Definition 1.1.

The term "naive categorical action" is used in this context since the natural isomorphisms in (1.2) are not explicitly defined. In contrast, a categorical $\mathfrak{sl}_2$-action specifies these natural isomorphisms through certain adjunctions. For a detailed explanation, please refer to [28, Definition 1.5].

The study of natural transformations between functors is a fundamental problem in higher representation theory, particularly in the context of categorical actions. Ideally, isomorphisms between functors, such as (1.2), should arise naturally from specific adjunction data. One solution to this problem in the case of $\mathfrak{sl}_2$-categorification was proposed by Chuang-Rouquier [20] and later extended to (simply-laced) Kac-Moody algebras $\mathfrak{g}$ by Khovanov-Lauda [29] [30] [31], and Rouquier [38].

Since then, there have been numerous developments and applications. In particular, people have extensively studied the categorical action of Lie algebra or quantum groups in several flavors. One of them is the notion of geometric categorical $\mathfrak{sl}_2$-action, which was introduced by Cautis-Kamnitzer-Licata in [16] by using Fourier-Mukai transformations. See [15], [17] for the applications.

1.2. Constructing Categorical Action via Derived Categories of Coherent Sheaves

1.2.1. Main results

Our work is motivated by previous studies on constructible sheaves, and we shift our focus to categories of coherent sheaves. This leads us to consider weight categories $\mathcal{K}(\lambda) = \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k,N))$, which are the bounded derived categories of coherent sheaves on the Grassmannian variety $\mathrm{Gr}(k,N)$.

In the coherent setting, $\mathrm{Fl}(k-1,k)$ naturally admits a line bundle given by the quotient of the tautological bundles $\mathcal{V}/\mathcal{V}'$, where $\mathcal{V}$ and $\mathcal{V}'$ are of rank k and $k-1$, respectively.

In contrast to the constructible setting, where the functors are given by a pull-push along the correspondence (1.3), in the coherent setting, we obtain a family of functors that depends on powers of the tautological line bundle $\mathcal{V}/\mathcal{V}'$ on $\mathrm{Fl}(k-1,k)$. In other words, we have a family of functors parameterized by the integers corresponding to the tensor powers of $\mathcal{V}/\mathcal{V}'$ and $(\mathcal{V}/\mathcal{V}')^{-1}$.

More precisely, we have for each $r \in \mathbb{Z}$

$$\mathsf{E}_r := p_{2*}(p_1^* \otimes (\mathcal{V}/\mathcal{V}')^{\otimes r}) : \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k,N)) \to \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k-1,N)) \tag{1.4}$$

and similarly for F_r in the opposite direction. The main goal of this article is to study the following problem.

Problem: Our goal is to investigate the algebraic structure arising from the action of the functors E_r and F_s defined in (1.4) on the category $\bigoplus_k \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k,N))$. This algebra is generated by the abstract symbols e_r and f_s, and its structure encodes important information about the categorical action on coherent sheaves. We may ask several natural questions about this algebraic structure, such as:

1. What are the commutator relations between E_r and F_s in the categorical sense?
2. What are the relations satisfied by the generators of the algebra after we take the Grothendieck group of the categories of coherent sheaves?
3. Assuming that we obtain the algebra in (2), can we define a categorical action of this algebra as in Definition 1.1 for $\mathfrak{sl}_2$?

This article provides answers to the above questions. Before we state the main result, we would like to make some remarks.

First, the algebra that we consider in this article has a close resemblance to the loop algebra $L\mathfrak{sl}_2 := \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$ (although they are not identical). The generators e_r, f_s bear a resemblance to $e \otimes t^r$, $f \otimes t^s$ respectively, where $r, s \in \mathbb{Z}$.

Second, the idea of constructing decategorified actions goes back to Nakajima's work [37], where twists by line bundles for the loop structure appear when moving from cohomology (or Borel-Moore homology) to K-theory.

Third, upon decategorification, we obtain an algebra with a presentation similar to the shifted quantum affine algebra defined by Finkelberg-Tsymbaliuk [22]. We call the resulting algebra *the shifted 0-affine algebra*, denoted by $\mathcal{U} = \dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_2)$. The term "zero" reflects that certain relations are obtained by setting $q = 0$ in the relations of the shifted quantum affine algebra. In Section 7, we also explore its connection to the affine 0-Hecke algebra.

Finally, answering these questions enables us to construct a categorical $\mathcal{U}$-action on $\bigoplus_k \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k, N))$. Furthermore, we extend our result to the $\mathfrak{sl}_n$ case, where the Grassmannians are replaced by the n-step partial flag varieties $\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)$ (see (1.12) for its definition). We summarize the main results of this article in the following theorem.

Theorem 1.2.

1. *For the values of r and s restrict to certain ranges, either $\mathsf{E}_{i,r}\mathsf{F}_{i,s}$ and $\mathsf{F}_{i,s}\mathsf{E}_{i,r}$ are isomorphic or there are non-split exact triangles relating them (Proposition 5.10).*
2. *The resulting algebra is a new algebra, the shifted 0-affine algebra $\mathcal{U}$ with generators and relations are given in Definition 2.6.*
3. *We give a definition of categorical $\mathcal{U}$-action (Definition 3.1). We prove that there is a categorical $\mathcal{U}$-action on $\bigoplus_{\underline{k}} \mathrm{D}^b\mathrm{Coh}(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ (Theorem 5.6).*

1.2.2. The difference from the constructible setting and other new features

In this subsection, we provide further details on our results, focusing specifically on the categorical commutator relations arising from the geometric setting as described in Theorem 1.2 (1).

We begin by explaining the restriction of the loop generators. This restriction stems primarily from the chosen presentation of the shifted 0-affine algebra $\mathcal{U}$, defined in Definition 2.6. Inspired by the Levendorskii presentation of the shifted quantum affine algebra introduced by Finkelberg and Tsymbaliuk [22] (see Definition 2.3), this presentation is particularly advantageous due to its simplicity. It employs a finite set of generators and relations for $\mathcal{U}$, enabling a straightforward definition of a categorical $\mathcal{U}$-action. More precisely, our algebra $\mathcal{U}$ is defined by the idempotent modification, and we only have the loop generators $e_r 1_{(k,N-k)}$ and $f_s 1_{(k,N-k)}$ for $-k \leq r \leq 0$ and $0 \leq s \leq N-k$.

In [22], Finkelberg-Tsymbaliuk proved the equivalence between the Levendorskii presentation and the usual loop presentation of the shifted quantum affine algebra (see Theorem 2.4). In Appendix A, we introduce a definition for the shifted 0-affine algebra using the loop presentation (see Definition A.1) and conjecture that the two presentations are equivalent (see Conjecture A.4).

To compare the compositions of functors $\mathsf{E}_r\mathsf{F}_s$ and $\mathsf{F}_s\mathsf{E}_r$ with E_r, F_s defined in (1.4), we use the language of Fourier-Mukai (FM) transformations. It translates the comparison between functor compositions to the comparison between convolutions of FM kernels. We denote $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$ to be the FM kernel for $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ and similarly $\mathcal{F}_s\mathbf{1}_{(k,N-k)}$ for $\mathsf{F}_s\mathbf{1}_{(k,N-k)}$ where $r, s \in \mathbb{Z}$.

Due to our presentation of $\mathcal{U}$, we only have to compare $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$ for $-k \leq r+s \leq N-k$, where we denote $*$ to be the convolution of FM kernels.

When $-k+1 \leq r+s \leq N-k-1$, we then obtain the following isomorphisms

$$(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)} \cong (\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}. \tag{1.5}$$

When $r+s = N-k$ and $r+s = -k$, we only have the two cases $(r, s) = (0, N-k)$ and $(r, s) = (-k, 0)$ respectively. Then, we have the following exact triangles in $\mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k, N) \times \mathrm{Gr}(k, N))$

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$$(\mathcal{F}_{N-k} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_0 * \mathcal{F}_{N-k})\mathbf{1}_{(k,N-k)} \to \Psi^+\mathbf{1}_{(k,N-k)}, \tag{1.6}$$

$$(\mathcal{E}_{-k} * \mathcal{F}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{F}_0 * \mathcal{E}_{-k})\mathbf{1}_{(k,N-k)} \to \Psi^-\mathbf{1}_{(k,N-k)}, \tag{1.7}$$

where $\Psi^+\mathbf{1}_{(k,N-k)}$, $\Psi^-\mathbf{1}_{(k,N-k)}$ are certain FM kernels (see Definition 5.5 for details).

Remark 1.3. By using the conjugation property for $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathsf{F}_s\mathbf{1}_{(k,N-k)}$ (see condition (7)(a) and (8)(a) in Definition 3.1), we obtain that the above exact triangles (1.6) and (1.7) also hold for $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$ when $r+s=N-k$ and $r+s=-k$ respectively (see (3.5)).

We highlight some key properties of the above results. Firstly, it's worth noting that the commutator between E_r and F_s depends only on the integer $r+s$ provided $-k \le r+s \le N-k$. Secondly, the isomorphisms (1.5) and exact triangles (1.6), (1.7) are closely related to the coherent sheaf cohomology $H^*(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(-r-s-k))$. Specifically, (1.5) corresponds to the vanishing $H^*(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(-r-s-k)) = 0$ for $-N+1 \le -r-s-k \le -1$, which in turn implies $[e_r, f_s]1_{(k,N-k)} = 0$ when $1-k \le r+s \le N-k-1$. Thirdly, the exact triangles (1.6), (1.7) are non-split, as we discuss in Remark 5.11, which is different from the corresponding constructible setting ((1.2) in Definition 1.1).

1.2.3. Toward a future study

Although our presentation of $\mathcal{U}$ does not cover all the loop generators, the FM kernels $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathcal{F}_s\mathbf{1}_{(k,N-k)}$ are defined for all $r, s \in \mathbb{Z}$. Therefore, it is of interest to explore the relations between $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$ for $r+s \ge N-k+1$ and $r+s \le -k-1$. In this article, we provide an initial exploration of the case where $r+s = N-k+1$ (and similarly $r+s=-k-1$), as discussed in Section 6.

Like (1.6) and (1.7), we obtain the following (non-split) exact triangles in $\mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N))$

$$(\mathcal{F}_{N-k+1} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_0 * \mathcal{F}_{N-k+1})\mathbf{1}_{(k,N-k)} \to (\Psi^+ * \mathcal{H}_1)\mathbf{1}_{(k,N-k)}, \tag{1.8}$$

$$(\mathcal{E}_{-k-1} * \mathcal{F}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{F}_0 * \mathcal{E}_{-k-1})\mathbf{1}_{(k,N-k)} \to (\Psi^- * \mathcal{H}_{-1})\mathbf{1}_{(k,N-k)}, \tag{1.9}$$

where $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ are certain FM kernels (see (6.7) and (6.8)).

The exact triangles (1.8) and (1.9) can be viewed as categorifications of the two commutators $[e_0, f_{N-k+1}]1_{(k,N-k)}$ and $[e_{-k-1}, f_0]1_{(k,N-k)}$ respectively. Although the elements $e_{-k-1}1_{(k,N-k)}$ and $f_{N-k+1}1_{(k,N-k)}$ are not included in the generators of $\mathcal{U}$, the conjugation property of the FM kernels $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathcal{F}_s\mathbf{1}_{(k,N-k)}$ (see (6.4)) suggests that we can define $e_{-k-1}1_{(k,N-k)}$ and $f_{N-k+1}1_{(k,N-k)}$ in $\mathcal{U}$ by using conjugation with $\psi^+1_{(k,N-k)}$ (see (6.5)). Finally, we define

$$h_1 1_{(k,N-k)} := (\psi^+)^{-1}[e_0, f_{N-k+1}]1_{(k,N-k)}, \; h_{-1}1_{(k,N-k)} := -(\psi^-)^{-1}[e_{-k-1}, f_0]1_{(k,N-k)},$$

which are elements of $\mathcal{U}$, ensuring that (1.8) and (1.9) categorify the desired commutator relations.

Let $\mathsf{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ denote the FM transformations with kernels given by $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$. These transformations categorify $h_{\pm 1}1_{(k,N-k)}$, which can be viewed as an analog of the loop-Cartan-type elements in the quantum loop algebra $\mathbf{U}_q(L\mathfrak{sl}_2)$. For the sake of completeness, it would be appropriate to include the 1-morphisms $\mathsf{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ and certain exact triangles, such as (1.8) and (1.9), as conditions in a categorical $\mathcal{U}$-action. However, this article does not include these conditions, as we have not yet identified any direct applications for the 1-morphisms $\mathsf{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$.

Although we do not include $\mathsf{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ and the conditions involving them in our definition of a categorical $\mathcal{U}$-action, we believe that studying the properties of $\mathsf{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ and their associated FM kernels $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ remains valuable, which is in Section 6. Moreover, such exploration can provide deeper insights into the structure and behavior of the categorical $\mathcal{U}$-action.

We summarize the results in the following proposition.

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Proposition 1.4.

1. *Assuming that certain exact triangles like (1.8) and (1.9) exist as a condition in a categorical $\mathcal{U}$-action, then $\mathsf{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ are biadjoint to each other up to conjugation by $\Psi^{\pm}\mathbf{1}_{(k,N-k)}$ (Lemma 6.1).*
2. *For the FM kernel $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$, there exists the following exact triangle:*

$$\Delta_*\mathcal{V} \to \mathcal{H}_1\mathbf{1}_{(k,N-k)} \to \Delta_*\mathbb{C}^N/\mathcal{V} \in \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k,N)\times \mathrm{Gr}(k,N))$$

   where $\Delta : \mathrm{Gr}(k,N) \to \mathrm{Gr}(k,N)\times\mathrm{Gr}(k,N)$ is the diagonal map. Moreover, $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ is neither isomorphic to $\Delta_*\mathbb{C}^N$ nor to $\Delta_*(\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})$, provided $k \neq 0, N$ (Proposition 6.2).
3. *We compute the convolutions between $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ and $\Psi^+\mathbf{1}_{(k,N-k)}$, $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$ in the simplest case where $k = 1$ and $N = 2$.*
   (a) *We have the non-isomorphism $(\mathcal{H}_1 * \Psi^+)\mathbf{1}_{(1,1)} \not\cong (\Psi^+ * \mathcal{H}_1)\mathbf{1}_{(1,1)}$, while $h_1\psi^+1_{(1,1)} = \psi^+h_1 1_{(1,1)}$ (Proposition 6.4).*
   (b) *There exists the exact triangle (Proposition 6.6)*

$$(\mathcal{H}_1 * \mathcal{E}_{-1})\mathbf{1}_{(1,1)} \to (\mathcal{E}_{-1} * \mathcal{H}_1)\mathbf{1}_{(1,1)} \to \mathcal{E}_0\mathbf{1}_{(1,1)} \oplus \mathcal{E}_0\mathbf{1}_{(1,1)}[1] \in \mathrm{D}^b\mathrm{Coh}(\mathbb{P}^1),$$

   which can be viewed as a nontrivial categorification of $[h_1, e_{-1}]1_{(1,1)} = 0$. Finally, we propose a conjecture (Conjecture 6.8) about the convolution relations between $\mathcal{H}_{\pm 1}\mathbf{1}_{(k,N-k)}$ and $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$, $\mathcal{F}_s\mathbf{1}_{(k,N-k)}$.

1.2.4. The Grothendieck groups

Like the categorical $\mathfrak{sl}_2$-action, as we expect, a categorical $\mathcal{U}$-action should also recover the action of $\mathcal{U}$ on the level of Grothendieck groups. We prove this in Lemma 3.5. Thus, by (3) in Theorem 1.2 (or Theorem 5.6), we obtain the following corollary.

Corollary 1.5 (Corollary 5.14 for $\mathfrak{sl}_n$ case). *There is an action of $\mathcal{U}$ on $\bigoplus_k K(\mathrm{Gr}(k,N))$.*

Although we do not include all the loop generators in our presentation of $\mathcal{U}$, with the help from the study of categorical $\mathcal{U}$-action on $\bigoplus_k \mathrm{D}^b\mathrm{Coh}(\mathrm{Gr}(k,N))$, we extend the commutator relation $[e_r, f_s]1_{(k,N-k)}$ on $K(\mathrm{Gr}(k,N))$ in Corollary 1.5 to all $r, s \in \mathbb{Z}$. Note that here the actions of $e_r1_{(k,N-k)}$ are given by the decategorified (or K-theoretic) FM transformations with kernels given by the classes of the line bundles $(\mathcal{V}/\mathcal{V}')^{\otimes r}$ in the Grothendieck group $K(\mathrm{Fl}(k-1,k))$, similarly for $f_s1_{(k,N-k)}$.

Corollary 1.6 (Corollary 6.10 for $\mathfrak{sl}_n$ case). *The commutator relations in the Grothendieck group $K(\mathrm{Gr}(k,N))$ for $e_r1_{(k,N-k)}$, $f_s1_{(k,N-k)}$ with $r, s \in \mathbb{Z}$ are given by*

$$[e_r, f_s]1_{(k,N-k)} = \begin{cases} \otimes(-1)^{N-k-1}[\det(\mathbb{C}^N/\mathcal{V})][\mathrm{Sym}^{r+s-N+k}(\mathbb{C}^N)] & \text{if } r+s \geq N-k \\ 0 & \text{if } -k+1 \leq r+s \leq N-k-1. \\ \otimes(-1)^k[\det(\mathcal{V})^{-1}][\mathrm{Sym}^{-r-s-k}(\mathbb{C}^N)^\vee] & \text{if } r+s \leq -k \end{cases}$$

1.3. Application to the Affine 0-Hecke Algebra

In the second part of this article, we apply the categorical $\mathcal{U}$-action to construct a categorical action of the affine 0-Hecke algebra, denoted by $\mathcal{H}_N(0)$, on the derived category of coherent sheaves on the full flag variety.

The affine 0-Hecke algebra $\mathcal{H}_N(0)$ arises as a specific degeneration of the affine Hecke algebra, obtained by setting $q = 0$ in its relations. It was first introduced in the work of Kostant and Kumar [32], where they provided a geometric realization of the affine 0-Hecke algebra via the G-equivariant K-theory of the product of full flag varieties, equipped with the convolution product.

The affine 0-Hecke algebra also plays a fundamental role in the study of mod p representations of p-adic reductive groups, where it appears under various names. For instance, He and Nie [24] use the

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term "affine pro-p Hecke algebra" in their investigation of its cocenter, while Abe [1] and Vignéras [39] refer to it as the "pro-p-Iwahori Hecke algebra" in their classification of irreducible representations.

Analogous to the categorification of affine Hecke algebras, such as Bezrukavnikov's two geometric realizations [11], it is natural to consider the categorification of affine 0-Hecke algebras. For example, see [4] for related developments. Building on the work of Kostant and Kumar, there is a natural action of the affine 0-Hecke algebra on the G-equivariant K-theory of the full flag variety, realized through the convolution product.

By forgetting the G-equivariance, our second main result of this article is to categorify the above action by lifting the action from K-theory to the derived category of coherent sheaves.

Let us elaborate on this in more detail. We fix $G = \mathrm{SL}_N(\mathbb{C})$ and let $B \subset G$ denote the Borel subgroup of upper triangular matrices. The full flag variety is described as follows:

$$G/B = \{0 = V_0 \subset V_1 \subset \ldots \subset V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for all } k\}, \tag{1.10}$$

Similarly, the partial flag variety is given by

$$G/P_i = \{0 \subset V_1 \subset V_2 \subset \ldots V_{i-1} \subset V_{i+1} \subset \ldots V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for } k \neq i\} \tag{1.11}$$

where P_i is a minimal parabolic subgroup for $1 \le i \le N-1$.

The Demazure operators, introduced by Demazure [21] and rooted in the fundamental works of Bernstein-Gelfand-Gelfand [10], are defined as

$$T_i := \pi_i^* \pi_{i*}, \text{ for all } 1 \le i \le N-1.$$

Here, $\pi_i : G/B \longrightarrow G/P_i$ denotes the natural projection, and π_i^* and π_{i*} represent the induced pullback and pushforward on the Grothendieck group, respectively, for all $1 \le i \le N-1$.

On the other hand, let $\mathcal{V}_i$ denote the tautological bundle of rank i on G/B for $0 \le i \le N$. Then, for $1 \le i \le N$, we have the natural line bundles $\mathscr{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ on G/B.

The Demazure operators T_i, together with the operators given by multiplication by the classes of the line bundles $[\mathscr{L}_j]$, generate the affine 0-Hecke algebra $\mathcal{H}_N(0)$. Consequently, these operators define an action of $\mathcal{H}_N(0)$ on $K(G/B)$. For the definition of $\mathcal{H}_N(0)$, we refer the readers to Definition 7.1.

The following is our second main result.

Theorem 1.7 (Theorem 7.3). *There is a categorical action of $\mathcal{H}_N(0)$ on $\mathrm{D}^b\mathrm{Coh}(G/B)$.*

One approach to proving this theorem is to directly lift the action by replacing the generators with FM transformations. However, verifying the relations for this action requires performing six convolutions of kernels and checking various exact triangles that relate them (see Theorem 7.3 for details). Instead of pursuing this direct but intricate method, we reinterpret the Demazure operators as elements of the shifted 0-affine algebra and leverage its categorical action to present a more concise and elegant proof.

We need to introduce more notations. For each $\underline{k} = (k_1, \ldots, k_n) \vDash N$, the n-step partial flag variety is defined by

$$\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N) := \{V_\bullet = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^N) \mid \dim V_i/V_{i-1} = k_i \text{ for all } i\}. \tag{1.12}$$

With this notation, the full flag variety G/B and partial flag varieties G/P_i in (1.10) and (1.11) have the following description

$$G/B = \mathrm{Fl}_{(1,1,\ldots,1)}(\mathbb{C}^N),\ G/P_i = \mathrm{Fl}_{(1,1,\ldots,1)+\alpha_i}(\mathbb{C}^N) = \mathrm{Fl}_{(1,1,\ldots,1)-\alpha_i}(\mathbb{C}^N)$$

where $\alpha_i = (0, \ldots, -1, 1, \ldots, 0)$ is the simple root with -1 in the ith position and we have the following diagram

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$$
\begin{tikzcd}
K(G/P_i = \mathrm{Fl}_{(1,1,\dots,1)-\alpha_i}(\mathbb{C}^N)) \arrow[r, shift left, "e_{i,r}"] & K(G/B = \mathrm{Fl}_{(1,1,\dots,1)}(\mathbb{C}^N)) \arrow[l, shift left, "f_{i,s}"] \arrow[r, shift left, "e_{i,r}"] \arrow[loop above, "T_i"] & K(G/P_i = \mathrm{Fl}_{(1,1,\dots,1)+\alpha_i}(\mathbb{C}^N)) \arrow[l, shift left, "f_{i,s}"]
\end{tikzcd}
\tag{1.13}
$$

By utilizing (1.13), it is possible to express the Demazure operators T_i as elements within $\mathcal{U}$. This implies that all the categorical relations we need to verify can be derived from conditions in the categorical $\mathcal{U}$-action, leading to a significant reduction in calculations.

We would like to mention the related works of Arkhipov and Kanstrup. In their series of papers [2], [3], [4], and [5], they introduced the concept of *Demazure descent data* on a triangulated category as an initial attempt to comprehend the higher categorical Beilinson-Bernstein localization, which was developed by Ben-Zvi and Nalder [9]. The categorified Demazure operators from our Theorem 1.7 provide Demazure descent data for the triangulated category $\mathrm{D}^b\mathrm{Coh}(G/B)$.

Lastly, for other potential applications of the categorical actions of the shifted 0-affine algebra that are not covered in this article, we recommend referring to [25] for constructing semiorthogonal decompositions of the weight categories, and to [26] for constructing pairs of complementary idempotents in the triangulated category of triangulated endofunctors for each weight category.

1.4. Related Works and Further Directions

We address the relations to other works and point out some possibly interesting further directions that we would like to study in the future.

1.4.1. Equivariant version

In this article, we construct a categorical $\mathcal{U}$-action on the *non-equivariant* derived category of coherent sheaves. Upgrading the result to the equivariant setting is a natural direction for future research. Two related works come to mind.

The first one is related to Nakajima's construction of the action of quantum loop algebras on the equivariant K-theory of cotangent bundles of n-step partial flag varieties [37]. Specifically, we have an action of $\mathbf{U}_q(L\mathfrak{sl}_n)$ on $\bigoplus_{\underline{k}} K^{\mathbb{C}^*}(T^*\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$, and since there are isomorphisms $K^{\mathbb{C}^*}(T^*\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \xrightarrow{\simeq} K^{\mathbb{C}^*}(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$, it would be interesting to explore the relationship between the two actions.

The second work is related to Arkhipov-Mazin's work [6], where they introduced an algebra $\mathfrak{U}$ called the $q = 0$ affine quantum group and constructed an action of $\mathfrak{U}$ on the $\mathrm{GL}_N(\mathbb{C})$-equivariant K-theory of n-step partial flag varieties. We expect that their algebra $\mathfrak{U}$ is the same as the one defined in Definition A.1 with loop presentation.

1.4.2. Other categorical relations and 2-representation

As previously mentioned, our chosen presentation allows us to define the categorical action using a finite number of generators and relations. Nonetheless, it is natural to consider other categorical relations that we do not explore in this article, especially those involving $\mathsf{E}_r\mathsf{F}_s\mathbf{1}_{(k,N-k)}$ and $\mathsf{F}_s\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ when $r + s \geq N - k + 1$ and $r + s \leq -k - 1$.

We would like to comment on the subject of 2-representations. Categorification of quantum groups has been extensively studied in the literature, as seen in [18], [29], [30], [31], and [38]. Many of these results lead to the construction of KLR (or quiver Hecke) algebras, which act as natural transformations on the generating 1-morphisms E_i, F_j and their compositions. Therefore, it is natural to consider higher relations, such as natural transformations between the functors in our categorical action. For example, it is expected that the exact triangles (1.6), (1.7) can be induced from certain natural transformations.

1.5. Organization

In Section 2, we define the shifted 0-affine algebras. We also mention the definition of shifted quantum affine algebra defined by Finkelberg-Tsymbaliuk [22].

In Section 3, we define the categorical action of the shifted 0-affine algebras.

In Section 4, we recall some background on the Fourier-Mukai transformations, which will be used in the next few sections to prove the categorical action.

In Section 5, we prove the main theorem of this article, that is, there is a categorical action of shifted 0-affine algebra on the bounded derived categories of coherent sheaves of Grassmannians and n-step partial flag varieties (Theorem 5.6).

In Section 6, we study the commutator relation between $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathsf{F}_s\mathbf{1}_{(k,N-k)}$ that is not covered in Section 5 for the first non-trivial case. Then, we discuss the properties of the 1-morphisms $\mathsf{H}_{\pm 1}$ and their FM kernels $\mathcal{H}_{\pm 1}$. Finally, we calculate the commutators of the loop generators at the level of the Grothendieck group.

In Section 7, we show that there is a categorical action of the affine 0-Hecke algebras on the bounded derived category of coherent sheaves on the full flag variety by interpreting the Demazure operators in terms of the elements in the shifted 0-affine algebra (Theorem 7.3).

2. Shifted 0-Affine Algebra

In this section, we first recall the definition of shifted quantum affine algebra from [22]. Then we define the shifted 0-affine algebra.

2.1. Shifted Quantum Affine Algebra

In this subsection, we recall the definition of shifted quantum affine algebras. Our main reference is [22, Section 5].

First, we fix some notations. Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $(\cdot,\cdot)$ be a non-degenerated invariant symmetric bilinear form on $\mathfrak{g}$. Let $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ be the simple roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$ be the simple coroots. Let $c_{ij} := 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ be the entries of the Cartan matrix and $d_i := \frac{(\alpha_i,\alpha_i)}{2}$ such that $d_ic_{ij} = d_jc_{ji}$ for any $i, j \in I$. We also fix the notations $q_i := q^{d_i}$, $[m]_q := \frac{q^m-q^{-m}}{q-q^{-1}}$ and $\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a-b+1]_q\dots[a]_q}{[1]_q\dots[b]_q}$.

Definition 2.1 [22, Subsection 5.1]. Given two coweights μ^+, μ^-, set $b_i^\pm := \alpha_i(\mu^\pm)$. Then *the shifted quantum affine algebra* (simply-connected version), denoted by $\mathcal{U}_{\mu^+,\mu^-}$, is an associated $\mathbb{C}(q)$ algebra generated by

$$\{e_{i,r},\ f_{i,r},\ (\psi^\pm_{i,\pm s_i^\pm}),\ (\psi^\pm_{i,\mp b_i^\pm})^{-1}\}_{i\in I}^{r\in\mathbb{Z},\ s_i^\pm\geq -b_i^\pm}$$

subject to the following relations (for all $i, j \in I$ and $\epsilon, \epsilon' \in \{\pm\}$)

$$[\psi_i^\epsilon(z), \psi_j^{\epsilon'}(w)] = 0,\ \psi^\pm_{i,\mp b_i^\pm}(\psi^\pm_{i,\mp b_i^\pm})^{-1} = (\psi^\pm_{i,\mp b_i^\pm})^{-1}\psi^\pm_{i,\mp b_i^\pm} = 1, \tag{U1}$$

$$(z - q_i^{c_{ij}}w)e_i(z)e_j(w) = (q_i^{c_{ij}}z - w)e_j(w)e_i(z), \tag{U2}$$

$$(q_i^{c_{ij}}z - w)f_i(z)f_j(w) = (z - q_i^{c_{ij}}w)f_j(w)f_i(z), \tag{U3}$$

$$(z - q_i^{c_{ij}}w)\psi_i^\epsilon(z)e_j(w) = (q_i^{c_{ij}}z - w)e_j(w)\psi_i^\epsilon(z), \tag{U4}$$

$$(q_i^{c_{ij}}z - w)\psi_i^\epsilon(z)f_j(w) = (z - q_i^{c_{ij}}w)f_j(w)\psi_i^\epsilon(z), \tag{U5}$$

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$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \delta(\frac{z}{w})(\psi_i^+(z) - \psi_i^-(z)), \tag{U6}$$

$$\mathrm{Sym}_{z_1,\ldots,z_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} e_i(z_1)\ldots.e_i(z_r)e_j(w)e_i(z_{r+1})\ldots e_i(z_{1-c_{ij}}) = 0, \tag{U7}$$

$$\mathrm{Sym}_{z_1,\ldots,z_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} f_i(z_1)\ldots.f_i(z_r)f_j(w)f_i(z_{r+1})\ldots f_i(z_{1-c_{ij}}) = 0, \tag{U8}$$

where $\mathrm{Sym}_{z_1,\ldots,z_s}$ stands for the symmetrization in $z_1,\ldots,z_s$ and the generating series are defined as follows

$$e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r}, f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r}z^{-r}, \psi_i^\pm(z) := \sum_{r\geq -b_i^\pm} \psi_{i,\pm r}^\pm z^{\mp r}, \delta(z) := \sum_{r\in\mathbb{Z}} z^r.$$

Let us introduce another set of Cartan generators $\{h_{i,r}\}_{i\in I}^{r>0}$ via

$$(\psi_{i,\mp b_i^\pm}^\pm z^{\pm b_i^\pm})^{-1}\psi_i^\pm(z) = \exp\Big(\pm(q_i - q_i^{-1})\sum_{r>0} h_{i,\pm r}z^{\mp r}\Big).$$

With this, the relations (U4), (U5) are equivalent to the following:

$$\psi_{i,\mp b_i^\pm}^\pm e_{j,s} = q_i^{\pm c_{ij}} e_{j,s}\psi_{i,\mp b_i^\pm}^\pm,\ [h_{i,r}, e_{j,s}] = \frac{[rc_{ij}]_{q_i}}{r} e_{j,r+s},$$
$$\psi_{i,\mp b_i^\pm}^\pm f_{j,s} = q_i^{\mp c_{ij}} f_{j,s}\psi_{i,\mp b_i^\pm}^\pm,\ [h_{i,r}, f_{j,s}] = -\frac{[rc_{ij}]_{q_i}}{r} f_{j,r+s}.$$

We mention some remarks about the properties of $\mathcal{U}_{\mu^+,\mu^-}$ that have been addressed in [22].

Remark 2.2. (1) The algebra $\mathcal{U}_{\mu^+,\mu^-}$ depends only on $\mu := \mu^+ + \mu^-$ up to isomorphism. We say that $\mathcal{U}_{\mu^+,\mu^-}$ is dominantly (resp. antidominantly) shifted if μ is a dominant (resp. antidominant) coweight.
(2) We have $\mathcal{U}_{0,0}/(\psi_{i,0}^+\psi_{i,0}^- - 1) \simeq \mathcal{U}_q(L\mathfrak{g})$-the standard quantum loop algebra of $\mathfrak{g}$.

When $\mathcal{U}_{\mu^+,\mu^-}$ is antidominantly shifted (i.e., $\mu = \mu^+ + \mu^-$ is antidominant), then it admits another presentation, which is the so-called Levendorskii type presentation.

Definition 2.3 [22, Subsection 5.5]. Given antidominant coweights μ_1, μ_2, set $\mu = \mu_1 + \mu_2$. Define $b_{1,i} := \alpha_i(\mu_1)$, $b_{2,i} := \alpha_i(\mu_2)$, $b_i = b_{1,i} + b_{2,i}$. Then we denote $\hat{\mathcal{U}}_{\mu_1,\mu_2}$ to be the associated $\mathbb{C}(q)$ algebra generated by

$$\{e_{i,r},\ f_{i,s},\ (\psi_{i,0}^+)^{\pm1},\ (\psi_{i,b_i}^-)^{\pm1},\ h_{i,\pm1} \mid i\in I,\ b_{2,i}-1\leq r\leq 0,\ b_{1,i}\leq s\leq 1\}$$

subject to the following relations

$$\{(\psi_{i,0}^+)^{\pm1},\ (\psi_{i,b_i}^-)^{\pm1},\ h_{i,\pm1}\}_{i\in I} \text{ pairwise commute}, \tag{U1'}$$

$$(\psi_{i,0}^+)^{\pm1}\cdot(\psi_{i,0}^+)^{\mp1} = (\psi_{i,b_i}^-)^{\pm1}\cdot(\psi_{i,b_i}^-)^{\mp1} = 1, \tag{U2'}$$

$$e_{i,r+1}e_{j,s} - q_i^{c_{ij}} e_{i,r}e_{j,s+1} = q_i^{c_{ij}} e_{j,s}e_{i,r+1} - e_{j,s+1}e_{i,r}, \tag{U3'}$$

$$q_i^{c_{ij}} f_{i,r+1}f_{j,s} - f_{i,r}f_{i,s+1} = f_{j,s}f_{i,r+1} - q_i^{c_{ij}} f_{j,s+1}f_{i,r}, \tag{U4'}$$

$$\psi_{i,0}^+e_{j,r} = q_i^{c_{ij}} e_{j,r}\psi_{i,0}^+,\ \psi_{i,b_i}^-e_{j,r} = q_i^{-c_{ij}} e_{j,r}\psi_{i,b_i}^-,\ [h_{i,\pm1}, e_{j,r}] = [c_{ij}]_{q_i} e_{j,r\pm1}, \tag{U5'}$$

$$\psi_{i,0}^+f_{j,s} = q_i^{-c_{ij}} f_{j,s}\psi_{i,0}^+,\ \psi_{i,b_i}^-f_{j,s} = q_i^{c_{ij}} f_{j,s}\psi_{i,b_i}^-,\ [h_{i,\pm1}, f_{j,s}] = -[c_{ij}]_{q_i} f_{j,s\pm1}, \tag{U6'}$$

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$$[e_{i,r}, f_{j,s}] = 0 \text{ if } i \neq j \quad\text{and}\quad [e_{i,r}, f_{i,s}] = \begin{cases} \psi^+_{i,0} h_{i,1} & \text{if } r+s=1 \\ \frac{\psi^+_{i,0} - \delta_{b_i,0}\psi^-_{i,b_i}}{q_i - q_i^{-1}} & \text{if } r+s=0 \\ 0 & \text{if } b_i+1 \leq r+s \leq -1, \\ \frac{-\psi^-_{i,b_i} + \delta_{b_i,0}\psi^-_{i,0}}{q_i - q_i^{-1}} & \text{if } r+s=b_i \\ \psi^-_{i,b_i} h_{i,-1} & \text{if } r+s=b_i-1 \end{cases} \tag{U7'}$$

$$\sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} e^r_{i,0} e_{j,0} e^{1-c_{ij}-r}_{i,0} = 0, \quad \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} f^r_{i,0} f_{j,0} f^{1-c_{ij}-r}_{i,0} = 0, \tag{U8'}$$

$$[h_{i,1}, [f_{i,1}, [h_{i,1}, e_{i,0}]]] = 0, \ [h_{i,-1}, [e_{i,b_{2,i}-1}, [h_{i,-1}, f_{i,b_{1,i}}]]] = 0, \tag{U9'}$$

for any $i, j \in I$ and r, s such that the above relations make sense.

With those generators, then we define inductively

$$e_{i,r} := [2]^{-1}_{q_i} \begin{cases} [h_{i,1}, e_{i,r-1}] & \text{if } r > 0 \\ [h_{i,-1}, e_{i,r+1}] & \text{if } r < b_{2,i} - 1, \end{cases}$$

$$f_{i,r} := -[2]^{-1}_{q_i} \begin{cases} [h_{i,1}, f_{i,r-1}] & \text{if } r > 1 \\ [h_{i,-1}, f_{i,r+1}] & \text{if } r < b_{1,i}, \end{cases}$$

$$\psi^+_{i,r} := (q_i - q_i^{-1})[e_{i,r-1}, f_{i,1}] \text{ for } r > 0,$$

$$\psi^-_{i,r} := (q_i^{-1} - q_i)[e_{i,r-b_{1,i}}, f_{i,b_{1,i}}] \text{ for } r < b_i.$$

Then we have the following theorem, which says that in the antidominantly shifted setting, the two presentations from Definition 2.1 and Definition 2.3 are equivalent.

Theorem 2.4 [22, Theorem 5.5] *There is a $\mathbb{C}(q)$-algebra isomorphism $\hat{\mathcal{U}}_{\mu_1,\mu_2} \longrightarrow \mathcal{U}_{0,\mu}$ such that*

$$e_{i,r} \mapsto e_{i,r}, \ f_{i,r} \mapsto f_{i,r}, \ \psi^\pm_{i,\pm s_i^\pm} \mapsto \psi^\pm_{i,\pm s_i^\pm} \ \textit{for } i \in I, \ r \in \mathbb{Z}, \ s_i^+ \geq 0, \ s_i^- \geq -b_i.$$

Remark 2.5. We list some relations explicitly for the readers when $\mathfrak{g} = \mathfrak{sl}_n$, which is the main type of Lie algebras that we will study for the shifted 0-affine algebra later. In this case, we have $c_{ij} = \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } |i-j|=1, \\ 0 & \text{if } |i-j| \geq 2 \end{cases}$ and $d_i = 1$ for all i. The Cartan matrix is given by

$$(c_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Then $q_i = q$ for all i and the numbers c_{ij} in the relations of the algebra $\hat{\mathcal{U}}_{\mu_1,\mu_2}$ in Definition 2.3 for $\mathfrak{g} = \mathfrak{sl}_n$ are known. For example, some of the relations in (U3') are $e_{i,r+1}e_{i,s} - q^2 e_{i,r}e_{i,s+1} = q^2 e_{i,s}e_{i,r+1} - e_{i,s+1}e_{i,r}$, similarly for (U4'). The relations in (U5') are $\psi^+_{i,0}e_{i,r} = q^2 e_{i,r}\psi^+_{i,0}$, $\psi^-_{i,b_i}e_{i,r} = q^{-2}e_{i,r}\psi^-_{i,b_i}$, and $[h_{i,\pm1}, e_{i,r}] = [2]_q e_{i,r\pm1}$, similarly for (U6'). The relation (U8') is just the (quantum) Serre relations, for example, $e_{i,0}e_{j,0}e_{i,0} = \frac{1}{[2]_q}(e^2_{i,0}e_{j,0} + e_{j,0}e^2_{i,0})$.

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2.2. Definition of the Shifted 0-Affine Algebras

In this section, we define the shifted 0-affine algebras. We define it by imitating Definition 2.3, which is by finite generators and relations. The main reason we use such a presentation is due to its simplicity and because we can easily define a categorical action for it (see next section).

On the other hand, we define another algebra in the appendix A by using the usual loop presentation (see Definition A.1). Similarly to Theorem 2.4, we conjecture that the two presentations, i.e., Definition 2.3 and Definition A.1, are equivalent (see Conjecture A.4).

In [8], they introduce the dot version (or idempotent modification) $\dot{\mathbf{U}}_q(\mathfrak{sl}_2)$ of $\mathbf{U}_q(\mathfrak{sl}_2)$, since our motivation comes from their geometric construction, the shifted 0-affine algebras we introduce below is also an idempotent version. This means that we replace the identity by the direct sum of a system of projectors, one for each element of the weight lattices. They are orthogonal idempotents for approximating the unit element. We refer to [36, Chapter 23] for details of such modification.

Throughout the rest of this article, we fix a positive integer $N \geq 2$. Let

$$C(n,N) := \{\underline{k} = (k_1, ..., k_n) \in (\mathbb{N} \cup \{0\})^n \mid k_1 + ... + k_n = N\}.$$

We regard each $\underline{k}$ as a weight for $\mathfrak{sl}_n$ via the identification of the weight lattice of $\mathfrak{sl}_n$ with the quotient $\mathbb{Z}^n/(1,1,...,1)$. We choose the simple root α_i to be $(0...0,-1,1,0...0)$ where the -1 is in the i-th position for $1 \leq i \leq n-1$. Then we introduce the shifted 0-affine algebra for $\mathfrak{sl}_n$, which is defined by using finite generators and relations.

Definition 2.6. We consider formal symbols of the form $1_\lambda x 1_\mu$ ($\lambda, \mu \in (\mathbb{N} \cup \{0\})^n$) and abbreviating $(1_{\lambda_1}x_1 1_{\mu_1})...(1_{\lambda_i}x_i 1_{\mu_i}) = 1_{\lambda_1}x_1...x_i 1_{\mu_i}$ if the product is nonzero. Then we define the *shifted 0-affine algebra*, denoted by $\mathcal{U} = \dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_n)$, to be the associative $\mathbb{C}$-algebra generated by

$$\bigcup_{\underline{k}\in C(n,N)} \{1_{\underline{k}},\ 1_{\underline{k}+\alpha_i} e_{i,r} 1_{\underline{k}},\ 1_{\underline{k}-\alpha_i} f_{i,s} 1_{\underline{k}},\ 1_{\underline{k}}(\psi_i^+)^{\pm 1} 1_{\underline{k}},\ 1_{\underline{k}}(\psi_i^-)^{\pm 1} 1_{\underline{k}}\}_{1\leq i\leq n-1}^{-k_i\leq r\leq 0,\ 0\leq s\leq k_{i+1}}$$

subject to the following relations

$$1_{\underline{k}} 1_{\underline{l}} = \delta_{\underline{k},\underline{l}} 1_{\underline{k}}, \tag{U01}$$

$$(\psi_i^+)^{\pm 1}(\psi_j^+)^{\pm 1} 1_{\underline{k}} = (\psi_j^+)^{\pm 1}(\psi_i^+)^{\pm 1} 1_{\underline{k}} \text{ for all } i, j, \tag{U02}$$

$$(\psi_i^+)^{\pm 1} \cdot (\psi_i^+)^{\mp 1} 1_{\underline{k}} = 1_{\underline{k}} = (\psi_i^-)^{\pm 1} \cdot (\psi_i^-)^{\mp 1} 1_{\underline{k}}, \tag{U03}$$

$$e_{i,r}e_{j,s}1_{\underline{k}} = \begin{cases} -e_{i,s+1}e_{i,r-1}1_{\underline{k}} & \text{if } j = i \\ e_{i+1,s}e_{i,r}1_{\underline{k}} - e_{i+1,s-1}e_{i,r+1}1_{\underline{k}} & \text{if } j = i+1 \\ e_{i,r+1}e_{i-1,s-1}1_{\underline{k}} - e_{i-1,s-1}e_{i,r+1}1_{\underline{k}} & \text{if } j = i-1 \\ e_{j,s}e_{i,r}1_{\underline{k}} & \text{if } |i-j| \geq 2 \end{cases}, \tag{U04}$$

$$f_{i,r}f_{j,s}1_{\underline{k}} = \begin{cases} -f_{i,s-1}f_{i,r+1}1_{\underline{k}} & \text{if } j = i \\ f_{i,r-1}f_{i+1,s+1}1_{\underline{k}} - f_{i+1,s+1}f_{i,r-1}1_{\underline{k}} & \text{if } j = i+1 \\ f_{i-1,s}f_{i,r}1_{\underline{k}} - f_{i-1,s+1}f_{i,r-1}1_{\underline{k}} & \text{if } j = i-1 \\ f_{j,s}f_{i,r}1_{\underline{k}} & \text{if } |i-j| \geq 2 \end{cases}, \tag{U05}$$

$$\psi_i^+ e_{j,r}1_{\underline{k}} = \begin{cases} -e_{i,r+1}\psi_i^+ 1_{\underline{k}} & \text{if } j = i \\ -e_{i+1,r-1}\psi_i^+ 1_{\underline{k}} & \text{if } j = i+1 \\ e_{i-1,r}\psi_i^+ 1_{\underline{k}} & \text{if } j = i-1 \\ e_{j,r}\psi_i^+ 1_{\underline{k}} & \text{if } |i-j| \geq 2 \end{cases}, \quad \psi_i^- e_{j,r}1_{\underline{k}} = \begin{cases} -e_{i,r+1}\psi_i^- 1_{\underline{k}} & \text{if } j = i \\ e_{i+1,r}\psi_i^- 1_{\underline{k}} & \text{if } j = i+1 \\ -e_{i-1,r-1}\psi_i^- 1_{\underline{k}} & \text{if } j = i-1 \\ e_{j,r}\psi_i^- 1_{\underline{k}} & \text{if } |i-j| \geq 2 \end{cases}, \tag{U06}$$

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$$\psi_i^+ f_{j,r} 1_{\underline{k}} = \begin{cases} -f_{i,r-1}\psi_i^+ 1_{\underline{k}} & \text{if } j=i \\ -f_{i+1,r+1}\psi_i^+ 1_{\underline{k}} & \text{if } j=i+1 \\ f_{i-1,r}\psi_i^+ 1_{\underline{k}} & \text{if } j=i-1 \\ f_{j,r}\psi_i^+ 1_{\underline{k}} & \text{if } |i-j|\geq 2 \end{cases}, \quad \psi_i^- f_{j,r} 1_{\underline{k}} = \begin{cases} -f_{i,r-1}\psi_i^- 1_{\underline{k}} & \text{if } j=i \\ f_{i+1,r}\psi_i^- 1_{\underline{k}} & \text{if } j=i+1 \\ -f_{i-1,r+1}\psi_i^- 1_{\underline{k}} & \text{if } j=i-1 \\ f_{j,r}\psi_i^- 1_{\underline{k}} & \text{if } |i-j|\geq 2 \end{cases}, \tag{U07}$$

$$[e_{i,r}, f_{j,s}]1_{\underline{k}} = 0 \text{ if } i \neq j \text{ and } [e_{i,r}, f_{i,s}]1_{\underline{k}} = \begin{cases} \psi_i^+ 1_{\underline{k}} & \text{if } (r,s) = (0, k_{i+1}) \\ 0 & \text{if } -k_i + 1 \leq r+s \leq k_{i+1}-1 \\ -\psi_i^- 1_{\underline{k}} & \text{if } (r,s) = (-k_i, 0) \end{cases}, \tag{U08}$$

for any $1 \leq i, j \leq n-1$ and r, s such that the above relations make sense.

We discuss a bit about the use of the notation $\dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_n)$, and why we call it the name shifted 0-affine algebra. We present the discussion as a list of remarks.

Remark 2.7. First, the 0 in the notation emphasizes that our algebra is a certain $q = 0$ degeneration of the shifted quantum affine algebra analogous to Hecke algebras degenerate to 0-Hecke algebras. However, here "$q = 0$" does not mean that we substitute $q = 0$ directly in the relations of the shifted quantum affine algebras since q^{-1} appears. Only some of them can be, from the relations (U3'), (U4'), in $\mathfrak{sl}_n$ case, we have $c_{ij} = 2$ when $i = j$. Taking $q = 0$, we can see that they become the relations (U04), (U05) when $i = j$.

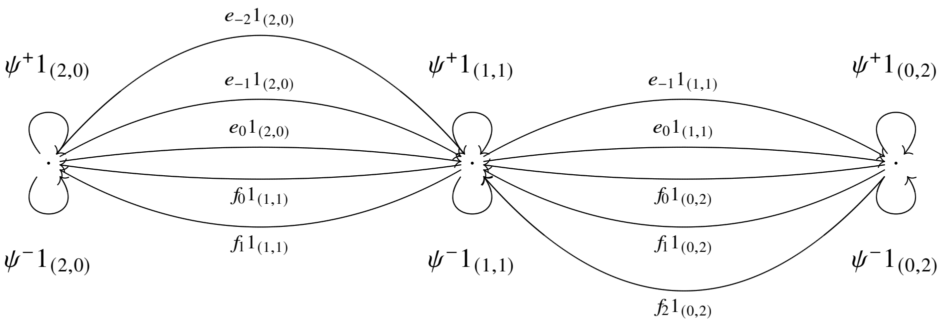
Remark 2.8. Second, the number N stands for the fact that our algebra depends on a choice of highest weight, for example, the highest weight $(0, N) = N\omega_1$ in the $\mathfrak{sl}_2$ case where ω_1 is the fundamental weight.

Remark 2.9. From Definition 2.6, the shifted 0-affine algebra is in fact a path algebra of the quiver with vertices indexed by $\underline{k} = (k_1, \ldots, k_n) \in C(n, N)$ and arrows given by $1_{\underline{k}}$, $e_{i,r}1_{\underline{k}}$, $f_{i,s}1_{\underline{k}}$, $(\psi_i^+)^{\pm 1}1_{\underline{k}}$, $(\psi_i^-)^{\pm 1}1_{\underline{k}}$ subjected to the relations (U01) to (U08) (see Example 2.10 below).

Thus, although the precise relation between our algebras and the shifted quantum affine algebras is still unclear, the above evidence suggests we call this algebra the name shifted 0-affine algebra.

Finally, we give an example of shifted 0-affine algebra.

Example 2.10. The shifted 0-affine algebra $\dot{\mathbf{U}}_{0,2}(L\mathfrak{sl}_2)$ is the path algebra of the following quiver



subject to the relations from (U01) to (U08). For example, we have $[e_0, f_1]1_{(1,1)} = \psi^+1_{(1,1)}$.

3. Categorical $\mathcal{U}$-Action

In this section, we define the categorical action for the shifted 0-affine algebra that is defined in Definition 2.6.

Before we give the definition, we have to mention that for the usual quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$, there are two presentations, one is the Kac-Moody presentation, and the other is the (Drinfeld-Jimbo) loop realization. The Kac-Moody presentation has the advantage that it is given by a finite number of generators and relations, while the loop realization is better for checking actions (on geometry) in practice. The categorical actions for the two presentations are also quite different.

While extensive research has been conducted on the categorical actions associated with the Kac-Moody presentation (e.g., [20], [29], [30], [31], and [38]); in contrast, the investigation into categorical actions of the loop realization is relatively limited, with a working definition provided in [14]. Although Definition 2.6 presents a slightly non-canonical form compared to the conventional loop presentation found in Appendix A, its appeal lies in the finite nature of its generators and relations. This characteristic serves as the primary motivation for utilizing this presentation and providing a definition for its categorical action.

We will use the notations $C(n,N)$ and α_i defined in Subsection 2.2. We also denote by $\langle\cdot,\cdot\rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ the standard pairing. Then we define the categorical action by imitating the definition of $(L\mathfrak{gl}_n,\theta)$-action in [14, Section 4], which originates from the definition of $(\mathfrak{g},\theta)$ action in [12, Section 2].

Definition 3.1. A *categorical* $\mathcal{U} = \dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_n)$*-action* consists of a target 2-category $\mathcal{K}$, which is triangulated, $\mathbb{C}$-linear and idempotent complete. The objects in $\mathcal{K}$ are

$$\mathrm{Ob}(\mathcal{K}) = \{\mathcal{K}(\underline{k}) \mid \underline{k} \in C(n,N)\}$$

where each $\mathcal{K}(\underline{k})$ is also a triangulated category, and each Hom space $\mathrm{Hom}(\mathcal{K}(\underline{k}),\mathcal{K}(\underline{l}))$ is also triangulated. On those objects $\mathcal{K}(\underline{k})$ we impose the following 1-morphisms:

$$\mathbf{1}_{\underline{k}},\ \mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} = \mathbf{1}_{\underline{k}+\alpha_i}\mathsf{E}_{i,r},\ \mathsf{F}_{i,s}\mathbf{1}_{\underline{k}} = \mathbf{1}_{\underline{k}-\alpha_i}\mathsf{F}_{i,s},\ (\Psi_i^{\pm})^{\pm1}\mathbf{1}_{\underline{k}} = \mathbf{1}_{\underline{k}}(\Psi_i^{\pm})^{\pm1},$$

where $1 \le i \le n-1$, $-k_i \le r \le 0$, $0 \le s \le k_{i+1}$. Here $\mathbf{1}_{\underline{k}}$ is the identity 1-morphism of $\mathcal{K}(\underline{k})$. Those 1-morphisms are subject to the following conditions.

1. The space of maps between any two 1-morphisms is finite-dimensional.
2. Suppose $i \neq j$. If $\mathbf{1}_{\underline{k}+\alpha_i}$ and $\mathbf{1}_{\underline{k}+\alpha_j}$ are nonzero, then $\mathbf{1}_{\underline{k}}$ and $\mathbf{1}_{\underline{k}+\alpha_i+\alpha_j}$ are also nonzero.
3. The left and right adjoints of $\mathsf{E}_{i,r}$ and $\mathsf{F}_{i,s}$ are given by conjugation of $\Psi_i^{\pm}$ up to homological shifts. More precisely,
   (a) $(\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}})^R \cong \mathbf{1}_{\underline{k}}(\Psi_i^+)^r\mathsf{F}_{i,k_{i+1}+1}(\Psi_i^+)^{-r-1}[-r]$ for all $1 \le i \le n-1$,
   (b) $(\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}})^L \cong \mathbf{1}_{\underline{k}}(\Psi_i^-)^{r+k_i-1}\mathsf{F}_{i,0}(\Psi_i^-)^{-r-k_i}[r+k_i]$ for all $1 \le i \le n-1$,
   (c) $(\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}})^R \cong \mathbf{1}_{\underline{k}}(\Psi_i^-)^{-s}\mathsf{E}_{i,-k_i-1}(\Psi_i^-)^{s-1}[s]$ for all $1 \le i \le n-1$,
   (d) $(\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}})^L \cong \mathbf{1}_{\underline{k}}(\Psi_i^+)^{-s+k_{i+1}-1}\mathsf{E}_{i,0}(\Psi_i^+)^{s-k_{i+1}}[-s+k_{i+1}]$ for all $1 \le i \le n-1$.
4. The 1-morphisms $\Psi_i^+\mathbf{1}_{\underline{k}}$ and $\Psi_i^-\mathbf{1}_{\underline{k}}$ satisfy the following

$$(\Psi_i^{\pm})^{\pm1}(\Psi_j^{\pm})^{\pm1}\mathbf{1}_{\underline{k}} \cong (\Psi_j^{\pm})^{\pm1}(\Psi_i^{\pm})^{\pm1}\mathbf{1}_{\underline{k}} \text{ for all } i,j,$$
$$(\Psi_i^{+})^{\pm1}(\Psi_i^{+})^{\mp1}\mathbf{1}_{\underline{k}} \cong (\Psi_i^{-})^{\pm1}(\Psi_i^{-})^{\mp1}\mathbf{1}_{\underline{k}} \cong \mathbf{1}_{\underline{k}} \text{ for all } i.$$

5. The relations between $\mathsf{E}_{i,r}$, $\mathsf{E}_{j,s}$ are given by the following
   (a)

$$\mathsf{E}_{i,r+1}\mathsf{E}_{i,s}\mathbf{1}_{\underline{k}} \cong \begin{cases} \mathsf{E}_{i,s+1}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}[-1] & \text{if } r-s \ge 1 \\ 0 & \text{if } r = s \\ \mathsf{E}_{i,s+1}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}[1] & \text{if } r-s \le -1. \end{cases}$$

   (b) $\mathsf{E}_{i,r}$, $\mathsf{E}_{i+1,s}$ are related by the following exact triangle

$$\mathsf{E}_{i+1,s}\mathsf{E}_{i,r+1}\mathbf{1}_{\underline{k}} \to \mathsf{E}_{i+1,s+1}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \to \mathsf{E}_{i,r}\mathsf{E}_{i+1,s+1}\mathbf{1}_{\underline{k}}.$$

(c)

$$\mathsf{E}_{i,r}\mathsf{E}_{j,s}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{j,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}, \text{ if } |i-j| \geq 2.$$

6. The relations between $\mathsf{F}_{i,r}$, $\mathsf{F}_{j,s}$ are given by the following
   (a)

$$\mathsf{F}_{i,r}\mathsf{F}_{i,s+1}\mathbf{1}_{\underline{k}} \cong \begin{cases} \mathsf{F}_{i,s}\mathsf{F}_{i,r+1}\mathbf{1}_{\underline{k}}[1] & \text{if } r-s \geq 1 \\ 0 & \text{if } r=s \\ \mathsf{F}_{i,s}\mathsf{F}_{i,r+1}\mathbf{1}_{\underline{k}}[-1] & \text{if } r-s \leq -1. \end{cases}$$

   (b) $\mathsf{F}_{i,r}$, $\mathsf{F}_{i+1,s}$ are related by the following exact triangles

$$\mathsf{F}_{i,r+1}\mathsf{F}_{i+1,s}\mathbf{1}_{\underline{k}} \to \mathsf{F}_{i,r}\mathsf{F}_{i+1,s+1}\mathbf{1}_{\underline{k}} \to \mathsf{F}_{i+1,s+1}\mathsf{F}_{i,r}\mathbf{1}_{\underline{k}}.$$

   (c)

$$\mathsf{F}_{i,r}\mathsf{F}_{j,s}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{j,s}\mathsf{F}_{i,r}\mathbf{1}_{\underline{k}}, \text{ if } |i-j| \geq 2.$$

7. The relations between $\mathsf{E}_{i,r}$, $\Psi^{\pm}_j$ are given by the following
   (a) $\Psi^{\pm}_i\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{i,r+1}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}[\mp 1]$.
   (b) For $|i-j|=1$, we have the following

$$\Psi^{\pm}_i\mathsf{E}_{i\pm 1,r}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{i\pm 1,r-1}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}[\pm 1],$$
$$\Psi^{\pm}_i\mathsf{E}_{i\mp 1,r}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{i\mp 1,r}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}.$$

   (c) $\Psi^{\pm}_i\mathsf{E}_{j,r}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{j,r}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}$, if $|i-j| \geq 2$.
8. The relations between $\mathsf{F}_{i,r}$, $\Psi^{\pm}_j$ are given by the following
   (a) $\Psi^{\pm}_i\mathsf{F}_{i,r}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{i,r-1}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}[\pm 1]$.
   (b) For $|i-j|=1$, we have the following

$$\Psi^{\pm}_i\mathsf{F}_{i\pm 1,r}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{i\pm 1,r+1}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}[\mp 1],$$
$$\Psi^{\pm}_i\mathsf{F}_{i\mp 1,r}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{i\mp 1,r}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}.$$

   (c) $\Psi^{\pm}_{i,}\mathsf{F}_{j,r}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{j,r}\Psi^{\pm}_i\mathbf{1}_{\underline{k}}$, if $|i-j| \geq 2$.
9. If $i \neq j$, then $\mathsf{E}_{i,r}\mathsf{F}_{j,s}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{j,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$.
10. The relation between the two compositions of 1-morphisms $\mathsf{E}_{i,r}\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}$ and $\mathsf{F}_{i,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \in \mathrm{Hom}(\mathcal{K}(\underline{k}),\mathcal{K}(\underline{k}))$ for $-k_i \leq r+s \leq k_{i+1}$ are given by
   (a) $\mathsf{F}_{i,k_{i+1}}\mathsf{E}_{i,0}\mathbf{1}_{\underline{k}} \to \mathsf{E}_{i,0}\mathsf{F}_{i,k_{i+1}}\mathbf{1}_{\underline{k}} \to \Psi^{+}_i\mathbf{1}_{\underline{k}}$,
   (b) $\mathsf{E}_{i,-k_i}\mathsf{F}_{i,0}\mathbf{1}_{\underline{k}} \to \mathsf{F}_{i,0}\mathsf{E}_{i,-k_i}\mathbf{1}_{\underline{k}} \to \Psi^{-}_i\mathbf{1}_{\underline{k}}$,
   (c) $\mathsf{F}_{i,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{i,r}\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}$, if $-k_i+1 \leq r+s \leq k_{i+1}-1$.
   for all r, s that make the above conditions make sense, and the isomorphisms between functors that appear in every condition are abstractly defined, i.e., we do not specify any 2-morphisms that induce those isomorphisms.

First, we give some remarks about this definition.

Remark 3.2. The 2-category $\mathcal{K}$ is called idempotent complete if, for any 2-morphism f with $f^2 = f$, the image of f is contained in $\mathcal{K}$.

Remark 3.3. Note that in our definition of categorical action, we do not have the linear maps

$$\mathrm{Span}\{\alpha_i \mid 1 \leq i \leq n-1\} \to \mathrm{End}^2(\mathbf{1}_{\underline{k}}),\ \underline{k} \in C(n,N)$$

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which is used to give the element θ in the definition of $(\mathfrak{g},\theta)$ or $(\hat{\mathfrak{g}},\theta)$ action in [12] or [18]. This is because usually, the geometry of the spaces that appear in our setting does not have a natural flat deformation; see Section 5 for our examples. The data of flat deformation can be used to obtain linear map $\mathrm{Span}\{\alpha_i \mid 1 \leq i \leq n-1\} \to \mathrm{End}^2(\mathbf{1}_{\underline{k}})$, which was shown in [12].

Remark 3.4. To establish a categorical action, we proceed by lifting the generators $e_{i,r}1_{\underline{k}}$, $f_{i,s}1_{\underline{k}}$, and $(\psi_i^{\pm})^{\pm 1}1_{\underline{k}}$ to 1-morphisms $\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$, $\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}$, and $(\Psi_i^{\pm})^{\pm 1}\mathbf{1}_{\underline{k}}$, respectively. Then, we check all the conditions in Definition 3.1. Note that our weight categories are assumed to be triangulated. Thus, when dealing with relations that involve three elements, i.e., (U04), (U05), and (U08), instead of employing a direct sum of 1-morphisms for the categorical action, we utilize exact triangles to lift relations that involve equality between three elements.

Next, we show that the above definition of categorical $\mathcal{U}$-action recovers an action of $\mathcal{U}$ on the vector spaces given by the Grothendieck groups of the weight categories.

Lemma 3.5. *Given a categorical $\mathcal{U}$-action $\mathcal{K}$, then there is an action of $\mathcal{U}$ on the Grothendieck groups $\bigoplus_{\underline{k}\in C(n,N)} K(\mathcal{K}(\underline{k}))$.*

Proof. Since there is a categorical $\mathcal{U}$-action on $\mathcal{K}$, we assign the action of the generators of $\mathcal{U}$ on $\bigoplus_{\underline{k}\in C(n,N)} K(\mathcal{K}(\underline{k}))$ via the following

$$1_{\underline{k}} \mapsto [\mathbf{1}_{\underline{k}}],\ e_{i,r}1_{\underline{k}} \mapsto [\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}],\ f_{i,s}1_{\underline{k}} \mapsto [\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}],\ \text{and}\ \psi_i^{\pm}1_{\underline{k}} \mapsto [\Psi_i^{\pm}\mathbf{1}_{(k,N-k)}] \tag{3.1}$$

where we denote $[T]$ to be the class of a 1-morphism T in the Grothendieck group.

Then, we need to verify that under the assignment (3.1), the generators satisfy relations (U01) to (U08). From the above discussion, it is clear to see that condition (4) implies (U02) and (U03), conditions (5) implies (U04), condition (6) implies (U05), conditions (7) implies (U06), condition (8) implies (U07), and conditions (9) and (10) imply (U08). Finally, relation (U01) is obvious from the definition of identity 1-morphisms $\mathbf{1}_{\underline{k}}$. □

Finally, since $\Psi_i^{+}\mathbf{1}_{\underline{k}}$ are invertible, from conditions (7)(a) and (8)(a) we have the following isomorphisms

$$\begin{aligned}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} &\cong (\Psi_i^+)^r\mathsf{E}_{i,0}(\Psi_i^+)^{-r}\mathbf{1}_{\underline{k}}[r],\ \text{for all } -k_i \leq r \leq 0,\\ \mathsf{F}_{i,s}\mathbf{1}_{\underline{k}} &\cong (\Psi_i^+)^{-s}\mathsf{F}_{i,0}(\Psi_i^+)^{s}\mathbf{1}_{\underline{k}}[s],\ \text{for all } 0 \leq s \leq k_{i+1}.\end{aligned}$$

Such a conjugation property suggests that, for a computational purpose and also for future applications, it is convenient to introduce the following 1-morphisms in a categorical $\mathcal{U}$-action

$$\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} := (\Psi_i^+)^r\mathsf{E}_{i,0}(\Psi_i^+)^{-r}\mathbf{1}_{\underline{k}}[r],\ \text{for all } r \in \mathbb{Z}, \tag{3.2}$$

$$\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}} := (\Psi_i^+)^{-s}\mathsf{F}_{i,0}(\Psi_i^+)^{s}\mathbf{1}_{\underline{k}}[s],\ \text{for all } s \in \mathbb{Z}. \tag{3.3}$$

Remark 3.6. Note that we also have the following isomorphisms by conditions (7) and (8)

$$\Psi_i^+(\Psi_i^-)^{-1}\mathsf{E}_{i,0}\mathbf{1}_{\underline{k}} \cong \mathsf{E}_{i,0}\Psi_i^+(\Psi_i^-)^{-1}\mathbf{1}_{\underline{k}},\ \Psi_i^+(\Psi_i^-)^{-1}\mathsf{F}_{i,0}\mathbf{1}_{\underline{k}} \cong \mathsf{F}_{i,0}\Psi_i^+(\Psi_i^-)^{-1}\mathbf{1}_{\underline{k}}.$$

Thus, we also obtain

$$\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \cong (\Psi_i^-)^r\mathsf{E}_{i,0}(\Psi_i^-)^{-r}\mathbf{1}_{\underline{k}}[r],\ \mathsf{F}_{i,s}\mathbf{1}_{\underline{k}} \cong (\Psi_i^-)^{-s}\mathsf{F}_{i,0}(\Psi_i^-)^{s}\mathbf{1}_{\underline{k}}[s] \tag{3.4}$$

for all $r, s \in \mathbb{Z}$.

With the general 1-morphisms in (3.2), (3.3), and the isomorphisms (3.4), by conjugating the exact triangles in conditions (10)(a) and (10)(b) with $(\Psi_i^+)^r\mathbf{1}_{\underline{k}}$ and $(\Psi_i^-)^{-s}\mathbf{1}_{\underline{k}}$, respectively, we obtain the following exact triangles

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$$\mathsf{F}_{i,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \to \mathsf{E}_{i,r}\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}} \to \Psi_i^+\mathbf{1}_{\underline{k}}, \text{ if } r+s=k_{i+1},$$
$$\mathsf{E}_{i,r}\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}} \to \mathsf{F}_{i,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}} \to \Psi_i^-\mathbf{1}_{\underline{k}}, \text{ if } r+s=-k_i. \tag{3.5}$$

Moreover, condition (10)(c) holds for all $r, s \in \mathbb{Z}$ such that $-1+k_i \le r+s \le k_{i+1}-1$.

4. Preliminaries on Coherent Sheaves and Fourier-Mukai Kernels

In this section, we briefly recall the notions about Fourier-Mukai transforms/kernels and other related tools that we would use for proofs in later sections. The readers can consult the book by Huybrechts [27] for details.

We will mostly work with the bounded derived category of coherent sheaves on an algebraic variety X, which we simply denote by $\mathrm{D}^b(X)$. Throughout this article, functors between derived categories are assumed to be derived functors, for example, we will write f^*, f_* instead of Lf^*, Rf_*, resp.

Let X and Y be two smooth projective varieties. A Fourier-Mukai kernel is any object $\mathcal{P}$ in the derived category of coherent sheaves on $X \times Y$. Given $\mathcal{P} \in \mathrm{D}^b(X \times Y)$, we define the associated Fourier-Mukai transform, which is the functor

$$\Phi_{\mathcal{P}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y),\ \mathcal{F} \mapsto \pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{P}).$$

We call $\Phi_{\mathcal{P}}$ the Fourier-Mukai transform with (Fourier-Mukai) kernel $\mathcal{P}$. For convenience, we would just write FM for Fourier-Mukai. The first property of FM transforms is that they have left and right adjoints that are themselves FM transforms.

Proposition 4.1 [27, Proposition 5.9]. *For $\Phi_{\mathcal{P}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is the FM transform with kernel $\mathcal{P}$, define*

$$\mathcal{P}_L = \mathcal{P}^\vee \otimes \pi_2^*\omega_Y[\dim Y],\ \mathcal{P}_R = \mathcal{P}^\vee \otimes \pi_1^*\omega_X[\dim X],$$

where $\mathcal{P}^\vee := R\mathcal{H}om(\mathcal{P}, \mathcal{O}_{X\times Y}) \in \mathrm{D}^b(X \times Y)$. Then

$$\Phi_{\mathcal{P}_L} : \mathrm{D}^b(Y) \to \mathrm{D}^b(X),\ \textit{and}\ \Phi_{\mathcal{P}_R} : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$$

are the left and right adjoints of $\Phi_{\mathcal{P}}$, respectively.

The second property is the composition of FM transforms is also a FM transform.

Proposition 4.2 [27, Proposition 5.10]. *Let X, Y, Z be smooth projective varieties over $\mathbb{C}$. Consider objects $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ and $\mathcal{Q} \in \mathrm{D}^b(Y \times Z)$. They define FM transforms $\Phi_{\mathcal{P}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, $\Phi_{\mathcal{Q}} : \mathrm{D}^b(Y) \to \mathrm{D}^b(Z)$. We would use $*$ to denote the operation for convolution, i.e.*

$$\mathcal{Q} * \mathcal{P} := \pi_{13*}(\pi_{12}^*(\mathcal{P}) \otimes \pi_{23}^*(\mathcal{Q})).$$

Then for $\mathcal{R} = \mathcal{Q} * \mathcal{P} \in \mathcal{D}^b(X \times Z)$, we have $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{R}}$.

Remark 4.3. Moreover by [27] remark 5.11, we have $(\mathcal{Q} * \mathcal{P})_L \cong (\mathcal{P})_L * (\mathcal{Q})_L$ and $(\mathcal{Q} * \mathcal{P})_R \cong (\mathcal{P})_R * (\mathcal{Q})_R$.

The next thing is about the derived pushforward of coherent sheaves. Let $\mathcal{V}$ be a vector bundle of rank n on a variety X, where $n \ge 2$. Then we can form the projective bundle $\mathbb{P}(\mathcal{V})$. We get in this way a $\mathbb{P}^{n-1}$-fibration $\pi : \mathbb{P}(\mathcal{V}) \to X$. Let $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(-1)$ be the relative tautological bundle and $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ be the dual bundle, and we define $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(i) := \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)^{\otimes i}$ for $i \in \mathbb{Z}$. Then we have the following result.

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Proposition 4.4 [23, Exercise 8.4 in Chapter 3].

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{V})}(i) \cong \begin{cases} \mathrm{Sym}^i(\mathcal{V}^\vee) & \text{if } i \geq 0 \\ 0 & \text{if } 1-n \leq i \leq -1 \\ \mathrm{Sym}^{-i-n}(\mathcal{V}) \otimes \det(\mathcal{V})[1-n] & \text{if } i \leq -n \end{cases}$$

in $\mathrm{D}^b(X)$, *where* $\mathcal{V}^\vee = R\mathcal{H}om(\mathcal{V}, \mathcal{O}_X) \in \mathrm{D}^b(X)$.

Remark 4.5. Note that the above result is slightly different from the result in [23]. More precisely, in this article $\mathbb{P}(\mathcal{V})$ parametrizes one-dimensional "sub-bundles" of $\mathcal{V}$ while in [23] $\mathbb{P}(\mathcal{V})$ parametrizes one-dimensional "quotient-bundles" of $\mathcal{V}$.

The final result we need is the push-pull of a complex under a closed embedding in the derived category.

Lemma 4.6 [27, Proposition 11.1 and Corollary 11.2]. *Let* $j : Y \hookrightarrow X$ *be a closed embedding. We assume* Y *is the zero loci of a regular section of a locally free sheaf of rank* $c = \mathrm{codim}(Y \subset X)$ *on* X. *Denote* $\mathcal{N}_{Y/X}$ *to be its normal bundle. Then, for any* $\mathcal{F}^\bullet \in \mathrm{D}^b(Y)$ *we have*

$$\mathcal{H}^l(j^* j_* \mathcal{F}^\bullet) \cong \bigoplus_{s-r=l} \bigwedge^r \mathcal{N}^\vee_{Y/X} \otimes \mathcal{H}^s(\mathcal{F}^\bullet).$$

5. A Geometric Example

In this section, we give a geometric example that satisfies Definition 3.1 of categorical $\mathcal{U}$-action. That means we have to define the categories $\mathcal{K}(\underline{k})$ and 1-morphisms $\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$, $\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}$, and $(\Psi_i^{\pm})^{\pm 1}\mathbf{1}_{\underline{k}}$ so that they satisfy the conditions.

5.1. An Overview

The utilization of FM transforms necessitates the introduction of additional geometric spaces (varieties) in conjunction with the spaces employed to define the categories $\mathcal{K}(\underline{k})$. These additional spaces play the role of correspondences, enabling us to define kernels for the respective 1-morphisms. Here, we give the readers an overview of the main ideas behind the definitions and proofs first.

It suffices to consider the $\mathfrak{sl}_2$ case, in which the categories are the bounded derived category of coherent sheaves on Grassmannians $\mathrm{D}^b(\mathrm{Gr}(k,N))$. The correspondence we use to define the 1-morphisms $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ is the 3-step partial flag variety $\mathrm{Fl}(k-1,k)$ in diagram (1.3), similar for $\mathsf{F}_s\mathbf{1}_{(k,N-k)}$.

To verify those conditions in Definition 3.1, we have to calculate many convolutions of FM kernels, and most of them are done by using standard tools like base change, projection formula, and Proposition 4.4.

The most technical and interesting ones are the categorical commutator relations between E_r and F_s, i.e., condition (10). The final step of convolution is pushforward induced from the projection π_{13} to the product of the first and third components. In the constructible/coherent setting, to calculate the convolutions of kernels for $\mathsf{EF}/\mathsf{E}_r\mathsf{F}_s$ and $\mathsf{FE}/\mathsf{F}_s\mathsf{E}_r$, we need to know what are the geometric spaces that the projection π_{13} really restrict to.

They are the following varieties

$$Z = \{(V, V', V'') \in \mathrm{Gr}(k,N) \times \mathrm{Gr}(k+1,N) \times \mathrm{Gr}(k,N) \mid V,\ V'' \overset{1}{\subset} V'\},$$

$$Z' = \{(V, V''', V'') \in \mathrm{Gr}(k,N) \times \mathrm{Gr}(k-1,N) \times \mathrm{Gr}(k,N) \mid V''' \overset{1}{\subset} V,\ V''\}$$

that naturally arise when calculating convolution of kernels for $\mathsf{EF}/\mathsf{E}_r\mathsf{F}_s$ and $\mathsf{FE}/\mathsf{F}_s\mathsf{E}_r$ respectively (see Lemma 5.9 for details). Restricting the projections to Z and Z', in order to distinguish them we use π'_{13} for Z'. Then we obtain the following diagram

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$$\begin{tikzcd} Z \arrow[dr, "\pi_{13}"'] & & Z' \arrow[dl, "\pi'_{13}"] \\ & Y & \end{tikzcd} \tag{5.1}$$

where Y is given by

$$Y = \{(V, V'') \in \mathrm{Gr}(k, N) \times \mathrm{Gr}(k, N) \mid \dim(V \cap V'') \geq k - 1\}. \tag{5.2}$$

Remark 5.1. Y is singular with 2 resolutions Z and Z'.

In the constructible setting, assume that $\lambda = N - 2k \geq 0$, then in the diagram (5.1) the map π'_{13} is a small resolution and π_{13} is not a small resolution. Using the theory about perverse sheaves (or IC sheaves), we obtain the isomorphism $\mathsf{EF}|_{\mathcal{C}(\lambda)} \cong \mathsf{FE}|_{\mathcal{C}(\lambda)} \bigoplus Id_{\mathcal{C}(\lambda)}^{\oplus \lambda}$ [2]. The extra term $Id_{\mathcal{C}(\lambda)}^{\oplus \lambda}$ is contributed from π_{13} since it is not small. Moreover, the extra term has a cohomological interpretation, which can be thought of as $Id_{\mathcal{C}(\lambda)} \otimes H^*_{sing}(\mathbb{P}^{\lambda-1}, \mathbb{C})$, see [16].

However, in the coherent setting, we do not have powerful tools like the decomposition theorem. We use the fibered product $X := Z' \times_Y Z$ which is defined as follows

$$X := \{(V''', V, V'', V') \in \mathrm{Gr}(k-1, N) \times \mathrm{Gr}(k, N) \times \mathrm{Gr}(k, N) \times \mathrm{Gr}(k+1, N) \mid V''' \subset V \subset V',\ V''' \subset V'' \subset V'\} \tag{5.3}$$

and denote $p : X \longrightarrow Y$ to be the natural projection. Moreover, there is a divisor $D \subset X$ which is the locus where $V = V''$ and cut out by the natural sections $\mathcal{V}''/\mathcal{V}''' \longrightarrow \mathcal{V}'/\mathcal{V}$ where $\mathcal{V}'''$, $\mathcal{V}$, $\mathcal{V}''$, $\mathcal{V}'$ are the natural tautological bundles on X. Thus we have the following short exact sequence

$$0 \to \mathcal{V}''/\mathcal{V}''' \to \mathcal{V}'/\mathcal{V} \to \mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \to 0. \tag{5.4}$$

To compare the kernels for $\mathsf{E}_r\mathsf{F}_s$ and $\mathsf{F}_s\mathsf{E}_r$, instead of directly pushing forward to Y, we pull them back to X and use the short exact sequence (5.4) together with Proposition 4.4 to help us prove the result (see the proof of Proposition 5.10).

The kernels for $\Psi^{\pm}\mathbf{1}_{(k,N-k)}$ are produced under the above comparison, and they are reflected by the non-vanishing of coherent sheaf cohomology $H^*(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(-r-s-k)) \neq 0$. Also, the kernels for $\Psi^{\pm}\mathbf{1}_{(k,N-k)}$ are just line bundles up to homological shifts.

5.2. *Correspondences and the Main Result*

First, we define those geometric spaces (varieties) that will be used to define the categories $\mathcal{K}(\underline{k})$ and FM kernels for the 1-morphisms $\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$, $\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}$, and $(\Psi_i^{\pm})^{\pm 1}\mathbf{1}_{\underline{k}}$. They can be viewed as a $\mathfrak{sl}_n$ generalization of the varieties mentioned in Subsection 5.1.

For each $\underline{k} \in C(n, N)$, we define the n-step partial flag variety

$$\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N) := \{V_\bullet = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^N) \mid \dim V_i/V_{i-1} = k_i \text{ for all } i\}. \tag{5.5}$$

We denote $Y(\underline{k}) = \mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)$ and $\mathrm{D}^b(Y(\underline{k}))$ to be the bounded derived categories of coherent sheaves on $Y(\underline{k})$. On $Y(\underline{k})$ we denote $\mathcal{V}_i$ to be the tautological bundle whose fiber over a point $(0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^N)$ is V_i. We define the following correspondence, which can be thought of as $\mathfrak{sl}_n$ generalization of $\mathrm{Fl}(k-1, k)$.

Definition 5.2. We define $W_i^1(\underline{k})$ to be the subvariety of $Y(\underline{k}) \times Y(\underline{k} + \alpha_i)$

$$W_i^1(\underline{k}) := \{(V_\bullet, V'_\bullet) \mid V_j = V'_j \text{ for } j \neq i,\ V'_i \subset V_i\} \subset Y(\underline{k}) \times Y(\underline{k} + \alpha_i)$$

[2]Here note that this is an isomorphism between kernels.

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and similarly its transpose correspondence

$$^{\mathsf{T}}W_i^1(\underline{k}) := \{(V_\bullet, V'_\bullet) \mid V_j = V'_j \text{ for } j \neq i,\ V_i \subset V'_i\} \subset Y(\underline{k}+\alpha_i) \times Y(\underline{k})$$

for all $\underline{k} \in C(n, N)$ and i.

We denote $\iota(\underline{k}) : W_i^1(\underline{k}) \hookrightarrow Y(\underline{k}) \times Y(\underline{k}+\alpha_i)$ and ${}^{\mathsf{T}}\iota(\underline{k}) : {}^{\mathsf{T}}W_i^1(\underline{k}) \hookrightarrow Y(\underline{k}+\alpha_i) \times Y(\underline{k})$ to be the natural inclusions. We also have the natural line bundle $\mathcal{V}_i/\mathcal{V}'_i$ on $W_i^1(\underline{k})$ and similarly $\mathcal{V}'_i/\mathcal{V}_i$ on ${}^{\mathsf{T}}W_i^1(\underline{k})$, where $1 \leq i \leq n$.

Next, we define the following varieties that can be thought as $\mathfrak{sl}_n$ generalization of X in (5.3).

Definition 5.3. We define $X_i(\underline{k})$ to be the subvariety of $Y(\underline{k}+\alpha_i) \times Y(\underline{k}) \times Y(\underline{k}) \times Y(\underline{k}-\alpha_i)$

$$X_i(\underline{k}) := \{(V'''_\bullet, V_\bullet, V''_\bullet, V'_\bullet) \mid V'''_i \subset V_i \subset V'_i,\ V'''_i \subset V''_i \subset V'_i,\ V'''_j = V_j = V''_j = V'_j\ \forall\ j \neq i\}$$

and the divisor $D_i(\underline{k}) \subset X_i(\underline{k})$ that is defined by

$$D_i(\underline{k}) := \{(V'''_\bullet, V_\bullet, V''_\bullet, V'_\bullet) \mid V'''_i \subset V_i = V''_i \subset V'_i,\ V'''_j = V_j = V''_j = V'_j\ \forall\ j \neq i\}.$$

Note $D_i(\underline{k})$ is cut out by the natural section of the line bundles $\mathcal{H}om(\mathcal{V}''_i/\mathcal{V}'''_i, \mathcal{V}'_i/\mathcal{V}_i)$. More precisely, we have $\mathcal{O}_{X_i(\underline{k})}(-D_i(\underline{k})) \cong \mathcal{V}''_i/\mathcal{V}'''_i \otimes (\mathcal{V}'_i/\mathcal{V}_i)^{-1}$ and the following short exact sequences

$$0 \to \mathcal{V}''_i/\mathcal{V}'''_i \to \mathcal{V}'_i/\mathcal{V}_i \to \mathcal{O}_{D_i(\underline{k})} \otimes \mathcal{V}'_i/\mathcal{V}_i \to 0.$$

We obtain similar results if we view $D_i(\underline{k})$ as the divisor that is cut out by the natural section of the line bundles $\mathcal{H}om(\mathcal{V}_i/\mathcal{V}'''_i, \mathcal{V}'_i/\mathcal{V}''_i)$.

Finally, we define the $\mathfrak{sl}_n$ generalization of Y in diagram (5.1).

Definition 5.4.

$$Y_i(\underline{k}) = \{(V_\bullet, V''_\bullet) \in Y(\underline{k}) \times Y(\underline{k}) \mid \dim V_i \cap V''_i \geq (\sum_{l=1}^{i} k_l) - 1,\ V_j = V''_j\ \forall\ j \neq i\}.$$

Let $\pi_i : Y(\underline{k}) \times Y(\underline{k}) \to Y(\underline{k})$ be the natural projection to the ith component for $i = 1, 2$. Let $p_i(\underline{k}) : X_i(\underline{k}) \to Y_i(\underline{k})$ be the natural projection defined by forgetting $V'''_\bullet$ and $V'_\bullet$, $t_i(\underline{k}) : Y_i(\underline{k}) \to Y(\underline{k}) \times Y(\underline{k})$ be the inclusion and $\Delta(\underline{k}) : Y(\underline{k}) \to Y(\underline{k}) \times Y(\underline{k})$ to be the diagonal map.

Then, we define the 1-morphisms by using FM transforms with kernels involving the geometric spaces we introduced above.

Definition 5.5. We define $\mathbf{1}_{\underline{k}}$, $\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$, $\mathbf{1}_{\underline{k}}\mathsf{F}_{i,s}$, and $(\Psi_i^{\pm})^{\pm 1}\mathbf{1}_{\underline{k}}$ to be FM transforms with the corresponding kernels

$$\begin{aligned}
\mathbf{1}_{\underline{k}} &:= \Delta(\underline{k})_* \mathcal{O}_{Y(\underline{k})} \in \mathrm{D}^b(Y(\underline{k}) \times Y(\underline{k})),\\
\mathcal{E}_{i,r}\mathbf{1}_{\underline{k}} &:= \iota(\underline{k})_*(\mathcal{V}_i/\mathcal{V}'_i)^{\otimes r} \in \mathrm{D}^b(Y(\underline{k}) \times Y(\underline{k}+\alpha_i)),\\
\mathbf{1}_{\underline{k}}\mathcal{F}_{i,r} &:= {}^{\mathsf{T}}\iota(\underline{k})_*(\mathcal{V}'_i/\mathcal{V}_i)^{\otimes r} \in \mathrm{D}^b(Y(\underline{k}+\alpha_i) \times Y(\underline{k})),\\
(\Psi_i^+)^{\pm 1}\mathbf{1}_{\underline{k}} &:= \Delta(\underline{k})_* \det(\mathcal{V}_{i+1}/\mathcal{V}_i)^{\pm 1}[\pm(1-k_{i+1})] \in \mathrm{D}^b(Y(\underline{k}) \times Y(\underline{k})),\\
(\Psi_i^-)^{\pm 1}\mathbf{1}_{\underline{k}} &:= \Delta(\underline{k})_* \det(\mathcal{V}_i/\mathcal{V}_{i-1})^{\mp 1}[\pm(1-k_i)] \in \mathrm{D}^b(Y(\underline{k}) \times Y(\underline{k})),
\end{aligned}$$

respectively.

Now, we can state the main result of this article, which is the following theorem.

Theorem 5.6. *Let $\mathcal{K}$ be the triangulated 2-categories whose nonzero objects are $\mathcal{K}(\underline{k}) = \mathcal{D}^b(Y(\underline{k}))$ where $\underline{k} \in C(n, N)$. The 1-morphisms are those Fourier-Mukai transformations $\mathbf{1}_{\underline{k}}$, $\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$, $\mathbf{1}_{\underline{k}}\mathsf{F}_{i,s}$, and*

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$(\Psi_i^{\pm})^{\pm 1}\mathbf{1}_{\underline{k}}$ *with kernels defined in Definition 5.5 and their compositions. The 2-morphisms are maps between kernels. Then, this gives a categorical* $\mathcal{U}$*-action.*

We devote the rest of this section to proof of this theorem.

5.3. The $\mathfrak{sl}_2$ Case

In this subsection, we prove there is a categorical $\mathcal{U}$-action on the derived category of coherent sheaves on Grassmannians first, which is the following theorem.

Theorem 5.7. *The data above define a categorical* $\mathcal{U}$*-action on* $\bigoplus_k \mathrm{D}^b(\mathrm{Gr}(k,N))$.

Now $n = 2$, to simplify the notations further, we drop i from all the functors with i in their notation, and thus the 1-morphisms are $\mathbf{1}_{(k,N-k)}$, $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$, $\mathsf{F}_s\mathbf{1}_{(k,N-k)}$, and $(\Psi^{\pm})^{\pm 1}\mathbf{1}_{(k,N-k)}$. Furthermore, for the weight $\underline{k}$, we will just write $(k, N-k)$ with $0 \le k \le N$.

To prove this, we need to prove the conditions (1), (3), (4), (5a), (6a), (7a), (8a), and (10) in Definition 3.1.

Note that condition (1) is obvious since the varieties are finite Grassmannians $\mathrm{Gr}(k,N)$, which are smooth and proper, and the Hom spaces are finite-dimensional. Before we check the rest, let us remark that since all functors above are defined by using kernels, we check the conditions in terms of kernels by Proposition 4.2.

Due to the repetition of arguments, the proofs we give below for most of the conditions are only for the functors $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ and $\Psi^+\mathbf{1}_{(k,N-k)}$. For example, we will only prove $\Psi^+\mathsf{E}_r\mathbf{1}_{(k,N-k)} \cong \mathsf{E}_{r+1}\Psi^+\mathbf{1}_{(k,N-k)}[-1]$ for condition (7)(a). Finally, in order to make calculations simple, we will omit $(k, N-k)$ for all the maps in Definition 5.5, i.e. we will just write ι, ${}^{\mathsf{T}}\iota$, Δ, t, p if there is no confusion.

It is helpful to keep the following picture.

$$\begin{tikzcd}
& \mathrm{Gr}(k,N)\times\mathrm{Gr}(k-1,N) \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & \\
& \mathrm{Fl}(k-1,k) \arrow[u, "\iota"] \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
\mathrm{Gr}(k,N) & & \mathrm{Gr}(k-1,N)
\end{tikzcd} \tag{5.6}$$

where $\iota : \mathrm{Fl}(k-1,k) \to \mathrm{Gr}(k,N)\times\mathrm{Gr}(k-1,N)$ is the natural inclusion and π_1, π_2, p_1, p_2 are the natural projections.

The first is the condition (3).

Lemma 5.8 (Condition (3)). $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ *and* $\mathsf{F}_s\mathbf{1}_{(k,N-k)}$ *are left and right adjoints up to homological shifts and twists of* $\Psi^{\pm}\mathbf{1}_{(k,N-k)}$*:*

$$(\mathcal{E}_r\mathbf{1}_{(k,N-k)})_R \cong \mathbf{1}_{(k,N-k)}(\Psi^+)^r * \mathcal{F}_{N-k+1} * (\Psi^+)^{-r-1}[-r].$$

Proof. By Proposition 4.1, the right adjoint of $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$ is given by

$$\{\iota_*(\mathcal{V}/\mathcal{V}')^{\otimes r}\}^{\vee} \otimes \pi_1^*\omega_{\mathrm{Gr}(k,N)}[\dim\mathrm{Gr}(k,N)],$$

where

$$\{\iota_*(\mathcal{V}/\mathcal{V}')^{\otimes r}\}^{\vee} \cong \iota_*((\mathcal{V}/\mathcal{V}')^{\otimes -r} \otimes \omega_{\mathrm{Fl}(k-1,k)}) \otimes \omega^{-1}_{\mathrm{Gr}(k,N)\times\mathrm{Gr}(k-1,N)}[-\mathrm{codim}\mathrm{Fl}(k-1,k)].$$

To calculate $\omega_{\mathrm{Fl}(k-1,k)}$, we have $\omega_{\mathrm{Fl}(k-1,k)} \cong \omega_{p_2} \otimes p_2^*\omega_{\mathrm{Gr}(k-1,N)}$ where ω_{p_2} is the relative canonical bundle. Since the relative cotangent bundle is $\mathcal{T}^{\vee}_{p_2} \cong \mathcal{V}/\mathcal{V}' \otimes (\mathbb{C}^N/\mathcal{V})^{\vee}$, we obtain $\omega_{p_2} \cong (\mathcal{V}/\mathcal{V}')^{\otimes(N-k)} \otimes \det(\mathbb{C}^N/\mathcal{V})^{-1}$. A calculation gives $\dim\mathrm{Gr}(k,N) - \mathrm{codim}\mathrm{Fl}(k-1,k) = N-k$.

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So summarizing above and use $\mathcal{V}/\mathcal{V}' \cong \det(\mathbb{C}^N/\mathcal{V}') \otimes \det(\mathbb{C}^N/\mathcal{V})^{-1}$ we have

$$\begin{aligned}
&\{\iota_*(\mathcal{V}/\mathcal{V}')^{\otimes r}\}^\vee \otimes \pi_1^*(\omega_{\mathrm{Gr}(k,N)})[\dim\mathrm{Gr}(k,N)] \cong \iota_*(\omega_{p_2} \otimes (\mathcal{V}/\mathcal{V}')^{\otimes -r})[N-k] \text{ (using projection formula)}\\
&\quad \cong \iota_*((\mathcal{V}/\mathcal{V}')^{\otimes(-r+N-k+1)} \otimes \det(\mathbb{C}^N/\mathcal{V}')^{-1})[N-k].\\
&\quad \cong \iota_*((\mathcal{V}/\mathcal{V}')^{\otimes(N-k+1)} \otimes \det(\mathbb{C}^N/\mathcal{V})^{\otimes r} \otimes \det(\mathbb{C}^N/\mathcal{V}')^{\otimes(-r-1)})[N-k]\\
&\quad \cong \iota_*((\mathcal{V}/\mathcal{V}')^{\otimes(N-k+1)} \otimes \det(\mathbb{C}^N/\mathcal{V})^{\otimes r}[r(1+k-N)] \otimes \det(\mathbb{C}^N/\mathcal{V}')^{\otimes(-r-1)}[(r+1)(N-k)])[-r]
\end{aligned}$$

Note that the kernel $(\Psi^+)^{-1}\mathbf{1}_{(k-1,N-k+1)}$ is defined by $\Delta_* \det(\mathbb{C}^N/\mathcal{V}')^{-1}[N-k]$, while $\Psi^+\mathbf{1}_{(k,N-k)}$ is defined by $\Delta_* \det(\mathbb{C}^N/\mathcal{V})[1+k-N]$. We conclude that it is isomorphic to $\mathbf{1}_{(k,N-k)}(\Psi^+)^r * \mathcal{F}_{N-k+1} * (\Psi^+)^{-r-1}[-r]$. □

Next, we consider conditions (5a), (6a), (7a), (8a), and (10) with $r = s = 0$. We summarize them in the following lemma.

Lemma 5.9.

1. $(\mathcal{E}_{r+1} * \mathcal{E}_s)\mathbf{1}_{(k,N-k)} \cong (\mathcal{E}_{s+1} * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}[-1]$ *if* $r - s \geq 1$ *(Conditions (5a), (6a)).*
2. $(\Psi^+ * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \cong (\mathcal{E}_{r+1} * \Psi^+)\mathbf{1}_{(k,N-k)}[-1]$ *(Conditions (7a), (8a)).*
3. $(\mathcal{E}_0 * \mathcal{F}_0)\mathbf{1}_{(k,N-k)} \cong (\mathcal{F}_0 * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$ *(Conditions (10) with* $r = s = 0$*).*

Proof. Since the argument is fairly standard by using base change, projection formula, and Proposition 4.4. We decide to prove (3) only and leave (1) and (2) to the readers.

By definition

$$(\mathcal{E}_0 * \mathcal{F}_0)\mathbf{1}_{(k,N-k)} \cong \pi_{13*}(\pi_{12}^* {}^{\top}\iota_* \mathcal{O}_{\mathrm{Fl}(k,k+1)} \otimes \pi_{23}^* \iota_* \mathcal{O}_{\mathrm{Fl}(k,k+1)}). \tag{5.7}$$

We will keep using the fibered product diagram, base change, and projection formula to calculate (5.7). Since the argument is pretty standard, we decided to omit the details and only mention the key steps in order to make the article short.

We have the following fibered product diagrams

$$\begin{tikzcd}
\mathrm{Fl}(k,k+1) \times \mathrm{Gr}(k,N) \arrow[r, "{}^{\top}\iota \times id"] \arrow[d, "a_1"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k+1,N) \times \mathrm{Gr}(k,N) \arrow[d, "\pi_{12}"] \\
\mathrm{Fl}(k,k+1) \arrow[r, "{}^{\top}\iota"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k+1,N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gr}(k,N) \times \mathrm{Fl}(k,k+1) \arrow[r, "id \times \iota"] \arrow[d, "a_2"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k+1,N) \times \mathrm{Gr}(k,N) \arrow[d, "\pi_{23}"] \\
\mathrm{Fl}(k,k+1) \arrow[r, "\iota"] & \mathrm{Gr}(k+1,N) \times \mathrm{Gr}(k,N)
\end{tikzcd}$$

where a_1, a_2 are the natural projections. Then we obtain

$$(5.7) \cong \pi_{13*}(({}^{\top}\iota \times id)_* \mathcal{O}_{\mathrm{Fl}(k,k+1)\times \mathrm{Gr}(k,N)} \otimes (id \times \iota)_* \mathcal{O}_{\mathrm{Gr}(k,N)\times\mathrm{Fl}(k,k+1)}). \tag{5.8}$$

Next, the following fibered product diagram

$$\begin{tikzcd}
Z \arrow[r, "b_1"] \arrow[d, "b_2"] & \mathrm{Fl}(k,k+1) \times \mathrm{Gr}(k,N) \arrow[d, "{}^{\top}\iota \times id"] \\
\mathrm{Gr}(k,N) \times \mathrm{Fl}(k,k+1) \arrow[r, "id \times \iota"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k+1,N) \times \mathrm{Gr}(k,N)
\end{tikzcd}$$

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where Z is the following variety

$$Z := \{(V, V', V'') \mid \dim V = \dim V'' = k,\ \dim V' = k+1,\ V \subset V',\ V'' \subset V'\}$$

gives us that

$$(5.8) \cong \pi_{13*}(id \times \iota)_* b_{2*}(\mathcal{O}_Z). \tag{5.9}$$

Finally, we have the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "j_1"] \arrow[d, "\pi_{13}|_Z"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k+1,N) \times \mathrm{Gr}(k,N) \arrow[d, "\pi_{13}"] \\ Y \arrow[r, "t"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N) \end{tikzcd}$$

where $Y = \pi_{13}(Z) = \{(V, V'') \mid \dim V \cap V'' \geq k-1\}$, and $j_1 = (id \times \iota) \circ b_2$, t are the inclusions.

Note that $Y \subset \mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N)$ is a Schubert variety, it has rational singularities. So $(\pi_{13}|_{Z*})(\mathcal{O}_Z) \cong \mathcal{O}_Y$ and thus

$$(\mathcal{E}_0 * \mathcal{F}_0)\mathbf{1}_{(k,N-k)} \cong (5.9) \cong t_*(\pi_{13}|_{Z*})(\mathcal{O}_Z) \cong t_*\mathcal{O}_Y.$$

Using the same method for calculating $(\mathcal{F}_0 * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$, we end up with the following commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "j_2"] \arrow[d, "\pi_{13'}|_{Z'}"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k-1,N) \times \mathrm{Gr}(k,N) \arrow[d, "\pi_{13'}"] \\ Y \arrow[r, "t"] & \mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N) \end{tikzcd}$$

where

$$Z' := \{(V, V''', V'') \mid \dim V = \dim V'' = k,\ \dim V''' = k-1,\ V''' \subset V,\ V''' \subset V''\}$$

and j_2 is the inclusion.

Again, we have $\pi_{13}(Z') = \{(V, V'') \mid \dim V \cap V'' \geq k-1\} = Y$, which is the same as we get when we calculate $(\mathcal{E}_0 * \mathcal{F}_0)\mathbf{1}_{(k,N-k)}$. Thus

$$(\mathcal{F}_0 * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \cong \pi_{13'*} j_{2*}(\mathcal{O}_{Z'}) \cong t_*(\pi_{13'}|_{Z'*})(\mathcal{O}_{Z'}) \cong t_*\mathcal{O}_Y$$

which prove the lemma. □

Then, we prove condition (10). Observe that from Lemma 5.9 (2), we have $(\Psi^+ * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \cong (\mathcal{E}_{r+1} * \Psi^+)\mathbf{1}_{(k,N-k)}[-1]$. Since $\Psi^+\mathbf{1}_{(k,N-k)}$ is invertible, we get

$$[\Psi^+ * \mathcal{E}_r * (\Psi^+)^{-1}]\mathbf{1}_{(k,N-k)} \cong \mathcal{E}_{r+1}\mathbf{1}_{(k,N-k)}[-1],$$

and apply this inductively, we obtain

$$[(\Psi^+)^r * \mathcal{E}_0 * (\Psi^+)^{-r}]\mathbf{1}_{(k,N-k)} \cong \mathcal{E}_r\mathbf{1}_{(k,N-k)}[-r] \tag{5.10}$$

where $(\Psi^+)^r$ means (Ψ^+) convolution with itself r times.

Similarly, for $\mathcal{F}_s\mathbf{1}_{(k,N-k)}$, we have

$$[(\Psi^+)^{-s} * \mathcal{F}_0 * (\Psi^+)^s]\mathbf{1}_{(k,N-k)} \cong \mathcal{F}_s\mathbf{1}_{(k,N-k)}[-s]. \tag{5.11}$$

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By (5.10) and (5.11), we obtain that

$$(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)} \cong [(\Psi^+)^r * \mathcal{E}_0 * \mathcal{F}_{s+r} * (\Psi^+)^{-r}]\mathbf{1}_{(k,N-k)},$$
$$(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)} \cong [(\Psi^+)^r * \mathcal{F}_{r+s} * \mathcal{E}_0 * (\Psi^+)^{-r}]\mathbf{1}_{(k,N-k)}.$$

Since $\Psi^+\mathbf{1}_{(k,N-k)}$ is invertible, in order to compare $(\mathcal{F}_s*\mathcal{E}_r)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{E}_r*\mathcal{F}_s)\mathbf{1}_{(k,N-k)}$, it suffices to compare $(\mathcal{F}_{r+s} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{E}_0 * \mathcal{F}_{r+s})\mathbf{1}_{(k,N-k)}$. Hence, we prove the following proposition.

Proposition 5.10 (Condition (10)(a)(c)). *We have the following (non-split) exact triangles in* $\mathrm{D}^b(\mathrm{Gr}(k,N)\times \mathrm{Gr}(k,N))$.

$$(\mathcal{F}_{N-k} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_0 * \mathcal{F}_{N-k})\mathbf{1}_{(k,N-k)} \to \Psi^+\mathbf{1}_{(k,N-k)}, \tag{5.12}$$

and the following isomorphisms

$$(\mathcal{F}_s * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \cong (\mathcal{E}_0 * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}, \text{ if } 0 \le s \le N-k-1. \tag{5.13}$$

To compare $(\mathcal{E}_0 * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$, by using the above fibered product diagrams, we have to compare the following two objects

$$\pi_{13*}(j_{1*}(\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong t_*(\pi_{13}|_{Z*})(\mathcal{V}'/\mathcal{V})^{\otimes s} \text{ and } \pi_{13'*}(j_{2*}(\mathcal{V}''/\mathcal{V}''')^{\otimes s}) \cong t_*(\pi_{13'}|_{Z'*})(\mathcal{V}''/\mathcal{V}''')^{\otimes s}$$

in the derived category $\mathrm{D}^b(\mathrm{Gr}(k,N)\times \mathrm{Gr}(k,N))$.

Note that both are pushforwards to Y. In order to handle the case where we tensor non-trivial line bundles, instead of directly pushing forward to Y, we lift the line bundles to a much larger space, i.e., their fibered product. The fibered product space $X := Z \times_Y Z'$, is given by

$$X = \{(V''', V, V'', V') \mid V''' \subset V \subset V',\ V''' \subset V'' \subset V'\}.$$

We have the following fibered product diagram

$$\begin{tikzcd} X = Z\times_Y Z' \arrow[r, "g_1"] \arrow[d, "g_2"] & Z \arrow[d, "\pi_{13}|_Z"] \\ Z' \arrow[r, "\pi_{13'}|_{Z'}"] & Y \end{tikzcd} \tag{5.14}$$

where g_1 and g_2 are the natural projections. We denote $p : X \to Y$ to be the natural projection.

On X, we have the divisor $D = \mathrm{Fl}(k-1,k,k+1) = \{(V''', V, V') \mid V''' \subset V \subset V'\}$ which is the locus where $V = V''$ and it is the vanishing locus of the natural sections $\mathcal{V}''/\mathcal{V}''' \to \mathcal{V}'/\mathcal{V}$. Thus, we have the following short exact sequence

$$0 \to \mathcal{V}''/\mathcal{V}''' \to \mathcal{V}'/\mathcal{V} \to \mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \to 0.$$

We can relate it to Proposition 4.4. More precisely, it is easy to see that the restriction of the line bundle $\mathcal{V}'/\mathcal{V}$ to D is the pullback of the relative tautological bundle $\mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1)$ on $\mathbb{P}(\mathbb{C}^N/\mathcal{V}) = \mathrm{Fl}(k,k+1) \subset Z$. Thus we obtain

$$0 \to \mathcal{V}''/\mathcal{V}''' \to \mathcal{V}'/\mathcal{V} \to \mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1) \to 0, \tag{5.15}$$

and we will use the above short exact sequence to help us prove Proposition 5.10.

Proof of Proposition 5.10. Note that $(\pi_{13}|_{Z*})(\mathcal{O}_Z) \cong \mathcal{O}_Y$. This implies that $(\pi_{13}|_{Z*})(\pi_{13}|_Z)^* \cong \mathrm{Id}_{\mathrm{D}^b(Y)}$. Using the fibered product diagram (5.14), since g_2 is the base change of $\pi_{13}|_Z$, we have $g_{2*}g_2^* \cong \mathrm{Id}_{\mathrm{D}^b(Z')}$. Similarly, $(\pi_{13'}|_{Z'*})(\mathcal{O}_{Z'}) \cong \mathcal{O}_Y$ implies that $g_{1*}g_1^* \cong \mathrm{Id}_{\mathrm{D}^b(Z)}$.

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So, from the above discussion, we obtain

$$
\begin{aligned}
(\mathcal{E}_0 * \mathcal{F}_s)\mathbf{1}_{(k,N-k)} &\cong t_*(\pi_{13}|_{Z*})(\mathcal{V}'/\mathcal{V})^{\otimes s} \cong t_*(\pi_{13}|_{Z*})g_{1*}g_1^*(\mathcal{V}'/\mathcal{V})^{\otimes s} \cong t_*p_*(\mathcal{V}'/\mathcal{V})^{\otimes s},\\
(\mathcal{F}_s * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} &\cong t_*(\pi_{13'}|_{Z'*})(\mathcal{V}''/\mathcal{V}''')^{\otimes s} \cong t_*(\pi_{13'}|_{Z'*})g_{2*}g_2^*(\mathcal{V}''/\mathcal{V}''')^{\otimes s} \cong t_*p_*(\mathcal{V}''/\mathcal{V}''')^{\otimes s}.
\end{aligned}
$$

Thus, at first, we have to compare $p_*(\mathcal{V}'/\mathcal{V})^{\otimes s}$ and $p_*(\mathcal{V}''/\mathcal{V}''')^{\otimes s}$ in $\mathrm{D}^b(Y)$.

Note that for each $n \geq 1$, we have the following short exact sequence on X

$$
0 \to (\mathcal{V}''/\mathcal{V}''')^{\otimes n} \to (\mathcal{V}'/\mathcal{V})^{\otimes n} \to \mathcal{O}_{nD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes n} \to 0. \tag{5.16}
$$

Tensoring (5.15) by $(\mathcal{V}''/\mathcal{V}''')^{\otimes(n-1)}$, we get

$$
0 \to (\mathcal{V}''/\mathcal{V}''')^{\otimes n} \to \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}''/\mathcal{V}''')^{\otimes(n-1)} \to \mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(n-1)} \to 0.
$$

Both of them are exact triangles in $\mathrm{D}^b(X)$, and they can be completed together to form the following diagram of morphisms between exact triangles

$$
\begin{tikzcd}
(\mathcal{V}''/\mathcal{V}''')^{\otimes n} \arrow[r] \arrow[d, "id"] & \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}''/\mathcal{V}''')^{\otimes(n-1)} \arrow[r] \arrow[d] & \mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(n-1)} \arrow[d] \\
(\mathcal{V}''/\mathcal{V}''')^{\otimes n} \arrow[r] \arrow[d] & (\mathcal{V}'/\mathcal{V})^{\otimes n} \arrow[r] \arrow[d] & \mathcal{O}_{nD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes n} \arrow[d] \\
0 \arrow[r] & \mathcal{O}_{(n-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes n} \arrow[r] & \mathcal{O}_{(n-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes n}
\end{tikzcd}
$$

So, we obtain the exact triangle

$$
\mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(n-1)} \to \mathcal{O}_{nD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes n} \to \mathcal{O}_{(n-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes n} \tag{5.17}
$$

for all $n \geq 1$ (here we take $\mathcal{O}_{(n-1)D}$ to be 0 when $n = 1$).

Note that the case $s = 0$ has already been proved in Lemma 5.9 (3). Now, we move to the case where s is nonzero. The first case is $1 \leq s \leq N - k - 1$. For (5.16) with $n = s$ we have

$$
0 \to (\mathcal{V}''/\mathcal{V}''')^{\otimes s} \to (\mathcal{V}'/\mathcal{V})^{\otimes s} \to \mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s} \to 0.
$$

Applying the derived pushforward p_*, we obtain the following exact triangle in $\mathrm{D}^b(Y)$

$$
p_*(\mathcal{V}''/\mathcal{V}''')^{\otimes s} \to p_*(\mathcal{V}'/\mathcal{V})^{\otimes s} \to p_*(\mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s})
$$

and it suffices to prove that $p_*(\mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong 0$ so that we can obtain $(\mathcal{E}_0 * \mathcal{F}_s)\mathbf{1}_{(k,N-k)} \cong (\mathcal{F}_s * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$ after applying t_*.

Using (5.17) with $n = s$, we have

$$
\mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(s-1)} \to \mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s} \to \mathcal{O}_{(s-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}.
$$

Applying the derived pushforward p_* to calculate $p_*(\mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(s-1)})$, we use the projection formula.

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$$\begin{aligned}
p_*(\mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(s-1)}) &\cong p_*(\mathcal{O}_D \otimes \mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(s-1)) \\
&\cong \pi_{13*}g_{1*}(\mathcal{O}_D \otimes g_1^*(\mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1)) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee)}(s-1)) \\
&\cong \pi_{13*}(\mathcal{O}_{\mathrm{Fl}(k,k+1)} \otimes \mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1) \otimes \mathrm{Sym}^{s-1}(\mathcal{V})) \\
&\cong \pi_{13*}(\mathcal{O}_{\mathrm{Fl}(k,k+1)} \otimes \mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1) \otimes \pi_{13}^*(\mathrm{Sym}^{s-1}(\mathcal{V}))) \\
&\cong \mathrm{Sym}^{s-1}(\mathcal{V}) \otimes \pi_{13*}(\mathcal{O}_{\mathbb{P}(\mathbb{C}^N/\mathcal{V})}(-1)) \cong 0.
\end{aligned}$$

Thus we get $p_*(\mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong p_*(\mathcal{O}_{(s-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s})$. Next, consider the exact triangle (5.17) with $n = s-1$, we have

$$\mathcal{O}_D \otimes \mathcal{V}'/\mathcal{V} \otimes (\mathcal{V}/\mathcal{V}''')^{\otimes(s-2)} \to \mathcal{O}_{(s-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(s-1)} \to \mathcal{O}_{(s-2)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(s-1)}.$$

Tensoring by $\mathcal{V}'/\mathcal{V}$ and then applying p_*, using the same argument as above, we obtain $p_*(\mathcal{O}_{(s-1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong p_*(\mathcal{O}_{(s-2)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s})$. Continuing this procedure, we will end up with

$$p_*(\mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong \dots \cong p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}). \tag{5.18}$$

For the exact triangle (5.17) with $n = 2$, tensoring by $(\mathcal{V}'/\mathcal{V})^{\otimes(s-2)}$, we get

$$\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(s-1)} \otimes \mathcal{V}/\mathcal{V}''' \to \mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s} \to \mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}.$$

Applying p_*, then we get $p_*(\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(s-1)} \otimes \mathcal{V}/\mathcal{V}''') \cong 0$ and $p_*(\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong 0$. The first isomorphism is via projection formula and Proposition 4.4 with $1 \leq s \leq N-k-1$, while the second one is via Proposition 4.4 with $1 \leq s \leq N-k-1$.

Hence we get $p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong 0$ and (5.18) implies that $p_*(\mathcal{O}_{sD} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes s}) \cong 0$, so we prove (5.13).

The remaining case is $s = N-k$. Applying p_* to (5.16) with $n = N-k$, we have

$$p_*(\mathcal{V}''/\mathcal{V}''')^{N-k} \to p_*(\mathcal{V}'/\mathcal{V})^{\otimes(N-k)} \to p_*(\mathcal{O}_{(N-k)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}).$$

It suffices to prove $p_*(\mathcal{O}_{(N-k)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) \cong j_* \det(\mathbb{C}^N/\mathcal{V})[1+k-N]$, where $j : \Delta = \mathrm{Gr}(k,N) \to Y$ is the natural inclusion. Note that we have the inclusion $t : Y \to \mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N)$, and so $\Delta = t \circ j$.

Again, using the similar argument as in the proof of the case $1 \leq s \leq N-k-1$, we also have

$$p_*(\mathcal{O}_{(N-k)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) \cong \dots \cong p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}).$$

Considering (5.17) with $n = 2$ and tensoring it with $(\mathcal{V}'/\mathcal{V})^{\otimes(N-k-2)}$, we get

$$\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)} \otimes \mathcal{V}/\mathcal{V}''' \to \mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)} \to \mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)},$$

applying p_*, we obtain $p_*(\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)} \otimes \mathcal{V}/\mathcal{V}''') \cong 0$ via projection formula, while $p_*(\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) \cong j_* \det(\mathbb{C}^N/\mathcal{V})[1+k-N]$ is by Proposition 4.4.

Hence we get

$$\begin{aligned}
p_*(\mathcal{O}_{(N-k)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) &\cong p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) \\
&\cong p_*(\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) \cong j_* \det(\mathbb{C}^N/\mathcal{V})[1+k-N]
\end{aligned}$$

and $t_*p_*(\mathcal{O}_{(N-k)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}) \cong \Delta_* \det(\mathbb{C}^N/\mathcal{V})[1+k-N]$, which proves (5.12). □

Remark 5.11. In this remark, we show that the exact triangle (5.12) in Proposition 5.10 is non-split. We prove this by showing that the Hom spaces of morphisms between objects are zero. Indeed, using adjunction (condition (3)) we have

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$$\begin{aligned}
&\mathrm{Hom}((\mathcal{E}_0 * \mathcal{F}_{N-k})\mathbf{1}_{(k,N-k)}, (\mathcal{F}_{N-k} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}) \\
&\cong \mathrm{Hom}((\mathcal{F}_{N-k})_L * \mathcal{E}_0\mathbf{1}_{(k-1,N-k+1)}, \mathcal{E}_0 * (\mathcal{F}_{N-k})_L\mathbf{1}_{(k-1,N-k+1)}) \\
&\cong \mathrm{Hom}(\mathcal{E}_0 * (\Psi^+)^{-1} * \mathcal{E}_0\mathbf{1}_{(k-1,N-k+1)}[1], \mathcal{E}_0 * (\Psi^+)^{-1} * \mathcal{E}_0\mathbf{1}_{(k-1,N-k+1)}) \\
&\cong \mathrm{Hom}(\mathcal{E}_0 * \mathcal{E}_{-1} * (\Psi^+)^{-1}\mathbf{1}_{(k-1,N-k+1)}[2], \mathcal{E}_0 * \mathcal{E}_{-1} * (\Psi^+)^{-1}\mathbf{1}_{(k-1,N-k+1)}[1]) \\
&\cong \mathrm{Hom}(0,0) \cong 0. \text{ (using condition (5)(a))}
\end{aligned}$$

Finally, since $\Psi^{\pm}\mathbf{1}_{(k,N-k)}$ are given by line bundles, condition (4) is standard to check. Thus, combining Lemma 5.8, Lemma 5.9, and Proposition 5.10, we prove Theorem 5.7.

5.4. *The Remaining Relations*

Since we have already proved the $\mathfrak{sl}_2$ case, many conditions in Definition 3.1 are direct generalization of the $\mathfrak{sl}_2$ version. To prove Theorem 5.6, we prove the remaining conditions for the $\mathfrak{sl}_n$ case in this section; this means that we prove those not addressed in the $\mathfrak{sl}_2$ case.

First, we note that condition (2) is apparent. Next, it is standard to check conditions (5c), (6c), (7c), and (8c) where $|i-j| \geq 2$, and condition (9) where $i \neq j$. So it remains to check the relations (4), (5b), (6b), (7b), and (8b). Again, condition (4) is standard to show since $\Psi_i^{\pm}\mathbf{1}_{\underline{k}}$ are just line bundles.

The next conditions we prove are conditions (5b) and (6b).

Lemma 5.12 (Conditions (5b) and (6b)). *We have the following exact triangle*

$$(\mathcal{E}_{i+1,s} * \mathcal{E}_{i,r+1})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i+1,s+1} * \mathcal{E}_{i,r})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i,r} * \mathcal{E}_{i+1,s+1})\mathbf{1}_{\underline{k}}.$$

Proof. A simple calculation gives that $(\mathcal{E}_{i+1,s+1} * \mathcal{E}_{i,r})\mathbf{1}_{\underline{k}}$ is given by

$$i_{1*}((\mathcal{V}_i/\mathcal{V}_i'')^{\otimes r} \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{\otimes(s+1)})$$

and $(\mathcal{E}_{i,r} * \mathcal{E}_{i+1,s+1})\mathbf{1}_{\underline{k}}$ is given by

$$i_{2*}((\mathcal{V}_i/\mathcal{V}_i'')^{\otimes r} \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{\otimes(s+1)}).$$

Here $i_1 : W_{i+1,i}^{1,1}(\underline{k}) \to Y(\underline{k}) \times Y(\underline{k}+\alpha_i+\alpha_{i+1})$, $i_2 : W_{i,i+1}^{1,1}(\underline{k}) \to Y(\underline{k}) \times Y(\underline{k}+\alpha_i+\alpha_{i+1})$ are the natural inclusions with the subvarieties $W_{i+1,i}^{1,1}(\underline{k}) \subset Y(\underline{k}) \times Y(\underline{k}+\alpha_i+\alpha_{i+1})$ and $W_{i,i+1}^{1,1}(\underline{k}) \subset Y(\underline{k}) \times Y(\underline{k}+\alpha_i+\alpha_{i+1})$ given by

$$\begin{aligned}
W_{i+1,i}^{1,1}(\underline{k}) &= \{(V_\bullet, V_\bullet'') \mid V_i'' \subset V_i,\ V_{i+1}'' \subset V_{i+1},\ V_j = V_j'' \text{ for } j \neq i, i+1\}, \\
W_{i,i+1}^{1,1}(\underline{k}) &= \{(V_\bullet, V_\bullet'') \mid V_i'' \subset V_i \subset V_{i+1}'' \subset V_{i+1},\ V_j = V_j'' \text{ for } j \neq i, i+1\}.
\end{aligned}$$

Note that $W_{i,i+1}^{1,1}(\underline{k}) \subset W_{i+1,i}^{1,1}(\underline{k})$ is a divisor that is cut out by the natural section of the line bundle $\mathcal{H}om(\mathcal{V}_i/\mathcal{V}_i'', \mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')$. This implies that

$$\mathcal{O}_{W_{i+1,i}^{1,1}(\underline{k})}(W_{i,i+1}^{1,1}(\underline{k})) \cong (\mathcal{V}_i/\mathcal{V}_i'')^\vee \otimes \mathcal{V}_{i+1}/\mathcal{V}_{i+1}''.$$

From the divisor short exact sequence

$$0 \to \mathcal{O}_{W_{i+1,i}^{1,1}(\underline{k})}(-W_{i,i+1}^{1,1}(\underline{k})) \cong \mathcal{V}_i/\mathcal{V}_i'' \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{-1} \to \mathcal{O}_{W_{i+1,i}^{1,1}(\underline{k})} \to \mathcal{O}_{W_{i,i+1}^{1,1}(\underline{k})} \to 0.$$

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Tensoring it with $(\mathcal{V}_i/\mathcal{V}_i'')^{\otimes r} \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{\otimes(s+1)}$ we get the following exact triangle

$$(\mathcal{V}_i/\mathcal{V}_i'')^{\otimes(r+1)} \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{\otimes s} \to (\mathcal{V}_i/\mathcal{V}_i'')^{\otimes r} \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{\otimes(s+1)}$$
$$\to \mathcal{O}_{W^{1,1}_{i,i+1}(\underline{k})} \otimes (\mathcal{V}_i/\mathcal{V}_i'')^{\otimes r} \otimes (\mathcal{V}_{i+1}/\mathcal{V}_{i+1}'')^{\otimes(s+1)}.$$

Applying i_{1*} and comparing the kernels, we get

$$(\mathcal{E}_{i+1,s} * \mathcal{E}_{i,r+1})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i+1,s+1} * \mathcal{E}_{i,r})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i,r} * \mathcal{E}_{i+1,s+1})\mathbf{1}_{\underline{k}}. \qquad \square$$

Next, we verify conditions (7b), (8b).

Lemma 5.13. (Conditions (7b), (8b))

$$(\Psi_i^+ * \mathcal{E}_{i+1,r})\mathbf{1}_{\underline{k}} \cong (\mathcal{E}_{i+1,r-1} * \Psi_i^+)\mathbf{1}_{\underline{k}}[1].$$

Proof. A direct calculation shows that $(\Psi_i^+ * \mathcal{E}_{i+1,r})\mathbf{1}_{\underline{k}}$ is given by

$$\iota_*((\mathcal{V}_{i+1}/\mathcal{V}_{i+1}')^{\otimes r} \otimes \det(\mathcal{V}_{i+1}'/\mathcal{V}_i))[2-k_{i+1}].$$

while $(\mathcal{E}_{i+1,r-1} * \Psi_i^+)\mathbf{1}_{\underline{k}}$ is given by

$$\iota_*(\mathcal{V}_{i+1}/\mathcal{V}_{i+1}')^{\otimes(r-1)} \otimes \det(\mathcal{V}_{i+1}/\mathcal{V}_i)[1-k_{i+1}].$$

Here $\iota : W^1_{i+1}(\underline{k}) \to Y(\underline{k}) \times Y(\underline{k}+\alpha_{i+1})$ is the inclusion. Note that the following short exact sequence

$$0 \to \mathcal{V}_{i+1}'/\mathcal{V}_i \to \mathcal{V}_{i+1}/\mathcal{V}_i \to \mathcal{V}_{i+1}/\mathcal{V}_{i+1}' \to 0$$

gives that $\det(\mathcal{V}_{i+1}'/\mathcal{V}_i) \otimes \mathcal{V}_{i+1}/\mathcal{V}_{i+1}' \cong \det(\mathcal{V}_{i+1}/\mathcal{V}_i)$.

Combining the above, we get

$$\iota_*(\mathcal{V}_{i+1}/\mathcal{V}_{i+1}')^{\otimes r} \otimes \det(\mathcal{V}_{i+1}'/\mathcal{V}_i)[2-k_{i+1}] \cong \iota_*(\mathcal{V}_{i+1}/\mathcal{V}_{i+1}')^{\otimes(r-1)} \otimes \det(\mathcal{V}_{i+1}/\mathcal{V}_i)[2-k_{i+1}]$$

which implies that $(\Psi_i^+ * \mathcal{E}_{i+1,r})\mathbf{1}_{\underline{k}} \cong (\mathcal{E}_{i+1,r-1} * \Psi_i^+)\mathbf{1}_{\underline{k}}[1]$. $\square$

Combining the above results in this section and Theorem 5.7, we prove Theorem 5.6.

Finally, the following corollary is a direct consequence by Lemma 3.5 and Theorem 5.6.

Corollary 5.14. *There is an action of* $\mathcal{U}$ *on* $\bigoplus_{\underline{k}} K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$.

6. Toward a Future Study of the Categorical $\mathcal{U}$-Action

Although we only have the categorical commutator relation (i.e., condition (10) in Definition 3.1) between $\mathsf{E}_{i,r}\mathsf{F}_{i,s}\mathbf{1}_{\underline{k}}$ and $\mathsf{F}_{i,s}\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$ for $-k_i \le r+s \le k_{i+1}$, from Theorem 5.6, the FM kernel $\mathcal{E}_{i,r}\mathbf{1}_{\underline{k}}$ for the 1-morphism $\mathsf{E}_{i,r}\mathbf{1}_{\underline{k}}$ in Definition 5.5 can be defined for all $r \in \mathbb{Z}$ (similarly for $\mathcal{F}_{i,r}\mathbf{1}_{\underline{k}}$). Thus, it is tempting to understand the relation between the two convolutions $(\mathcal{E}_{i,r} * \mathcal{F}_{i,s})\mathbf{1}_{\underline{k}}$ and $(\mathcal{F}_{i,s} * \mathcal{E}_{i,r})\mathbf{1}_{\underline{k}}$ for $r+s \ge k_{i+1}+1$ and $r+s \le -k_i-1$.

In this section, we try to provide a short study for the first non-trivial case where $r+s = k_{i+1}+1$ (and similarly for $r+s=-k_i-1$). Again, by reducing to the $\mathfrak{sl}_2$ case and using the conjugation properties (5.10), (5.11), it suffices to relate $(\mathcal{E}_0 * \mathcal{F}_{N-k+1})\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_{N-k+1} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$.

With the help from the proof in Proposition 5.10, we have the following exact triangle in $\mathrm{D}^b(Y)$

$$p_*(\mathcal{V}''/\mathcal{V}''')^{\otimes(N-k+1)} \to p_*(\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)} \to p_*(\mathcal{O}_{(N-k+1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \tag{6.1}$$

which is obtained by applying p_* to (5.16) with $n = N-k+1$.

By the same argument in the proof in Proposition 5.10, we obtain the following isomorphisms

$$p_*(\mathcal{O}_{(N-k+1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \cong \ldots \cong p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}).$$

Then applying t_* to (6.1), we get the following exact triangle

$$(\mathcal{F}_{N-k+1} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_0 * \mathcal{F}_{N-k+1})\mathbf{1}_{(k,N-k)} \to t_* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}). \tag{6.2}$$

Finally, the above argument can also be applied to the first non-trivial case on the other side, i.e. $r + s = -k - 1$ and we have the following exact triangle

$$(\mathcal{E}_{-k-1} * \mathcal{F}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{F}_0 * \mathcal{E}_{-k-1})\mathbf{1}_{(k,N-k)} \to t_* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(-k-1)}). \tag{6.3}$$

6.1. The New Kernels $\mathcal{H}_{\pm1}\mathbf{1}_{(k,N-k)}$

When we pass to the Grothendieck groups, the two exact triangles (6.2) and (6.3) should give the commutator relations $[e_0, f_{N-k+1}]1_{(k,N-k)}$ and $[e_{-k-1}, f_0]1_{(k,N-k)}$ respectively.

Note that from Definition 2.6, we do not have $e_{-k-1}1_{(k,N-k)}$ and $f_{N-k+1}1_{(k,N-k)}$ in the generators of $\mathcal{U}$. However, we have the following isomorphisms that can be deduced from (5.10) and (5.11),

$$\begin{aligned} \mathcal{E}_{-k-1}\mathbf{1}_{(k,N-k)} &\cong (\Psi^+)^{-1} * \mathcal{E}_{-k} * \Psi^+\mathbf{1}_{(k,N-k)}[-1], \\ \mathcal{F}_{N-k+1}\mathbf{1}_{(k,N-k)} &\cong (\Psi^+)^{-1} * \mathcal{F}_{N-k} * \Psi^+\mathbf{1}_{(k,N-k)}[1]. \end{aligned} \tag{6.4}$$

Thus, it suggests that we can define $e_{-k-1}1_{(k,N-k)}$ and $f_{N-k+1}1_{(k,N-k)}$ in $\mathcal{U}$ by using the conjugation property, i.e.

$$\begin{aligned} e_{-k-1}1_{(k,N-k)} &:= -(\psi^+)^{-1} e_{-k}\psi^+ 1_{(k,N-k)}, \\ f_{N-k+1}1_{(k,N-k)} &:= -(\psi^+)^{-1} f_{N-k}\psi^+ 1_{(k,N-k)}. \end{aligned} \tag{6.5}$$

By comparing to the shifted quantum affine algebra in Definition 2.3, it is reasonable to define the following two new generators in $\mathcal{U}$

$$\begin{aligned} h_1 1_{(k,N-k)} &:= (\psi^+)^{-1}[e_0, f_{N-k+1}]1_{(k,N-k)} \\ h_{-1} 1_{(k,N-k)} &:= -(\psi^-)^{-1}[e_{-k-1}, f_0]1_{(k,N-k)} \end{aligned} \tag{6.6}$$

such that (6.2) and (6.3) categorify the relations $[e_0, f_{N-k+1}]1_{(k,N-k)} = \psi^+ h_1 1_{(k,N-k)}$ and $[e_{-k-1}, f_0]1_{(k,N-k)} = -\psi^- h_{-1}1_{(k,N-k)}$ respectively.

As a consequence, we define the following two FM kernels in $\mathrm{D}^b(\mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N))$

$$\mathcal{H}_1\mathbf{1}_{(k,N-k)} := (\Psi^+\mathbf{1}_{(k,N-k)})^{-1} * t_* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}), \tag{6.7}$$

$$\mathcal{H}_{-1}\mathbf{1}_{(k,N-k)} := (\Psi^-\mathbf{1}_{(k,N-k)})^{-1} * t_* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(-k-1)}), \tag{6.8}$$

such that their associated FM transforms categorify $h_{\pm1}1_{(k,N-k)}$ respectively. Finally, we should mention that the above FM kernels $\mathcal{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ can be generalized to $\mathfrak{sl}_n$ case.

6.2. New Conditions and Properties of $\mathcal{H}_{\pm1}\mathbf{1}_{(k,N-k)}$

With the introduction of the new elements $e_{-k-1}1_{(k,N-k)}, f_{N-k+1}1_{(k,N-k)}, h_{\pm1}1_{(k,N-k)} \in \mathcal{U}$ from (6.5) and (6.6), we define the corresponding new 1-morphisms $\mathsf{E}_{-k-1}\mathbf{1}_{(k,N-k)}$, $\mathsf{F}_{N-k+1}\mathbf{1}_{(k,N-k)}$, and $\mathsf{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ in a categorical $\mathcal{U}$-action.

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Then, from (6.4) we should include the following as conditions in a categorical $\mathcal{U}$-action

$$\mathsf{E}_{-k-1}\mathbf{1}_{(k,N-k)} := (\Psi^+)^{-1}\mathsf{E}_{-k}\Psi^+\mathbf{1}_{(k,N-k)}[-1],$$
$$\mathsf{F}_{N-k+1}\mathbf{1}_{(k,N-k)} := (\Psi^+)^{-1}\mathsf{F}_{N-k}\Psi^+\mathbf{1}_{(k,N-k)}[1].$$

Similarly, from (6.2), (6.3), (6.7) and (6.8), we should include the following two exact triangles as conditions in a categorical $\mathcal{U}$-action

$$\mathsf{F}_{N-k+1}\mathsf{E}_0\mathbf{1}_{(k,N-k)} \to \mathsf{E}_0\mathsf{F}_{N-k+1}\mathbf{1}_{(k,N-k)} \to \Psi^+\mathsf{H}_1\mathbf{1}_{(k,N-k)} \tag{6.9}$$

$$\mathsf{E}_{-k-1}\mathsf{F}_0\mathbf{1}_{(k,N-k)} \to \mathsf{F}_0\mathsf{E}_{-k-1}\mathbf{1}_{(k,N-k)} \to \Psi^-\mathsf{H}_{-1}\mathbf{1}_{(k,N-k)}. \tag{6.10}$$

In this subsection, we will provide an adjunction property about the 1-morphisms $\mathsf{H}_{\pm1}\mathbf{1}_{(k,N-k)}$. Moreover, we will also study the FM kernels $\mathcal{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ from (6.7) and (6.8).

6.2.1. Adjunctions

The first property is about the adjunctions between $\mathsf{H}_{\pm1}\mathbf{1}_{(k,N-k)}$.

Lemma 6.1. *Assume we have the two exact triangles (6.9) and (6.10) in a categorical $\mathcal{U}$-action. Then, the 1-morphisms $\mathsf{H}_1\mathbf{1}_{(k,N-k)}$ and $\mathsf{H}_{-1}\mathbf{1}_{(k,N-k)}$ are biadjoint up to conjugations of $\Psi^{\pm}\mathbf{1}_{(k,N-k)}$. More precisely,*

$$(\mathsf{H}_1\mathbf{1}_{(k,N-k)})^L \cong (\Psi^-)^k\mathsf{H}_{-1}(\Psi^-)^{-k}\mathbf{1}_{(k,N-k)} \cong (\Psi^+)^k\mathsf{H}_{-1}(\Psi^+)^{-k}\mathbf{1}_{(k,N-k)},$$
$$(\mathsf{H}_1\mathbf{1}_{(k,N-k)})^R \cong (\Psi^-)^{-N+k}\mathsf{H}_{-1}(\Psi^-)^{N-k}\mathbf{1}_{(k,N-k)} \cong (\Psi^+)^{-N+k}\mathsf{H}_{-1}(\Psi^+)^{N-k}\mathbf{1}_{(k,N-k)}.$$

Proof. We prove the case for the left adjoint, and the proof for the right adjoint is similar.

Taking the left adjoint of the following exact triangle

$$\mathsf{F}_{N-k+1}\mathsf{E}_0\mathbf{1}_{(k,N-k)} \to \mathsf{E}_0\mathsf{F}_{N-k+1}\mathbf{1}_{(k,N-k)} \to \Psi^+\mathsf{H}_1\mathbf{1}_{(k,N-k)}$$

we obtain

$$\mathsf{H}_1^L(\Psi^+)^L\mathbf{1}_{(k,N-k)} \to \mathsf{F}_{N-k+1}^L\mathsf{E}_0^L\mathbf{1}_{(k,N-k)} \to \mathsf{E}_0^L\mathsf{F}_{N-k+1}^L\mathbf{1}_{(k,N-k)}. \tag{6.11}$$

Using $\mathsf{F}_{N-k+1}\mathbf{1}_{(k,N-k)} = (\Psi^+)^{-1}\mathsf{F}_{N-k}\Psi^+\mathbf{1}_{(k,N-k)}[1]$, we get

$$\begin{aligned}(\mathsf{F}_{N-k+1}\mathbf{1}_{(k,N-k)})^L &\cong (\Psi^+)^{-1}(\mathsf{F}_{N-k}\mathbf{1}_{(k,N-k)})^L\Psi^+\mathbf{1}_{(k,N-k)}[-1] \\ &\cong (\Psi^+)^{-2}\mathsf{E}_0\Psi^+\mathbf{1}_{(k,N-k)}[-1] \text{ (by condition (3)(d))}.\end{aligned}$$

Thus

$$\begin{aligned}\mathsf{F}_{N-k+1}^L\mathsf{E}_0^L\mathbf{1}_{(k,N-k)} &\cong (\Psi^+)^{-2}\mathsf{E}_0(\Psi^+)(\Psi^-)^k\mathsf{F}_0(\Psi^-)^{-k-1}\mathbf{1}_{(k,N-k)}[k] \text{ (by conditions (3)(b)(d))} \\ &\cong \mathsf{E}_{-2}(\Psi^+)^{-1}(\Psi^-)^k\mathsf{F}_0(\Psi^-)^{-k-1}\mathbf{1}_{(k,N-k)}[k+2] \text{ (by conditions (7)(a))} \\ &\cong \mathsf{E}_{-2}(\Psi^-)^k\mathsf{F}_1(\Psi^+)^{-1}(\Psi^-)^{-k-1}\mathbf{1}_{(k,N-k)}[k+1] \text{ (by conditions (8)(a))} \\ &\cong \mathsf{E}_{-2}(\Psi^-)^{k-1}\mathsf{F}_0(\Psi^+)^{-1}(\Psi^-)^{-k}\mathbf{1}_{(k,N-k)}[k] \text{ (by conditions (8)(a))} \\ &\cong (\Psi^-)^{k-1}\mathsf{E}_{-k-1}\mathsf{F}_0(\Psi^-)^{-k}(\Psi^+)^{-1}\mathbf{1}_{(k,N-k)}[1] \text{ (by conditions (7)(a))}.\end{aligned}$$

A similar calculation shows that

$$\mathsf{E}_0^L\mathsf{F}_{N-k+1}^L\mathbf{1}_{(k,N-k)} \cong (\Psi^-)^{k-1}\mathsf{F}_0\mathsf{E}_{-k-1}(\Psi^-)^{-k}(\Psi^+)^{-1}\mathbf{1}_{(k,N-k)}[1].$$

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Since $\Psi^+\mathbf{1}_{(k,N-k)}$ is invertible, $(\Psi^+)^L\mathbf{1}_{(k,N-k)} \cong (\Psi^+)^{-1}\mathbf{1}_{(k,N-k)}$. Thus (6.11) becomes

$$\mathsf{H}_1^L(\Psi^+)^{-1}\mathbf{1}_{(k,N-k)} \to (\Psi^-)^{k-1}\mathsf{E}_{-k-1}\mathsf{F}_0(\Psi^-)^{-k}(\Psi^+)^{-1}\mathbf{1}_{(k,N-k)}[1] \to (\Psi^-)^{k-1}\mathsf{F}_0\mathsf{E}_{-k-1}(\Psi^-)^{-k}(\Psi^+)^{-1}\mathbf{1}_{(k,N-k)}[1]. \tag{6.12}$$

Applying $(\Psi^-)^{-k+1}-$, $-(\Psi^-)^k(\Psi^+)$ and homological shift by one to (6.12), we get

$$\mathsf{E}_{-k-1}\mathsf{F}_0\mathbf{1}_{(k,N-k)} \to \mathsf{F}_0\mathsf{E}_{-k-1}\mathbf{1}_{(k,N-k)} \to (\Psi^-)^{-k+1}\mathsf{H}_1^L(\Psi^-)^k\mathbf{1}_{(k,N-k)}.$$

Comparing to the exact triangle (6.10), there is an isomorphism

$$(\Psi^-)^{-k+1}\mathsf{H}_1^L(\Psi^-)^k\mathbf{1}_{(k,N-k)} \cong \Psi^-\mathsf{H}_{-1}\mathbf{1}_{(k,N-k)}$$

and hence we conclude that $\mathsf{H}_1^L\mathbf{1}_{(k,N-k)} \cong (\Psi^-)^k\mathsf{H}_{-1}(\Psi^-)^{-k}\mathbf{1}_{(k,N-k)}$.

To obtain the second isomorphism, note that from conditions (7)(a) and (8)(a)

$$\Psi^+(\Psi^-)^{-1}\mathsf{E}_r\mathsf{F}_s\mathbf{1}_{(k,N-k)} \cong \mathsf{E}_r\mathsf{F}_s\Psi^+(\Psi^-)^{-1}\mathbf{1}_{(k,N-k)},$$

for all r and s. This implies that $\Psi^+(\Psi^-)^{-1}\mathsf{H}_1\mathbf{1}_{(k,N-k)} \cong \mathsf{H}_1\Psi^+(\Psi^-)^{-1}\mathbf{1}_{(k,N-k)}$. Thus $(\Psi^-)^{-1}\mathsf{H}_1\Psi^-\mathbf{1}_{(k,N-k)} \cong (\Psi^+)^{-1}\mathsf{H}_1\Psi^+\mathbf{1}_{(k,N-k)}$ and repeating this process we get the second isomorphism. □

6.2.2. Non-triviality

The second property is about the geometric property of the FM kernels $\mathcal{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ defined in (6.7) and (6.8).

It suffices to consider $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ since the argument for $\mathcal{H}_{-1}\mathbf{1}_{(k,N-k)}$ is similar. A simple calculation shows that the FM kernel $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ for $\mathsf{H}_1\mathbf{1}_{(k,N-k)}$ is

$$\mathcal{H}_1\mathbf{1}_{(k,N-k)} \cong t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \otimes \pi_2^*\det(\mathbb{C}^N/\mathcal{V})^{-1}[N-k-1] \in \mathrm{D}^b(\mathrm{Gr}(k,N)\times\mathrm{Gr}(k,N)) \tag{6.13}$$

where $p : X \to Y$ is the projection and $t : Y \to \mathrm{Gr}(k,N)\times\mathrm{Gr}(k,N)$ is the inclusion with X, Y defined in (5.3), (5.2) respectively.

On the other hand, tensoring (5.17) with $n = 2$ by $(\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)}$, we get

$$\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)} \otimes \mathcal{V}/\mathcal{V}''' \to \mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)} \to \mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}. \tag{6.14}$$

By using the projection formula and Proposition 4.4, after applying t_*p_* to (6.14), we get the following exact triangle

$$\Delta_*\mathcal{V} \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N] \to t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \to \Delta_*\mathbb{C}^N/\mathcal{V} \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N]. \tag{6.15}$$

Tensoring (6.15) with $\pi_2^*\det(\mathbb{C}^N/\mathcal{V})^{-1}$ and homologically shifted by $[N-k-1]$, by the projection formula and (6.13), we obtain the following exact triangle

$$\Delta_*\mathcal{V} \to \mathcal{H}_1\mathbf{1}_{(k,N-k)} \to \Delta_*\mathbb{C}^N/\mathcal{V} \tag{6.16}$$

since $\pi_2 \circ \Delta = id$. This implies that $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ is determined by $\mathrm{Ext}^1_{\mathrm{Gr}(k,N)\times\mathrm{Gr}(k,N)}(\Delta_*\mathbb{C}^N/\mathcal{V}, \Delta_*\mathcal{V})$. The following result shows that, in the non-extreme cases, the exact triangle (6.16) is not isomorphic to the derived pushforward of the tautological exact triangles on $\mathrm{Gr}(k,N)$ under the diagonal map $\Delta : \mathrm{Gr}(k,N) \to \mathrm{Gr}(k,N)\times\mathrm{Gr}(k,N)$.

Proposition 6.2. *When $k \neq 0$, N, the Fourier-Mukai kernel $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ is neither isomorphic to $\Delta_*\mathbb{C}^N$ nor to $\Delta_*(\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})$.*

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Proof. From the definition of $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$, it is equivalent to show $t_* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)})$ is neither isomorphic to $\Delta_*\Big(\mathbb{C}^N \otimes \det(\mathbb{C}^N/\mathcal{V})\Big)[1+k-N]$ nor to $\Delta_*\Big((\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})\otimes\det(\mathbb{C}^N/\mathcal{V})\Big)[1+k-N]$.

We have the following diagram

$$\begin{tikzcd} D = \mathrm{Fl}(k-1,k,k+1) \arrow[r, hook, "i"] \arrow[d, "p|_D"] & X \arrow[d, "p"] & \\ \mathrm{Gr}(k,N) \arrow[r, hook, "j"] & Y \arrow[r, hook, "t"] & \mathrm{Gr}(k,N)\times\mathrm{Gr}(k,N) \end{tikzcd}$$

where the left square is a fibered product and $t \circ j = \Delta$ is the diagonal map.

Since t_* is exact and $\Delta_* = t_* j_*$, it is equivalent to show $p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)})$ is neither isomorphic to $j_*\Big(\mathbb{C}^N \otimes \det(\mathbb{C}^N/\mathcal{V})\Big)[1+k-N]$ nor to $j_*\Big((\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})\otimes\det(\mathbb{C}^N/\mathcal{V})\Big)[1+k-N]$.

Assume otherwise, then applying j^* we have

$$\begin{aligned} j^* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) &\cong j^* j_*\Big(\mathbb{C}^N \otimes \det(\mathbb{C}^N/\mathcal{V})\Big)[1+k-N] \\ &\quad \text{or } j^* j_*\Big((\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})\otimes\det(\mathbb{C}^N/\mathcal{V})\Big)[1+k-N]. \end{aligned} \tag{6.17}$$

Taking the sheaf cohomology $\mathcal{H}^l$ to both sides of (6.17), by Lemma 4.6, we obtain

$$\mathcal{H}^l(j^* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)})) \cong \bigoplus_{s-r=l} \bigwedge^r \mathcal{N}_j^\vee \otimes \mathcal{H}^s(\mathbb{C}^N \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N]) \tag{6.18}$$

$$\text{or } \bigoplus_{s-r=l} \bigwedge^r \mathcal{N}_j^\vee \otimes \mathcal{H}^s((\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})\otimes\det(\mathbb{C}^N/\mathcal{V})[1+k-N]). \tag{6.19}$$

Since $\mathbb{C}^N \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N]$ and $(\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})\otimes\det(\mathbb{C}^N/\mathcal{V})[1+k-N]$ are complex concentrate at degree $N-k-1$, the only value s such that $\mathcal{H}^s \neq 0$ in (6.18) and (6.19) is $s = N-k-1$. Thus, if we choose $l = N-k-1$, then $r = s-l = 0$ and

$$\mathcal{H}^{N-k-1}(j^* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)})) \cong \mathbb{C}^N \otimes \det(\mathbb{C}^N/\mathcal{V}) \text{ or } (\mathcal{V}\oplus\mathbb{C}^N/\mathcal{V})\otimes\det(\mathbb{C}^N/\mathcal{V}). \tag{6.20}$$

On the other hand, using base change, the left-hand side of (6.17) becomes

$$j^* p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \cong p|_{D*} i^*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}).$$

We tensor (5.16) with $n = 2$ by $(\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)}$ such that it becomes

$$(\mathcal{V}''/\mathcal{V}''')^{\otimes 2} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)} \xrightarrow{c\otimes id} (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)} \to \mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}$$

where we denote $c : (\mathcal{V}''/\mathcal{V}''')^{\otimes 2} \to (\mathcal{V}'/\mathcal{V})^{\otimes 2}$ to be the natural inclusion. Applying the pullback i^*, since we have $V = V''$ on D, the map c becomes 0 and thus

$$i^*((\mathcal{V}''/\mathcal{V}''')^{\otimes 2} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)}) \xrightarrow{0} i^*((\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \to i^*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}),$$

which is again an exact triangle. Thus,

$$i^*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}) \cong i^*((\mathcal{V}''/\mathcal{V}''')^{\otimes 2} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)})[1] \bigoplus i^*((\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}).$$

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Using projection formula and Proposition 4.4, we conclude that

$$p|_{D*}i^*(\mathcal{O}_{2D}\otimes(\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)})\cong\mathbb{C}^N/\mathcal{V}\otimes\det(\mathbb{C}^N/V)[1+k-N].$$

When $l=N-k-1$,

$$\begin{aligned}\mathcal{H}^{N-k-1}(j^*p_*(\mathcal{O}_{2D}\otimes(\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}))&\cong\mathcal{H}^{N-k-1}(p|_{D*}i^*(\mathcal{O}_{2D}\otimes(\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)}))\\&\cong\mathcal{H}^{N-k-1}(\mathbb{C}^N/\mathcal{V}\otimes\det(\mathbb{C}^N/\mathcal{V})[1+k-N])\\&\cong\mathbb{C}^N/\mathcal{V}\otimes\det(\mathbb{C}^N/\mathcal{V})\end{aligned}$$

which is not isomorphic to (6.20). Hence, we obtain a contradiction. □

Remark 6.3. A similar argument shows that $\mathcal{H}_{-1}\mathbf{1}_{(k,N-k)}$ is neither isomorphic to $\Delta_*(\mathbb{C}^N)^\vee$ nor to $\Delta_*(\mathcal{V}^\vee\oplus(\mathbb{C}^N/\mathcal{V})^\vee)$.

6.3. Calculation of Convolutions and a Conjecture

Although the application of $\mathsf{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ is unclear to us, it is still interesting to know the categorical commutator relations between $\mathsf{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ and other 1-morphisms.

We will only consider the relations between $\mathsf{H}_1\mathbf{1}_{(k,N-k)}$ and $\Psi^+\mathbf{1}_{(k,N-k)}$, $\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ since the arguments for the others are similar. Then we have to calculate the convolutions of the FM kernels between $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ and $\mathbf{\Psi}^+\mathbf{1}_{(k,N-k)}$, $\mathcal{E}_r\mathbf{1}_{(k,N-k)}$.

By Proposition 6.2, the kernel $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ is complicated. Thus, the computation of convolutions involving $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ is generally difficult to carry out. In this subsection, we provide the calculation in the simplest case $N=2$.

When $N=2$, $\mathrm{Gr}(0,2)=\mathrm{Gr}(2,2)=pt$ and $\mathrm{Gr}(1,2)=\mathbb{P}^1$. The most interesting case is when $k=1$ where the fibered product variety X in (5.3) is given by

$$X=\{(0,V,V'',\mathbb{C}^2)\in\mathrm{Gr}(0,2)\times\mathbb{P}^1\times\mathbb{P}^1\times\mathrm{Gr}(2,2)\mid 0\subset V\subset\mathbb{C}^2,\ 0\subset V''\subset\mathbb{C}^2\}$$

and similarly the variety Y from diagram (5.1) is given by

$$Y=\{(V,V'')\in\mathbb{P}^1\times\mathbb{P}^1\mid\ \dim(V\cap V'')\geq 0\}.$$

Thus we obtain $X=Y=\mathbb{P}^1\times\mathbb{P}^1$. Both the projection map $p:X\longrightarrow Y$ and the inclusion map $t:Y\longrightarrow\mathbb{P}^1\times\mathbb{P}^1$ are the identity map. Let $\pi_i:\mathbb{P}^1\times\mathbb{P}^1\longrightarrow\mathbb{P}^1$ be the natural projection to the i-th copy where $i=1,2$. We also denote $\mathcal{V}$ and $\mathcal{V}''$ as the two tautological line bundles on the first and second copies of $\mathbb{P}^1$. Then we have the two tautological line bundles $\pi_1^*\mathcal{V}$, $\pi_2^*\mathcal{V}''$ on $\mathbb{P}^1\times\mathbb{P}^1$. Let $\mathbb{C}^2$ be the trivial bundle of rank two on $\mathbb{P}^1$ and denote $\Delta:\mathbb{P}^1\longrightarrow\mathbb{P}^1\times\mathbb{P}^1$ as the diagonal map. By abusing notations, we still denote its image as Δ. Then the associated line bundle is $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta)=\pi_2^*\mathcal{V}''\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{-1}$. Then, from (6.7), we have $\mathcal{H}_1\mathbf{1}_{(1,1)}=\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}$.

First, we compute the convolutions between $\mathcal{H}_1\mathbf{1}_{(1,1)}$ and $\mathbf{\Psi}^+\mathbf{1}_{(1,1)}$ and show that they are non-isomorphic.

Proposition 6.4. $(\mathbf{\Psi}^+*\mathcal{H}_1)\mathbf{1}_{(1,1)}\ncong(\mathcal{H}_1*\mathbf{\Psi}^+)\mathbf{1}_{(1,1)}$.

Proof. It is standard to check that $(\mathbf{\Psi}^+*\mathcal{H}_1)\mathbf{1}_{(1,1)}=\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes2}$. Now, we calculate $(\mathcal{H}_1*\mathbf{\Psi}^+)\mathbf{1}_{(1,1)}$. Using base-change with respect to

$$\begin{tikzcd} \mathbb{P}^1\times\mathbb{P}^1 \arrow[r, "\Delta\times id"] \arrow[d, "\pi_1"] & \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 \arrow[d, "\pi_{12}"] \\ \mathbb{P}^1 \arrow[r, "\Delta"] & \mathbb{P}^1\times\mathbb{P}^1, \end{tikzcd}$$

we have

$$\begin{aligned}(\mathcal{H}_1 * \Psi^+)\mathbf{1}_{(1,1)} &= \pi_{13*}\Big(\pi_{12}^*\Delta_*(\mathbb{C}^2/\mathcal{V})\otimes\pi_{23}^*\Big(\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\Big)\Big)\\ &\cong \pi_{13*}\Big((\Delta\times id)_*\pi_1^*(\mathbb{C}^2/\mathcal{V})\otimes\pi_{23}^*\Big(\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\Big)\Big)\\ &\cong (\pi_{13}\circ(\Delta\times id))_*\Big(\pi_1^*(\mathbb{C}^2/\mathcal{V})\otimes(\pi_{23}\circ(\Delta\times id))^*\Big(\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\Big)\Big)\\ &= \mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 3}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\end{aligned}$$

since $\pi_{13}\circ(\Delta\times id) = id$, $\pi_{23}\circ(\Delta\times id) = id$.

To see that $(\Psi^+ * \mathcal{H}_1)\mathbf{1}_{(1,1)} \not\cong (\mathcal{H}_1 * \Psi^+)\mathbf{1}_{(1,1)}$, we apply the pushforward π_{1*} to them and using the projection formula we obtain

$$\begin{aligned}\pi_{1*}(\Psi^+ * \mathcal{H}_1)\mathbf{1}_{(1,1)} &= \pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 2}) = \pi_{1*}(\mathcal{O}_{2\Delta})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2},\\ \pi_{1*}(\mathcal{H}_1 * \Psi^+)\mathbf{1}_{(1,1)} &= \pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 3}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})\\ &= \pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 3}.\end{aligned}$$

To calculate $\pi_{1*}(\mathcal{O}_{2\Delta})$, we use the following exact triangle

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2\Delta)\to\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\to\mathcal{O}_{2\Delta}. \tag{6.21}$$

Note that $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-\Delta) = \pi_2^*\mathcal{V}''\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{-1}$, thus

$$\pi_{1*}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2\Delta) = \pi_{1*}(\pi_2^*(\mathcal{V}'')^{\otimes 2}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}) \cong \pi_{1*}\pi_2^*(\mathcal{V}'')^{\otimes 2}\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes(-2)} \cong (\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}[-1]$$

since $H^*(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-2)) = \mathbb{C}[-1]$. Then, after applying π_{1*} to (6.21), it becomes

$$\mathcal{O}_{\mathbb{P}^1}\to\pi_{1*}\mathcal{O}_{2\Delta}\to(\mathbb{C}^2/\mathcal{V})^{\otimes(-2)} \tag{6.22}$$

and $\pi_{1*}(\mathcal{O}_{2\Delta})$ is determined by $\mathrm{Ext}^1((\mathbb{C}^2/\mathcal{V})^{\otimes(-2)},\mathcal{O}_{\mathbb{P}^1}) = \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-2),\mathcal{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2)) = 0$.
Thus

$$\pi_{1*}(\mathcal{O}_{2\Delta}) = (\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}\oplus\mathcal{O}_{\mathbb{P}^1}$$

and we obtain

$$\pi_{1*}(\Psi^+ * \mathcal{H}_1)\mathbf{1}_{(1,1)} = \pi_{1*}(\mathcal{O}_{2\Delta})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2} = (\mathbb{C}^2/\mathcal{V})^{\otimes 2}\oplus\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}. \tag{6.23}$$

On the other hand, to calculate $\pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})$, we tensor (6.21) with $\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}$ to get

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2\Delta)\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\to\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\to\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}.$$

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Then, applying π_{1*}, we calculate

$$\begin{aligned}\pi_{1*}\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1} &\cong 0\\ \pi_{1*}(\pi_2^*(\mathcal{V}'')^{\otimes 2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}) &\cong \pi_{1*}(\pi_2^*(\mathcal{V}'')^{\otimes 2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}\\ &\cong((\mathbb{C}^2/\mathcal{V})^{\otimes(-2)})^{\oplus 2}[-1]\end{aligned}$$

since $H^*(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-1))=0$ and $H^*(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-3))=\mathbb{C}^2[-1]$.

Thus,

$$\pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})=((\mathbb{C}^2/\mathcal{V})^{\otimes(-2)})^{\oplus 2} \tag{6.24}$$

which implies

$$\pi_{1*}(\mathcal{H}_1*\mathbf{\Psi}^+)\mathbf{1}_{(1,1)}=\pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 3}=(\mathbb{C}^2/\mathcal{V})^{\oplus 2}=\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \tag{6.25}$$

which is not isomorphic to (6.23). □

Remark 6.5. Using the same argument, we can also show that $(\mathcal{H}_1*\mathbf{\Psi}^-)\mathbf{1}_{(1,1)}\ncong(\mathbf{\Psi}^-*\mathcal{H}_1)\mathbf{1}_{(1,1)}$.

Next, we compute the convolutions between $\mathcal{H}_1\mathbf{1}_{(1,1)}$ and $\mathcal{E}_r\mathbf{1}_{(1,1)}$. Note that besides $r=-1,-0$, we also have the FM kernel $\mathcal{E}_{-2}\mathbf{1}_{(1,1)}$ since we have defined the new 1-morphism $\mathsf{E}_{-2}\mathbf{1}_{(1,1)}$.

Proposition 6.6. *We have the following exact triangle in* $\mathrm{D}^b(\mathrm{Gr}(1,2)\times\mathrm{Gr}(0,2))=\mathrm{D}^b(\mathbb{P}^1)$

$$(\mathcal{H}_1*\mathcal{E}_{-1})\mathbf{1}_{(1,1)}\to(\mathcal{E}_{-1}*\mathcal{H}_1)\mathbf{1}_{(1,1)}\to\mathcal{E}_0\mathbf{1}_{(1,1)}\oplus\mathcal{E}_0\mathbf{1}_{(1,1)}[1]$$

and $(\mathcal{H}_1*\mathcal{E}_0)\mathbf{1}_{(1,1)}\cong(\mathcal{E}_0*\mathcal{H}_1)\mathbf{1}_{(1,1)}$, $(\mathcal{H}_1*\mathcal{E}_{-2})\mathbf{1}_{(1,1)}\cong(\mathcal{E}_{-2}*\mathcal{H}_1)\mathbf{1}_{(1,1)}$.

Proof. First, we calculate $(\mathcal{H}_1*\mathcal{E}_r)\mathbf{1}_{(1,1)}$. To know $\mathcal{H}_1\mathbf{1}_{(0,2)}$, from Proposition 5.10, we have the following exact triangle

$$(\mathcal{F}_3*\mathcal{E}_0)\mathbf{1}_{(0,2)}\to(\mathcal{E}_0*\mathcal{F}_3)\mathbf{1}_{(0,2)}\to(\mathbf{\Psi}^+*\mathcal{H}_1)\mathbf{1}_{(0,2)}.$$

Since $(\mathcal{F}_3*\mathcal{E}_0)\mathbf{1}_{(0,2)}=0$ and $\mathbf{\Psi}^+\mathbf{1}_{(0,2)}=\Delta_*\mathbb{C}[-1]\in\mathrm{D}^b(\mathrm{Gr}(0,2)\times\mathrm{Gr}(0,2))$, a direct calculation shows that $(\mathcal{E}_0*\mathcal{F}_3)\mathbf{1}_{(0,2)}\cong\mathbb{C}^2[-1]\in\mathrm{D}^b(\mathrm{Gr}(0,2)\times\mathrm{Gr}(0,2))$.

So, $\mathcal{H}_1\mathbf{1}_{(0,2)}=((\mathbf{\Psi}^+)^{-1}*\mathcal{E}_0*\mathcal{F}_3)\mathbf{1}_{(0,2)}\cong\mathbb{C}^2$, and $(\mathcal{H}_1*\mathcal{E}_r)\mathbf{1}_{(1,1)}=\mathcal{O}_{\mathbb{P}^1}(-r)^{\oplus 2}$ since $\pi_{12}=\pi_{23}=\pi_{13}=id$.

Next, we calculate $(\mathcal{E}_r*\mathcal{H}_1)\mathbf{1}_{(1,1)}$, which is given by

$$\begin{aligned}(\mathcal{E}_r*\mathcal{H}_1)\mathbf{1}_{(1,1)}&=\pi_{13*}(\pi_{12}^*\mathcal{H}_1\mathbf{1}_{(1,1)}\otimes\pi_{23}^*\mathcal{E}_r\mathbf{1}_{(1,1)})\\ &=\pi_{1*}(\mathcal{H}_1\mathbf{1}_{(1,1)}\otimes\pi_2^*\mathcal{E}_r\mathbf{1}_{(1,1)})\\ &=\pi_{1*}\Big(\mathcal{O}_{2\Delta}\otimes\pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes 2}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\otimes\pi_2^*(\mathcal{V}'')^{\otimes r}\Big)\\ &=\pi_{1*}\Big(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\otimes\pi_2^*(\mathcal{V}'')^{\otimes r}\Big)\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2}.\end{aligned}$$

From (6.3) and (6.24), we know that

$$\begin{aligned}(\mathcal{E}_0*\mathcal{H}_1)\mathbf{1}_{(1,1)}&=\pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2}=((\mathbb{C}^2/\mathcal{V})^{\otimes(-2)})^{\oplus 2}\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2}=\mathcal{O}_{\mathbb{P}^1}^{\oplus 2},\\ (\mathcal{E}_{-1}*\mathcal{H}_1)\mathbf{1}_{(1,1)}&=\pi_{1*}(\mathcal{O}_{2\Delta}\otimes\pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1}\otimes\pi_2^*(\mathcal{V}'')^{-1})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2}\\ &=\pi_{1*}(\mathcal{O}_{2\Delta})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2}=((\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}\oplus\mathcal{O}_{\mathbb{P}^1})\otimes(\mathbb{C}^2/\mathcal{V})^{\otimes 2}=\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}.\end{aligned}$$

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Finally, to calculate

$$
\begin{aligned}
(\mathcal{E}_{-2} * \mathcal{H}_1)\mathbf{1}_{(1,1)} &= \pi_{1*}(\mathcal{O}_{2\Delta} \otimes \pi_2^*(\mathbb{C}^2/\mathcal{V}'')^{-1} \otimes \pi_2^*(\mathcal{V}'')^{\otimes(-2)}) \otimes (\mathbb{C}^2/\mathcal{V})^{\otimes 2} \\
&= \pi_{1*}(\mathcal{O}_{2\Delta} \otimes \pi_2^*(\mathcal{V}'')^{-1}) \otimes (\mathbb{C}^2/\mathcal{V})^{\otimes 2},
\end{aligned}
$$

we need to calculate $\pi_{1*}(\mathcal{O}_{2\Delta} \otimes \pi_2^*(\mathcal{V}'')^{-1})$.

Tensoring (6.21) with $\pi_2^*(\mathcal{V}'')^{-1}$ and applying π_{1*}, we get $\pi_{1*}(\pi_2^*\mathcal{V}'' \otimes \pi_1^*(\mathbb{C}^2/\mathcal{V})^{\otimes(-2)}) \cong \pi_{1*}\pi_2^*\mathcal{V}'' \otimes (\mathbb{C}^2/\mathcal{V})^{\otimes(-2)} = 0$ by projection formula and $\pi_{1*}\pi_2^*(\mathcal{V}'')^{-1} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. So we have

$$
(\mathcal{E}_{-2} * \mathcal{H}_1)\mathbf{1}_{(1,1)} = \pi_{1*}(\mathcal{O}_{2\Delta} \otimes \pi_2^*(\mathcal{V}'')^{-1}) \otimes (\mathbb{C}^2/\mathcal{V})^{\otimes 2} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \otimes (\mathbb{C}^2/\mathcal{V})^{\otimes 2} = \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2}.
$$

As a consequence, we have the following isomorphisms

$$
\begin{aligned}
(\mathcal{H}_1 * \mathcal{E}_0)\mathbf{1}_{(1,1)} &= \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} = (\mathcal{E}_0 * \mathcal{H}_1)\mathbf{1}_{(1,1)}, \\
(\mathcal{H}_1 * \mathcal{E}_{-2})\mathbf{1}_{(1,1)} &= \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2} \cong \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 2} = (\mathcal{E}_{-2} * \mathcal{H}_1)\mathbf{1}_{(1,1)}.
\end{aligned}
$$

On the other hand, for $r = -1$, we have the following exact triangle

$$
(\mathcal{H}_1 * \mathcal{E}_{-1})\mathbf{1}_{(1,1)} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to (\mathcal{E}_{-1} * \mathcal{H}_1)\mathbf{1}_{(1,1)} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}[1] \oplus \mathcal{O}_{\mathbb{P}^1} = \mathcal{E}_0\mathbf{1}_{(1,1)} \oplus \mathcal{E}_0\mathbf{1}_{(1,1)}[1].
$$

□

Remark 6.7. Like Proposition 6.4 and Proposition 6.6, similar results also hold for the convolutions of $\mathcal{H}_{-1}\mathbf{1}_{(1,1)}$, which is given by

$$
\mathcal{H}_{-1}\mathbf{1}_{(1,1)} = \mathcal{O}_{2\Delta} \otimes \pi_1^*\mathcal{V}^{\otimes(-2)} \otimes \pi_2^*\mathcal{V}''.
$$

Finally, based on the above observation, we formulate a conjecture, which is the $\mathfrak{sl}_n$ version of Proposition 6.6.

Conjecture 6.8.

1. *We have the following exact triangles in* $\mathrm{D}^b(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N) \times \mathrm{Fl}_{\underline{k}+\alpha_i}(\mathbb{C}^N))$

$$
\begin{aligned}
&(\mathcal{H}_{i,1} * \mathcal{E}_{i,-k_i})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i,-k_i} * \mathcal{H}_{i,1})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i,-k_i+1} \bigoplus \mathcal{E}_{i,-k_i+1}[1])\mathbf{1}_{\underline{k}}, \\
&(\mathcal{E}_{i,-k_i} * \mathcal{H}_{i,-1})\mathbf{1}_{\underline{k}} \to (\mathcal{H}_{i,-1} * \mathcal{E}_{i,-k_i})\mathbf{1}_{\underline{k}} \to (\mathcal{E}_{i,-k_i-1} \bigoplus \mathcal{E}_{i,-k_i-1}[1])\mathbf{1}_{\underline{k}},
\end{aligned}
$$

   and

$$
(\mathcal{H}_{i,\pm 1} * \mathcal{E}_{i,r})\mathbf{1}_{\underline{k}} \cong (\mathcal{E}_{i,r} * \mathcal{H}_{i,\pm 1})\mathbf{1}_{\underline{k}}
$$

   for $-k_i - 1 \le r \le 0$ *with* $r \neq -k_i$.

2. *We have the following exact triangles in* $\mathrm{D}^b(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N) \times \mathrm{Fl}_{\underline{k}-\alpha_i}(\mathbb{C}^N))$

$$
\begin{aligned}
&(\mathcal{F}_{i,k_{i+1}} * \mathcal{H}_{i,1})\mathbf{1}_{\underline{k}} \to (\mathcal{H}_{i,1} * \mathcal{F}_{i,k_{i+1}})\mathbf{1}_{\underline{k}} \to (\mathcal{F}_{i,k_{i+1}+1} \bigoplus \mathcal{F}_{i,k_{i+1}+1}[1])\mathbf{1}_{\underline{k}}, \\
&(\mathcal{H}_{i,-1} * \mathcal{F}_{i,k_{i+1}})\mathbf{1}_{\underline{k}} \to (\mathcal{F}_{i,k_{i+1}} * \mathcal{H}_{i,-1})\mathbf{1}_{\underline{k}} \to (\mathcal{F}_{i,k_{i+1}-1} \bigoplus \mathcal{F}_{i,k_{i+1}-1}[1])\mathbf{1}_{\underline{k}},
\end{aligned}
$$

   and

$$
(\mathcal{H}_{i,\pm 1} * \mathcal{F}_{i,s})\mathbf{1}_{\underline{k}} \cong (\mathcal{F}_{i,s} * \mathcal{H}_{i,\pm 1})\mathbf{1}_{\underline{k}}
$$

   for $0 \le s \le k_{i+1} + 1$ *with* $s \neq k_{i+1}$.

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6.4. The Commutator Relation on the Grothendieck Groups for All Loop Generators

From Corollary 5.14, we know that there is an action of $\mathcal{U}$ on $\bigoplus_{\underline{k}} K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ which comes from decategorifying the categorical $\mathcal{U}$-action in Theorem 5.6. Since the 1-morphisms act by FM transforms with kernels in Definition 5.5, the generators of $\mathcal{U}$ act on $\bigoplus_{\underline{k}} K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ by the K-theoretic FM transform. More precisely,

$$e_{i,r}1_{\underline{k}} : K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}+\alpha_i}(\mathbb{C}^N)),\ x \mapsto \pi_{2*}(\pi_1^*(x) \otimes [\iota(\underline{k})_*(\mathcal{V}_i/\mathcal{V}_i')^{\otimes r}])$$

where we denote $[\iota(\underline{k})_*(\mathcal{V}_i/\mathcal{V}_i')^{\otimes r}]$ to be the class of $\iota(\underline{k})_*(\mathcal{V}_i/\mathcal{V}_i')^{\otimes r}$ in $K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N) \times \mathrm{Fl}_{\underline{k}+\alpha_i}(\mathbb{C}^N))$, and similarly for $f_{i,s}1_{\underline{k}}$ in the opposite direction. For other generators, it is direct to see that

$$\begin{aligned} \psi_i^+ 1_{\underline{k}} &: K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)),\ x \mapsto x \otimes (-1)^{k_{i+1}-1}[\det(\mathcal{V}_{i+1}/\mathcal{V}_i)], \\ \psi_i^- 1_{\underline{k}} &: K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)),\ x \mapsto x \otimes (-1)^{k_i-1}[\det(\mathcal{V}_i/\mathcal{V}_{i-1})^{-1}]. \end{aligned}$$

As we mentioned at the beginning of this section, the FM kernels $\mathcal{E}_{i,r}\mathbf{1}_{\underline{k}}$, $\mathcal{F}_{i,r}\mathbf{1}_{\underline{k}}$ are defined for all $r \in \mathbb{Z}$. Thus, the action of $e_{i,r}1_{\underline{k}}$, $f_{i,r}1_{\underline{k}}$ by K-theoretic FM transforms are also defined for all $r \in \mathbb{Z}$. In this subsection, we want to understand the commutator relation $[e_{i,r}, f_{i,s}]1_{\underline{k}}$ for all $r, s \in \mathbb{Z}$.

It suffices to consider the $\mathfrak{sl}_2$ case where the action is on $\bigoplus_k K(\mathrm{Gr}(k,N))$. Then, condition (10) gives the following commutator relation

$$\begin{aligned} [e_0, f_{N-k}]1_{(k,N-k)} &= \psi^+ 1_{(k,N-k)} = \otimes(-1)^{N-k-1}[\det(\mathbb{C}^N/\mathcal{V})], \\ [e_r, f_s]1_{(k,N-k)} &= 0, \text{ if } -k+1 \le r+s \le N-k-1, \\ [e_{-k}, f_0]1_{(k,N-k)} &= \psi^- 1_{(k,N-k)} = \otimes(-1)^k[\det(\mathcal{V})^{-1}]. \end{aligned} \tag{6.26}$$

Moreover, the two exact triangles (6.2) and (6.3) give the following commutator relations

$$\begin{aligned} [e_0, f_{N-k+1}]1_{(k,N-k)} &= \psi^+ h_1 1_{(k,N-k)}, \\ [e_{-k-1}, f_0]1_{(k,N-k)} &= -\psi^- h_{-1} 1_{(k,N-k)}. \end{aligned} \tag{6.27}$$

To know the action of $h_{\pm1}1_{(k,N-k)}$, although their FM kernels $\mathcal{H}_{\pm1}\mathbf{1}_{(k,N-k)}$ are complicate from (6.7) and (6.8), from Subsection 6.2 we know that $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ fits in the exact triangle (6.16).

Passing to the Grothendieck group, we have the following equality in $K(\mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N))$

$$[\mathcal{H}_1\mathbf{1}_{(k,N-k)}] = [\Delta_*\mathcal{V}] + [\Delta_*\mathbb{C}^N/\mathcal{V}] = [\Delta_*\mathbb{C}^N].$$

Thus the action of $h_1 1_{(k,N-k)}$ is given by

$$h_1 1_{(k,N-k)} : K(\mathrm{Gr}(k,N)) \to K(\mathrm{Gr}(k,N)),\ x \mapsto x \otimes [\mathbb{C}^N].$$

A similar argument also shows that the action of $h_{-1}1_{(k,N-k)}$ is given by

$$h_{-1}1_{(k,N-k)} : K(\mathrm{Gr}(k,N)) \to K(\mathrm{Gr}(k,N)),\ x \mapsto x \otimes [(\mathbb{C}^N)^\vee].$$

Note that even though the convolutions between $\mathcal{H}_1\mathbf{1}_{(k,N-k)}$ and $\Psi^+\mathbf{1}_{(k,N-k)}$ are non-isomorphic in general when we pass to the Grothendieck group, they are commutative, i.e., $\psi^+ h_1 1_{(k,N-k)} = h_1\psi^+ 1_{(k,N-k)}$ (similarly for $h_{-1}1_{(k,N-k)}$ and $\psi^- 1_{(k,N-k)}$).

Next, it is standard to check that we have the following conjugation property

$$\begin{aligned} e_r 1_{(k,N-k)} &= (-1)^r(\psi^+)^r e_0(\psi^+)^{-r}1_{(k,N-k)} = (-1)^r(\psi^-)^r e_0(\psi^-)^{-r}1_{(k,N-k)}, \\ f_r 1_{(k,N-k)} &= (-1)^r(\psi^+)^r f_0(\psi^+)^{-r}1_{(k,N-k)} = (-1)^r(\psi^-)^r f_0(\psi^-)^{-r}1_{(k,N-k)}, \end{aligned}$$

for all $r \in \mathbb{Z}$ (see (3.2), (3.3), and (3.4) for their categorical version).

The above conjugation property can help us to know other commutator relations. For example, we can write

$$\begin{aligned}[e_r, f_s]1_{(k,N-k)} &= e_r f_s 1_{(k,N-k)} - f_s e_r 1_{(k,N-k)} = (\psi^+)^r e_0 f_{r+s}(\psi^+)^{-r} - (\psi^+)^r f_{r+s} e_0 (\psi^+)^{-r} 1_{(k,N-k)} \\ &= (\psi^+)^r [e_0, f_{r+s}](\psi^+)^{-r} 1_{(k,N-k)}.\end{aligned}$$

Thus, by applying suitable conjugation to (6.26) and (6.27), we obtain

$$[e_r, f_s]1_{(k,N-k)} = \begin{cases} \otimes(-1)^{N-k-1}[\det(\mathbb{C}^N/\mathcal{V})][\mathbb{C}^N] & \text{if } r+s = N-k+1 \\ \otimes(-1)^{N-k-1}[\det(\mathbb{C}^N/\mathcal{V})] & \text{if } r+s = N-k \\ 0 & \text{if } -k+1 \leq r+s \leq N-k-1 \\ \otimes(-1)^{k}[\det(\mathcal{V})^{-1}] & \text{if } r+s = -k \\ \otimes(-1)^{k}[\det(\mathcal{V})^{-1}][(\mathbb{C}^N)^\vee] & \text{if } r+s = -k-1 \end{cases}.$$

For the rest of the cases, we have to understand the categorical commutator relation between $\mathsf{E}_r\mathsf{F}_s\mathbf{1}_{(k,N-k)}$ and $\mathsf{F}_s\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ (or more precisely $(\mathcal{E}_r * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_r)\mathbf{1}_{(k,N-k)}$) for $r+s \geq N-k+2$ and $r+s \leq -k-2$.

Consider first case $r+s = N-k+2$, similarly like Proposition 5.10, it suffices to compare $(\mathcal{E}_0 * \mathcal{F}_s)\mathbf{1}_{(k,N-k)}$ and $(\mathcal{F}_s * \mathcal{E}_0)\mathbf{1}_{(k,N-k)}$ for $s = N-k+2$.

Then like the exact triangle (6.1) we have the following exact triangle

$$p_*(\mathcal{V}''/\mathcal{V}''')^{\otimes(N-k+2)} \to p_*(\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)} \to p_*(\mathcal{O}_{(N-k+2)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}) \tag{6.28}$$

by applying p_* to (5.16) with $n = N-k+2$. Again, a similar argument as in the proof of Proposition 5.10 tells us that

$$p_*(\mathcal{O}_{(N-k+2)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}) \cong p_*(\mathcal{O}_{(N-k+1)D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}) \cong \ldots \cong p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}),$$

and applying t_* to (6.28) we obtain the following exact triangle

$$(\mathcal{F}_{N-k+2} * \mathcal{E}_0)\mathbf{1}_{(k,N-k)} \to (\mathcal{E}_0 * \mathcal{F}_{N-k+2})\mathbf{1}_{(k,N-k)} \to t_*p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}). \tag{6.29}$$

Thus the first question is we have to study $t_*p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)})$. Tensoring (5.17) with $n = 3$ by $(\mathcal{V}'/\mathcal{V})^{\otimes(N-k-1)}$, we obtain

$$\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k)} \otimes (\mathcal{V}/\mathcal{V}''')^2 \to \mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)} \to \mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}$$

applying t_*p_*, using the projection formula and Proposition 4.4, we get the following exact triangle

$$\Delta_*\mathrm{Sym}^2(\mathcal{V}) \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N] \to t_*p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes N-k+2}) \to t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes N-k+2}). \tag{6.30}$$

Next, tensoring (5.17) with $n = 2$ by $(\mathcal{V}'/\mathcal{V})^{\otimes(N-k)}$, we obtain

$$\mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+1)} \otimes (\mathcal{V}/\mathcal{V}''') \to \mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)} \to \mathcal{O}_D \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}$$

again applying t_*p_*, using the projection formula and Proposition 4.4, we get the following exact triangle

$$\begin{aligned}&\Delta_*\mathcal{V} \otimes \mathbb{C}^N/\mathcal{V} \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N] \to t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}) \\ &\quad \to \Delta_*\mathrm{Sym}^2(\mathbb{C}^N/\mathcal{V}) \otimes \det(\mathbb{C}^N/\mathcal{V})[1+k-N].\end{aligned} \tag{6.31}$$

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Thus, understanding $t_*p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)})$ is equivalent to understanding the two exact triangles (6.30) and (6.31). Moreover, we define

$$\begin{aligned}\mathcal{H}_2\mathbf{1}_{(k,N-k)} &:= (\Psi^+\mathbf{1}_{(k,N-k)})^{-1} * [t_*p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)})] \\ &= t_*p_*(\mathcal{O}_{3D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)}) \otimes \pi_2^* \det(\mathbb{C}^N/\mathcal{V})^{-1}[N-k-1] \in \mathrm{D}^b(\mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N))\end{aligned} \tag{6.32}$$

which can be thought of as an FM kernel for a certain (undefined) 1-morphism $\mathsf{H}_2\mathbf{1}_{(k,N-k)}$. Then $\mathcal{H}_2\mathbf{1}_{(k,N-k)}$ is build up from the following two exact triangles in $\mathrm{D}^b(\mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N))$

$$\Delta_*\mathrm{Sym}^2(\mathcal{V}) \to \mathcal{H}_2\mathbf{1}_{(k,N-k)} \to (\Psi^+\mathbf{1}_{(k,N-k)})^{-1} * [t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)})], \tag{6.33}$$

$$\Delta_*\mathcal{V} \otimes \mathbb{C}^N/\mathcal{V} \to (\Psi^+\mathbf{1}_{(k,N-k)})^{-1} * [t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)})] \to \Delta_*\mathrm{Sym}^2(\mathbb{C}^N/\mathcal{V}). \tag{6.34}$$

Remark 6.9. Another natural question to ask is what is the categorical commutator relation between $\mathsf{H}_2\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathsf{E}_r\mathsf{H}_2\mathbf{1}_{(k,N-k)}$ (or exact triangle relates $\mathcal{H}_2 * \mathcal{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathcal{E}_r * \mathcal{H}_2\mathbf{1}_{(k,N-k)}$). Due to the difficulty and complication, we will like to address those questions in the future and similar for other comparisons between $\mathsf{H}_n\mathsf{E}_r\mathbf{1}_{(k,N-k)}$ and $\mathsf{E}_r\mathsf{H}_n\mathbf{1}_{(k,N-k)}$ with $n \geq 3$.

However, the above discussions can help us to understand what exactly are the commutators for the loop generators $e_r 1_{(k,N-k)}, f_s 1_{(k,N-k)}$ at the level of Grothendieck groups. From (6.33) and (6.34), we have the following equality in $K(\mathrm{Gr}(k,N) \times \mathrm{Gr}(k,N))$

$$\begin{aligned}[\mathcal{H}_2\mathbf{1}_{(k,N-k)}] &= [\Delta_*\mathrm{Sym}^2(\mathcal{V})] + [(\Psi^+\mathbf{1}_{(k,N-k)})^{-1} * t_*p_*(\mathcal{O}_{2D} \otimes (\mathcal{V}'/\mathcal{V})^{\otimes(N-k+2)})] \\ &= [\Delta_*\mathrm{Sym}^2(\mathcal{V})] + [\Delta_*\mathcal{V} \otimes \mathbb{C}^N/\mathcal{V}] + [\Delta_*\mathrm{Sym}^2(\mathbb{C}^N/\mathcal{V})] \\ &= [\mathrm{Sym}^2(\mathcal{V} \oplus \mathbb{C}^N/\mathcal{V})] = [\mathrm{Sym}^2(\mathbb{C}^N)].\end{aligned}$$

Thus, by the definition of $\mathcal{H}_2\mathbf{1}_{(k,N-k)}$ and the exact triangle (6.29), we get the following commutator relation after a suitable conjugation

$$[e_r, f_s]1_{(k,N-k)} = \otimes(-1)^{N-k-1}[\mathrm{Sym}^2(\mathbb{C}^N)][\det(\mathbb{C}^N/\mathcal{V})], \text{ if } r+s = N-k+2.$$

The above argument can be applied to other cases where $r+s \geq N-k+3$ and similarly $r+s \leq -k-2$. Moreover, the result can be generalized to the $\mathfrak{sl}_n$ case directly. In conclusion, we have the following result.

Corollary 6.10. *The commutator relations at the level of Grothendieck groups $K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ for all $e_{i,r}1_{\underline{k}}$, $f_{i,s}1_{\underline{k}}$ with $r, s \in \mathbb{Z}$ is given by*

$$[e_{i,r}, f_{i,s}]1_{\underline{k}} = \begin{cases} \otimes(-1)^{k_{i+1}-1}[\det(\mathcal{V}_{i+1}/\mathcal{V}_i)][\mathrm{Sym}^{r+s-k_{i+1}}(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})] & \text{if } r+s \geq k_{i+1} \\ 0 & \text{if } -k_i+1 \leq r+s \leq k_{i+1}-1 \\ \otimes(-1)^{k_i}[\det(\mathcal{V}_i/\mathcal{V}_{i-1})^{-1}][\mathrm{Sym}^{-r-s-k_i}(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})^\vee] & \text{if } r+s \leq -k_i \end{cases}.$$

Finally, we give a remark.

Remark 6.11. It would be interesting to find an integral basis for $K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ and compute the matrix elements of the generators $e_{i,r}1_{\underline{k}}$ and $f_{i,s}1_{\underline{k}}$.

7. Categorical Action of the Affine 0-Hecke Algebras

In this final section, we apply the categorical action of the shifted 0-affine algebra, i.e., Theorem 5.6, to construct categorical action of the affine 0-Hecke algebra on the derived category of coherent sheaves on the full flag variety.

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7.1. Affine 0-Hecke Algebras

We begin with the definition of the affine 0-Hecke algebras of type A.

Definition 7.1. The *affine 0-Hecke algebra* $\mathcal{H}_N(0)$ is defined to be the unital associative $\mathbb{C}$-algebra generated by the elements $T_1, ..., T_{N-1}$ and the polynomial algebra $\mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$ subject to the following relations

$$T_i^2 = T_i, \tag{H01}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \; T_i T_j = T_j T_i, \text{ if } |i-j| \geq 2, \tag{H02}$$

$$T_i X_j = X_j T_i \text{ if } j \neq i, i+1, \tag{H03}$$

$$X_{i+1} T_i = T_i X_i + X_{i+1}, \tag{H04}$$

$$X_i T_i = T_i X_{i+1} - X_{i+1}. \tag{H05}$$

We give a remark about the affine 0-Hecke algebra.

Remark 7.2. The affine 0-Hecke algebra is the affine Hecke algebra with variable q specialized at 0.

Next, we recall the action of $\mathcal{H}_N(0)$ on the Grothendieck group of the full flag variety. Let $G = \mathrm{SL}_N(\mathbb{C})$ and $B \subset G$ be the Borel of upper triangular matrices. Then, the full flag variety is defined by

$$G/B := \{0 \subset V_1 \subset V_2 \subset ... \subset V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for all } k\}. \tag{7.1}$$

The Weyl group W is isomorphic to the symmetric group, denoted by S_N. Let $\{s_1, ..., s_{N-1}\}$ be the simple reflections that generate the Weyl group $W = S_N$. Then, for each $1 \leq i \leq N-1$, there is the minimal parabolic subgroup $P_i = B \cup B s_i B$. Similarly, the quotient G/P_i, which is called a partial flag variety, can be identified with the following space

$$G/P_i := \{0 \subset V_1 \subset V_2 \subset ... \subset V_{i-1} \subset V_{i+1} \subset ... \subset V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for } k \neq i\}. \tag{7.2}$$

There are natural projections $\pi_i : G/B \to G/P_i$ where $1 \leq i \leq N-1$, which obviously are $\mathbb{P}^1$-fibrations. Those maps induce pushforwards $\pi_{i_*} : K(G/B) \to K(G/P_i)$ and pullbacks $\pi_i^* : K(G/P_i) \to K(G/B)$ where $1 \leq i \leq N-1$. Then the fundamental construction of the push-pull operators gives the *Demazure operators* $T_i := \pi_i^* \pi_{i_*} : K(G/B) \to K(G/B)$ for all $1 \leq i \leq N-1$.

On G/B, we denote $\mathcal{V}_i$ to be the tautological bundle of rank i for all $1 \leq i \leq N$. Let $a_i = [\mathcal{V}_i/\mathcal{V}_{i-1}]$ be the class of the tautological line bundle $\mathcal{V}_i/\mathcal{V}_{i-1}$ in $K(G/B)$, where $1 \leq i \leq N$. Then, $K(G/B)$ admits a presentation, which is called the Borel presentation.

$$K(G/B) \cong \mathbb{C}[a_1^{\pm}, ..., a_N^{\pm}] \Big/ \left\langle e_i - \binom{N}{i} \right\rangle \tag{7.3}$$

where $\langle e_i - \binom{N}{i} \rangle$ is the ideal generated by $\{e_i - \binom{N}{i}\}_{i=1}^N$, and e_i is the i-th elementary symmetric polynomial in $a_1, a_2, ..., a_N$.

Under the presentation (7.3), T_i has the following explicit description

$$T_i = \frac{a_{i+1} - a_i s_i}{a_{i+1} - a_i}$$

where s_i is the simple reflection that permutes a_i and a_{i+1} for all $1 \leq i \leq N-1$. Finally, we define the operator $X_j : K(G/B) \to K(G/B)$ to be multiplication by a_j for all $1 \leq j \leq N$.

Then it is easy to check that those operators T_i, $X_j^{\pm 1}$ defined in such a way satisfy relations relations (H01) to (H05) in Definition 7.1. Thus we get an action of $\mathcal{H}_N(0)$ on $K(G/B)$.

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It is natural to lift this action to the categorical level. More precisely, we replace $K(G/B)$ by the derived category of coherent sheaves $\mathrm{D}^b(G/B)$ and we need to define certain functors $\mathbb{T}_i : \mathrm{D}^b(G/B) \to \mathrm{D}^b(G/B)$ and $\mathbb{X}_j : \mathrm{D}^b(G/B) \to \mathrm{D}^b(G/B)$ that satisfy categorical version of the relations (H01) to (H05). Again, we would use the tools of FM transforms/kernels from Section 4 to define the categorical action.

We know that $T_i := \pi_i^* \pi_{i_*}$ are defined by using the projection maps $\pi_i : G/B \to G/P_i$. Since π_i also induces functors on derived categories of coherent sheaves, i.e. the derived pushforward $\pi_{i*} : \mathrm{D}^b(G/B) \to \mathrm{D}^b(G/P_i)$ and the derived pullback $\pi_i^* : \mathrm{D}^b(G/P_i) \to \mathrm{D}^b(G/B)$, it is natural to define the functors $\mathbb{T}_i := \pi_i^* \pi_{i_*} : \mathrm{D}^b(G/B) \to \mathrm{D}^b(G/B)$ for all $1 \leq i \leq N-1$.

In fact, $\mathbb{T}_i$ are also FM transforms. Recall that for a morphism $f : X \to Y$ between smooth projective varieties, the derived pushforward $f_* : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is isomorphic to the FM transform with kernel $\mathcal{O}_{\Gamma_f}$, where $\Gamma_f = (id \times f)(X)$ is the graph of f in $X \times Y$. Similarly, the derived pullback f^* is isomorphic to the FM transform with kernel $\mathcal{O}_{(f \times id)(X)}$ in $\mathrm{D}^b(Y \times X)$.

Thus, a simple calculation of the convolution of FM kernels shows that

$$\mathbb{T}_i \cong \Phi_{\mathcal{O}_{(\pi_i \times id)(G/B)}} \circ \Phi_{\mathcal{O}_{(id \times \pi_i)(G/B)}} \cong \Phi_{\mathcal{T}_i}$$

where $\mathcal{T}_i \cong \mathcal{O}_{G/B \times_{G/P_i} G/B}$ for all $1 \leq i \leq N-1$.

The fibered product variety $G/B \times_{G/P_i} G/B$ is called the *Bott-Samelson variety for s_i*. Thus, the FM transform with kernel given by structure sheaf of the Bott-Samelson variety for s_i categorifies the Demazure operator T_i for all $1 \leq i \leq N-1$.

Next, we define the functors $\mathbb{X}_j$. Since X_j is the map given by multiplication with the element $a_j = [\mathcal{V}_j/\mathcal{V}_{j-1}]$ on the Grothendieck group, it is natural to define $\mathbb{X}_j$ to be the functor that is given by tensoring the line bundle $\mathcal{V}_j/\mathcal{V}_{j-1}$. Thus its FM kernel is given by $\mathcal{X}_j = \Delta_*(\mathcal{V}_j/\mathcal{V}_{j-1}) \in \mathrm{D}^b(G/B \times G/B)$ for all $1 \leq j \leq N$, where $\Delta : G/B \to G/B \times G/B$ is the diagonal map. Since line bundles are invertible, we can define its inverse functor $\mathbb{X}_j^{-1}$ with FM kernel given by $\mathcal{X}_j^{-1} = \Delta_*((\mathcal{V}_j/\mathcal{V}_{j-1})^{-1}) \in \mathrm{D}^b(G/B \times G/B)$.

With all the above settings, we can prove the following theorem, which says that there is a categorical action of the affine 0-Hecke algebra on $\mathrm{D}^b(G/B)$. More precisely, we lift the relations (H01) to (H05) in Definition 7.1 to the categorical level, which are (7.4) to (7.8), respectively in Theorem 7.3. The proof will be given in the next subsection.

Theorem 7.3. *There is a categorical action of the affine 0-Hecke algebra $\mathcal{H}_N(0)$ on $\mathrm{D}^b(G/B)$. More precisely, if we define the FM kernels $\mathcal{T}_i = \mathcal{O}_{G/B \times_{G/P_i} G/B}$ and $\mathcal{X}_j = \Delta_*(\mathcal{V}_j/\mathcal{V}_{j-1})$ for $1 \leq i \leq N-1$, $1 \leq j \leq N$, then we have the following categorical relations*

$$\mathcal{T}_i * \mathcal{T}_i \cong \mathcal{T}_i, \tag{7.4}$$

$$\mathcal{T}_i * \mathcal{T}_{i+1} * \mathcal{T}_i \cong \mathcal{T}_{i+1} * \mathcal{T}_i * \mathcal{T}_{i+1}, \ \mathcal{T}_i * \mathcal{T}_j \cong \mathcal{T}_j * \mathcal{T}_i \text{ if } |i-j| \geq 2, \tag{7.5}$$

$$\mathcal{T}_i * \mathcal{X}_j \cong \mathcal{X}_j * \mathcal{T}_i \text{ if } j \neq i, i+1, \tag{7.6}$$

We have the following exact triangles in $\mathrm{D}^b(G/B \times G/B)$

$$\mathcal{T}_i * \mathcal{X}_i \to \mathcal{X}_{i+1} * \mathcal{T}_i \to \mathcal{X}_{i+1}, \tag{7.7}$$

$$\mathcal{X}_i * \mathcal{T}_i \to \mathcal{T}_i * \mathcal{X}_{i+1} \to \mathcal{X}_{i+1}. \tag{7.8}$$

7.2. Application by the Shifted 0-Affine Algebra

In this section, instead of proving Theorem 7.3 by direct computation of convolution of kernels, we use the theory of categorical action of the shifted 0-affine algebra, which was developed in Section 3 and Section 5 to help us prove the theorem.

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To use the results of categorical action, i.e., Theorem 5.6 and Definition 3.1, the main idea is to interpret the Demazure operators in terms of elements in the shifted 0-affine algebra.

Recall that the Demazure operators are given by $T_i := \pi_i^*\pi_{i*} : K(G/B) \to K(G/B)$ where $\pi_i : G/B \to G/P_i$ is the natural $\mathbb{P}^1$-fibration with $1 \leq i \leq N-1$. With the notation of n-step partial flag variety (5.5), the full flag variety in (7.1) can be written as $G/B = \mathrm{Fl}_{(1,1,\ldots,1)}(\mathbb{C}^N)$. For the partial flag varieties in (7.2), we observe that

$$\begin{aligned} V_{i-1} \overset{2}{\subset} V_{i+1} &= V_{i-1} \overset{0}{\subset} V_{i-1} \overset{2}{\subset} V_{i+1} \\ &= V_{i-1} \overset{2}{\subset} V_{i+1} \overset{0}{\subset} V_{i+1} \end{aligned}$$

where the number above the inclusions is the jump of dimensions. Thus they can be written as $G/P_i = \mathrm{Fl}_{(1,1,..,1)+\alpha_i}(\mathbb{C}^N) = \mathrm{Fl}_{(1,1,..,1)-\alpha_i}(\mathbb{C}^N)$ where $1 \leq i \leq N-1$.

We have the following diagram

$$\begin{tikzcd} K(G/P_i = \mathrm{Fl}_{(1,1,\ldots,1)-\alpha_i}(\mathbb{C}^N)) \arrow[r, shift left, "e_{i,r}"] & K(G/B = \mathrm{Fl}_{(1,1,\ldots,1)}(\mathbb{C}^N)) \arrow[l, shift left, "f_{i,s}"] \arrow[r, shift left, "e_{i,r}"] \arrow[loop above, "T_i"] & K(G/P_i = \mathrm{Fl}_{(1,1,\ldots,1)+\alpha_i}(\mathbb{C}^N)) \arrow[l, shift left, "f_{i,s}"] \end{tikzcd}$$

where $e_{i,r}1_{(1,1,\ldots,1)}$, $e_{i,r}1_{(1,1,\ldots,1)-\alpha_i}$, $f_{i,s}1_{(1,1,\ldots,1)}$, and $f_{i,s}1_{(1,1,\ldots,1)+\alpha_i}$ are elements in $\dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_N)$. Thus we can try to interpret T_i in terms of elements in $\dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_N)$.

Since we define both the action of $\dot{\mathrm{U}}_{0,N}(L\mathfrak{sl}_N)$ and the action of the Demazure operators T_i on derived categories by using FM transforms/kernels, to relate the 1-morphisms in the categorical $\dot{\mathrm{U}}_{0,N}(L\mathfrak{sl}_N)$-action to $\mathbb{T}_i$, we have to compare their kernels.

By definition, we have $\mathcal{E}_{i,0}1_{(1,1,..,1)} = \iota_*\mathcal{O}_{W_i^1((1,1,\ldots,1))}$, where

$$W_i^1((1,1,\ldots,1)) := \{(V_\bullet, V'_\bullet) \in \mathrm{Fl}_{(1,1,..,1)}(\mathbb{C}^N) \times \mathrm{Fl}_{(1,1,..,1)+\alpha_i}(\mathbb{C}^N) \mid V_j = V'_j \text{ for } j \neq i, \text{ and } V'_i \subset V_i\}$$

and $\iota : W_i^1((1,1,\ldots,1)) \to \mathrm{Fl}_{(1,1,..,1)}(\mathbb{C}^N) \times \mathrm{Fl}_{(1,1,..,1)+\alpha_i}(\mathbb{C}^N) = G/B \times G/P_i$ is the natural inclusion. Note that by definition $V'_i = V_{i-1}$, so the condition $V'_i \subset V_i$ is automatically satisfied. Then $W_i^1((1,1,\ldots,1)) = (id \times \pi_i)(G/B)$. Thus we have $\mathcal{E}_{i,0}\mathbf{1}_{(1,1,..,1)} = \iota_*\mathcal{O}_{(id\times\pi_i)(G/B)}$, which is the kernel for the derived pushforward π_{i*}. Using the same argument, we can show that $\mathcal{F}_{i,0}\mathbf{1}_{(1,1,\ldots,1)}$ is also the same as the kernel for π_{i*} and $\mathcal{E}_{i,0}\mathbf{1}_{(1,1,..,1)-\alpha_i} \cong \mathcal{F}_{i,0}\mathbf{1}_{(1,1,..,1)+\alpha_i}$ is the same as the kernel for π_i^*.

Since the categorified Demazure operators are defined by $\mathbb{T}_i := \pi_i^*\pi_{i*}$, by the above argument, we obtain the following isomorphisms

$$\mathcal{T}_i \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \tag{7.9}$$

for all $1 \leq i \leq N-1$. The second isomorphism can be seen from Proposition 5.10. Next, we make a simple modification to the isomorphisms in (7.9) so that it becomes more “natural” (which will be explained later). From condition (8)(a) in Definition 3.1, we have

$$((\Psi_i^+)^{-1} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)}[1] \cong (\mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}. \tag{7.10}$$

Observe that by Definition 5.5, the kernel $\mathbf{\Psi}_i^+\mathbf{1}_{(1,1,\ldots,1)-\alpha_i}$ is given by $\mathcal{O}_\Delta[1]$ where Δ is the diagonal in $G/P_i \times G/P_i$. Thus the left hand side in (7.10) is isomorphic to $\mathcal{F}_{i,0}\mathbf{1}_{(1,1,\ldots,1)}$ and we obtain

$$\mathcal{F}_{i,0}\mathbf{1}_{(1,1,\ldots,1)} \cong (\mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}. \tag{7.11}$$

Similarly, using condition (7)(a) and the same argument, we obtain

$$\mathcal{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)} \cong (\mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)}. \tag{7.12}$$

With (7.11) and (7.12), (7.9) can be written as

$$\mathcal{T}_i \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)} \cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)}. \tag{7.13}$$

We prefer to use (7.13) over (7.9) for relating the categorical action of Demazure operators to the categorical $\dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_N)$-action. So after passing to the K-theory (decategorifying), we obtain

$$T_i = e_{i,0}f_{i,1}(\psi_i^+)^{-1}1_{(1,1,\ldots,1)} = f_{i,0}e_{i,-1}(\psi_i^-)^{-1}1_{(1,1,\ldots,1)}$$

for all $1 \le i \le N-1$, which interprets the Demazure operators in terms of elements in $\dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_N)$.

Finally, for the kernel $\mathcal{X}_i$, it is easy to see that $\mathcal{X}_1 \cong (\Psi_1^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$, $\mathcal{X}_i \cong \Psi_{i-1}^+\mathbf{1}_{(1,1,\ldots,1)} \cong (\Psi_i^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$ for all $2 \le i \le N-1$, and $\mathcal{X}_N \cong \Psi_{N-1}^+\mathbf{1}_{(1,1,\ldots,1)}$. This implies that $X_1 = (\psi_1^-)^{-1}1_{(1,1,\ldots,1)}$, $X_i = \psi_{i-1}^+\mathbf{1}_{(1,1,\ldots,1)} = (\psi_i^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$ for all $2 \le i \le N-1$, and $X_N = \psi_{N-1}^+\mathbf{1}_{(1,1,\ldots,1)}$.

Before we give the proof, we need the following result, which is similar to Serre relations. This can be deduced from the conditions in Definition 3.1, and the proof is left to the readers.

Lemma 7.4.

$$\begin{aligned}
&(\mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{E}_{i+1,0})\mathbf{1}_{\underline{k}} \cong (\mathcal{E}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i,0})\mathbf{1}_{\underline{k}},\\
&(\mathcal{E}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i,0})\mathbf{1}_{\underline{k}} \cong (\mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{\underline{k}},\\
&(\mathcal{F}_{i+1,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{\underline{k}} \cong (\mathcal{F}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{\underline{k}},\\
&(\mathcal{F}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{F}_{i,0})\mathbf{1}_{\underline{k}} \cong (\mathcal{F}_{i,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{\underline{k}}.
\end{aligned}$$

We are in a position to prove the Theorem 7.3.

Proof of Theorem 7.3. For the proof of most relations, we will use the isomorphism $\mathcal{T}_i \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}$ from (7.13), since the argument for the other is similar.

First, we prove the relation (7.4). Then, we obtain

$$\begin{aligned}
\mathcal{T}_i * \mathcal{T}_i &\cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1} * \mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}\\
&\cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * \mathcal{E}_{i,-1} * (\Psi_i^+)^{-1} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}[1] \text{ (by condition (7)(a))}\\
&\cong (\mathcal{E}_{i,0} * \Psi_i^+[-1] * (\Psi_i^+)^{-1} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}[1] \text{ (by condition (10)(a) with conjugation)}\\
&\cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)} \cong \mathcal{T}_i.
\end{aligned}$$

Next, for relation (7.6), we let $j \neq i, i+1$. Then, we use $\mathcal{X}_j \cong \Psi_{j-1}^+\mathbf{1}_{(1,1,\ldots,1)}$ if $i+2 \le j \le N$ and $\mathcal{X}_j \cong (\Psi_j^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$ if $1 \le j \le i-1$. The result follows from conditions (4), (7)(b)(c), and (8)(b)(c) in Definition 3.1.

For relation (7.7) and (7.8), by conditions (10)(a)(b), we have the following exact triangles

$$(\mathcal{F}_{i,1} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \to (\mathcal{E}_{i,0} * \mathcal{F}_{i,1})\mathbf{1}_{(1,1,\ldots.,1)} \to \Psi_i^+\mathbf{1}_{(1,1,\ldots,1)}, \tag{7.14}$$

$$(\mathcal{E}_{i,-1} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots.,1)} \to (\mathcal{F}_{i,0} * \mathcal{E}_{i,-1})\mathbf{1}_{(1,1,\ldots,1)} \to \Psi_i^-\mathbf{1}_{(1,1,\ldots,1)}. \tag{7.15}$$

Note that $(\mathcal{E}_{i,0} * \mathcal{F}_{i,1})\mathbf{1}_{(1,1,\ldots.,1)} \cong \mathcal{T}_i * \mathcal{X}_{i+1}$ since $\mathcal{T}_i \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}$ and $\mathcal{X}_{i+1} \cong \Psi_i^+\mathbf{1}_{(1,1,\ldots,1)}$. On the other hand, $\mathcal{T}_i \cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)}$ by (7.9) and $\mathcal{X}_i \cong (\Psi_i^-)^{-1}\mathbf{1}_{(1,1,\ldots,1)}$, thus

$$\mathcal{X}_i * \mathcal{T}_i \cong ((\Psi_i^-)^{-1} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \cong (\mathcal{F}_{i,1} * (\Psi_i^-)^{-1} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)}[1] \cong (\mathcal{F}_{i,1} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)}$$

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where we use condition (8)(a) in the second isomorphism and the third isomorphism comes from the fact that the kernel $\Psi_i^-\mathbf{1}_{(1,1,\ldots,1)+\alpha_i}$ is given by $\mathcal{O}_\Delta[1]$. With these facts, the exact triangle (7.14) is precisely the following

$$\mathcal{X}_i * \mathcal{T}_i \to \mathcal{T}_i * \mathcal{X}_{i+1} \to \mathcal{X}_{i+1}$$

which is the relation (7.8). A similar argument with the exact triangle (7.15) gives the relation (7.7).

Finally, we prove the braid relation in (7.5).

For the non-adjacent relation, it follows from conditions (4), (5)(c), (6)(c), (7)(c), (8)(c), and (9) in Definition 3.1.

Since $\mathcal{T}_i \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)}$ by (7.9), a direct calculation shows that

$$\begin{aligned}
\mathcal{T}_i * \mathcal{T}_{i+1} * \mathcal{T}_i &\cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \\
&\cong (\mathcal{E}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by condition (9))} \\
&\cong (\mathcal{E}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by condition (10)(c))} \\
&\cong (\mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by Lemma 7.4)} \\
&\cong (\mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)}. \text{ (by condition (10)(c))}
\end{aligned} \tag{7.16}$$

Next, we show that $(\mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}$. First, we observe that $(\mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} \cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}$. Then by condition (7)(a), we get

$$\begin{aligned}
(\mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} &\cong (\mathcal{F}_{i,0} * \Psi_i^- * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}[-1] \\
&\cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}
\end{aligned}$$

where the second isomorphism comes from the fact that $\Psi_i^-\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}+\alpha_i}$ is given by $\mathcal{O}_\Delta[1]$. Thus

$$\begin{aligned}
&(\mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} \\
&\cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1} * \mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} \\
&\cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * \mathcal{F}_{i,1} * (\Psi_i^-)^{-1} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}[1] \text{ (by condition (8)(a))} \\
&\cong (\mathcal{F}_{i,0} * \Psi_i^- * (\Psi_i^-)^{-1} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}[1][-1] \text{ (by condition (10)(b) with conjugation)} \\
&\cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} \cong (\mathcal{F}_{i,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}} \cong (\mathcal{E}_{i,0} * \mathcal{F}_{i,0})\mathbf{1}_{(1,1,\ldots,1)-\alpha_{i+1}}.
\end{aligned}$$

So (7.16) becomes

$$\begin{aligned}
(\mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)} &\cong (\mathcal{E}_{i+1,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)} \\
&\cong (\mathcal{F}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)}.
\end{aligned} \tag{7.17}$$

Using the same argument, we can show that $(\mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)+\alpha_i} \cong (\mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)+\alpha_i}$. Hence (7.17) becomes

$$\begin{aligned}
&(\mathcal{F}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \\
&\cong (\mathcal{F}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by condition (10)(c))} \\
&\cong (\mathcal{F}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by condition (10)(c))} \\
&\cong (\mathcal{F}_{i+1,0} * \mathcal{F}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{E}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by Lemma 7.4)} \\
&\cong (\mathcal{F}_{i+1,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i,0} * \mathcal{E}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by condition (10)(c))} \\
&\cong (\mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0} * \mathcal{F}_{i,0} * \mathcal{E}_{i,0} * \mathcal{F}_{i+1,0} * \mathcal{E}_{i+1,0})\mathbf{1}_{(1,1,\ldots,1)} \text{ (by condition (9))} \\
&\cong \mathcal{T}_{i+1} * \mathcal{T}_i * \mathcal{T}_{i+1} \text{ (by (7.9))}. \qquad \square
\end{aligned}$$

We give a few remarks to conclude this section.

Remark 7.5. We explain why we prefer (7.13) over (7.9) for writing the categorified Demazure operators in terms of 1-morphisms in the categorical $\mathcal{U}$-action. The main reason comes from the naturality of adjoint functors, i.e., condition (3).

By condition (3), we obtain that

$$\begin{aligned}\mathrm{Hom}(\mathcal{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)-\alpha_i}, \mathcal{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)-\alpha_i}) &\cong \mathrm{Hom}(\mathbf{1}_{(1,1,\ldots,1)}\mathcal{E}_{i,0} * (\mathcal{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)-\alpha_i})_R, \mathbf{1}_{(1,1,\ldots,1)}) \\ &\cong \mathrm{Hom}((\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}, \mathbf{1}_{(1,1,\ldots,1)}).\end{aligned}$$

Thus, $(\mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\Psi_i^+)^{-1})\mathbf{1}_{(1,1,\ldots,1)}$ naturally appears as the kernel for the composition of adjoint functors $\mathsf{E}_{i,0} \circ (\mathsf{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)-\alpha_i})^R\mathbf{1}_{(1,1,\ldots,1)}$. Similarly, we also have $(\mathcal{F}_{i,0} * \mathcal{E}_{i,-1} * (\Psi_i^-)^{-1})\mathbf{1}_{(1,1,\ldots,1)}$ naturally appearing as the kernel for the composition of adjoint functors $\mathsf{F}_{i,0} \circ (\mathsf{F}_{i,0}\mathbf{1}_{(1,1,\ldots,1)+\alpha_i})^R\mathbf{1}_{(1,1,\ldots,1)}$.

Moreover, we expect that the morphisms in the exact triangles (7.14) and (7.15) should be determined to come from the identity morphisms $Id : \mathcal{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)-\alpha_i} \to \mathcal{E}_{i,0}\mathbf{1}_{(1,1,\ldots,1)-\alpha_i}$ and $Id : \mathcal{F}_{i,0}\mathbf{1}_{(1,1,\ldots,1)+\alpha_i} \to \mathcal{F}_{i,0}\mathbf{1}_{(1,1,\ldots,1)+\alpha_i}$, respectively.

Remark 7.6. In [26], we generalize the interpretation of categorified Demazure operators in terms of the 1-morphisms in the categorical $\mathcal{U}$-action, i.e. (7.13), from the full flag variety G/B to the n-step partial flag varieties $\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)$. More precisely, we construct two families of pairs of complementary idempotents in $\mathrm{D}^b(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N) \times \mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ acting on $\mathrm{D}^b(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$. Moreover, those complementary idempotents provide short semiorthogonal decompositions of $\mathrm{D}^b(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$, and we determine the generators of the component categories in the Grassmannians case. Finally, we also give another proof of the braid relation (7.5) in Theorem 7.3 which is shorter than the one in this article.

Remark 7.7. We try to relate categorical actions of affine 0-Hecke algebras to the notion of Demazure descent data on a triangulated category defined in [2], [3].

From (7.5) in Theorem 7.3, we know that those FM kernels $\{\mathcal{T}_i\}_{1\leq i\leq N-1}$ give a weak braid monoid action on $\mathrm{D}^b(G/B)$. Moreover, (7.4) in Theorem 7.3 implies that there is a co-projector structure on $\mathcal{T}_i$ for all i. Thus $\{\mathcal{T}_i\}_{1\leq i\leq N-1}$ gives a Demazure descent data on $\mathrm{D}^b(G/B)$.

A. A Conjectural Presentation

In this appendix, we define another algebra which can be viewed as a second definition of the shifted 0-affine algebra. It has the so-called loop presentation that given by using generating series. Thus, in this definition, we have infinite generators and relations.

Then, we construct an action of this algebra on the Grothendieck groups of n-step partial flag varieties $\bigoplus_{\underline{k}} K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$.

Finally, we conjecture that the loop presentation we defined for the shifted 0-affine algebra is equivalent to the presentation defined in Definition 2.6.

Definition A.1. Defining the associative $\mathbb{C}$-algebra $\dot{\mathbf{U}}'_{0,N}(L\mathfrak{sl}_n)$ that is generated by

$$\bigcup_{\underline{k}\in C(n,N)} \{1_{\underline{k}},\ 1_{\underline{k}+\alpha_i}e_{i,r}1_{\underline{k}},\ 1_{\underline{k}-\alpha_i}f_{i,r}1_{\underline{k}},\ 1_{\underline{k}}\psi^{\pm}_{i,\pm s_i^{\pm}}1_{\underline{k}},\ 1_{\underline{k}}(\psi^+_{i,k_{i+1}})^{-1}1_{\underline{k}},\ 1_{\underline{k}}(\psi^-_{i,-k_i})^{-1}1_{\underline{k}}\}^{r\in\mathbb{Z},\ s_i^+\geq k_{i+1},\ s_i^-\geq k_i}_{1\leq i\leq n-1}$$

with the following relations (for all $1 \leq i, j \leq n-1$, $\underline{k} \in C(n,N)$, $\epsilon, \epsilon' \in \{\pm\}$ and $s_i^+ \geq k_{i+1}$, $s_i^- \geq k_i$).

$$1_{\underline{k}}1_{\underline{l}} = \delta_{\underline{k},\underline{l}}1_{\underline{k}}, \tag{A.1}$$

$$[\psi^{\epsilon}_{i,\underline{k}}(z), \psi^{\epsilon'}_{j,\underline{k}}(w)]1_{\underline{k}} = 0,\ (\psi^+_{i,k_{i+1}})^{\pm1}\cdot(\psi^+_{i,k_{i+1}})^{\mp1}1_{\underline{k}} = 1_{\underline{k}} = (\psi^-_{i,-k_i})^{\pm1}\cdot(\psi^-_{i,-k_i})^{\mp1}1_{\underline{k}}. \tag{A.2}$$

$$\begin{aligned} &ze_{i,\underline{k}+\alpha_i}(z)e_{i,\underline{k}}(w)1_{\underline{k}} = -we_{i,\underline{k}+\alpha_i}(w)e_{i,\underline{k}}(z)1_{\underline{k}}, \\ &we_{i,\underline{k}+\alpha_{i+1}}(z)e_{i+1,\underline{k}}(w)1_{\underline{k}} = (w-z)e_{i+1,\underline{k}+\alpha_i}(w)e_{i,\underline{k}}(z)1_{\underline{k}}, \\ &(z-w)e_{i,\underline{k}+\alpha_j}(z)e_{j,\underline{k}}(w)1_{\underline{k}} = (z-w)e_{j,\underline{k}+\alpha_i}(w)e_{i,\underline{k}}(z)1_{\underline{k}},\ \text{if } |i-j|\geq 2. \end{aligned} \tag{A.3}$$

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$$\begin{aligned}
&-wf_{i,\underline{k}-\alpha_i}(z)f_{i,\underline{k}}(w)1_{\underline{k}} = zf_{i,\underline{k}-\alpha_i}(w)f_{i,\underline{k}}(z)1_{\underline{k}},\\
&(w-z)f_{i,\underline{k}-\alpha_{i+1}}(z)f_{i+1,\underline{k}}(w)1_{\underline{k}} = wf_{i+1,\underline{k}-\alpha_i}(w)f_{i,\underline{k}}(z)1_{\underline{k}},\\
&(z-w)f_{i,\underline{k}-\alpha_j}(z)f_{j,\underline{k}}(w)1_{\underline{k}} = (z-w)f_{j,\underline{k}-\alpha_i}(w)f_{i,\underline{k}}(z)1_{\underline{k}},\ \text{if } |i-j|\geq 2.
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
&z\psi^+_{i,\underline{k}+\alpha_i}(z)e_{i,\underline{k}}(w)1_{\underline{k}} = -we_{i,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\\
&\frac{-w}{z}(\sum_{s\geq 0}(\frac{w}{z})^s)\psi^+_{i,\underline{k}+\alpha_{i+1}}(z)e_{i+1,\underline{k}}(w)1_{\underline{k}} = e_{i+1,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^+_{i,\underline{k}+\alpha_{i-1}}(z)e_{i-1,\underline{k}}(w)1_{\underline{k}} = (\sum_{s\geq 0}(\frac{w}{z})^s)e_{i-1,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^+_{i,\underline{k}+\alpha_j}(z)e_{j,\underline{k}}(w)1_{\underline{k}} = e_{j,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\ \text{if } |i-j|\geq 2.
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&z\psi^-_{i,\underline{k}+\alpha_i}(z)e_{i,\underline{k}}(w)1_{\underline{k}} = -we_{i,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\\
&(\sum_{s\geq 0}(\frac{z}{w})^s)\psi^-_{i,\underline{k}+\alpha_{i+1}}(z)e_{i+1,\underline{k}}(w)1_{\underline{k}} = e_{i+1,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^-_{i,\underline{k}+\alpha_{i-1}}(z)e_{i-1,\underline{k}}(w)1_{\underline{k}} = \frac{-z}{w}(\sum_{s\geq 0}(\frac{z}{w})^s)e_{i-1,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^-_{i,\underline{k}+\alpha_j}(z)e_{j,\underline{k}}(w)1_{\underline{k}} = e_{j,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\ \text{if } |i-j|\geq 2.
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
&-w\psi^+_{i,\underline{k}-\alpha_i}(z)f_{i,\underline{k}}(w)1_{\underline{k}} = zf_{i,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^+_{i,\underline{k}-\alpha_{i+1}}(z)f_{i+1,\underline{k}}(w)1_{\underline{k}} = \frac{-w}{z}(\sum_{s\geq 0}(\frac{w}{z})^s)f_{i+1,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\\
&(\sum_{s\geq 0}(\frac{w}{z})^s)\psi^+_{i,\underline{k}-\alpha_{i-1}}(z)f_{i-1,\underline{k}}(w)1_{\underline{k}} = f_{i-1,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^+_{i,\underline{k}-\alpha_j}(z)f_{j,\underline{k}}(w)1_{\underline{k}} = f_{j,\underline{k}}(w)\psi^+_{i,\underline{k}}(z)1_{\underline{k}},\ \text{if } |i-j|\geq 2.
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
&-w\psi^-_{i,\underline{k}-\alpha_i}(z)f_{i,\underline{k}}(w)1_{\underline{k}} = zf_{i,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^-_{i,\underline{k}-\alpha_{i+1}}(z)f_{i+1,\underline{k}}(w)1_{\underline{k}} = (\sum_{s\geq 0}(\frac{z}{w})^s)f_{i+1,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\\
&\frac{-z}{w}(\sum_{s\geq 0}(\frac{z}{w})^s)\psi^-_{i,\underline{k}-\alpha_{i-1}}(z)f_{i-1,\underline{k}}(w)1_{\underline{k}} = f_{i-1,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\\
&\psi^-_{i,\underline{k}-\alpha_j}(z)f_{j,\underline{k}}(w)1_{\underline{k}} = f_{j,\underline{k}}(w)\psi^-_{i,\underline{k}}(z)1_{\underline{k}},\ \text{if } |i-j|\geq 2.
\end{aligned} \tag{A.8}$$

$$e_{i,\underline{k}-\alpha_i}(z)f_{i,\underline{k}}(w)1_{\underline{k}} - f_{i,\underline{k}+\alpha_i}(w)e_{i,\underline{k}}(z)1_{\underline{k}} = \delta_{ij}\delta(\frac{z}{w})(\psi^+_{i,\underline{k}}(z)-\psi^-_{i,\underline{k}}(z))1_{\underline{k}}. \tag{A.9}$$

where the generating series are defined as follows

$$\begin{aligned}
&e_{i,\underline{k}}(z) := \sum_{r\in\mathbb{Z}} e_{i,r}1_{\underline{k}}z^{-r},\ f_{i,\underline{k}}(z) := \sum_{r\in\mathbb{Z}} f_{i,r}1_{\underline{k}}z^{-r},\\
&\psi^+_{i,\underline{k}}(z) := \sum_{r\geq k_{i+1}} \psi^+_{i,r}1_{\underline{k}}z^{-r},\ \psi^-_{i,\underline{k}}(z) := \sum_{r\geq k_i}\psi^-_{i,-r}1_{\underline{k}}z^r,\ \delta(z) := \sum_{r\in\mathbb{Z}} z^r.
\end{aligned}$$

Although the notion of categorical action of $\dot{\mathbf{U}}'_{0,N}(L\mathfrak{sl}_n)$ is not easy to define, we can construct an action on the Grothendieck groups of n-step partial flag varieties $\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)$.

To construct such action, we have to define the action of those generators in $\dot{\mathbf{U}}'_{0,N}(L\mathfrak{sl}_n)$. First, we define $e_{i,r}1_{\underline{k}}$ and $f_{i,s}1_{\underline{k}}$ for all $r,s\in\mathbb{Z}$ via using the K-theoretic FM transforms, i.e.,

$$e_{i,r}1_{\underline{k}} : K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}+\alpha_i}(\mathbb{C}^N)),\ x\mapsto \pi_{2*}(\pi_1^*(x)\otimes[\iota(\underline{k})_*(\mathcal{V}_i/\mathcal{V}'_i)^{\otimes r}])$$

for all $r \in \mathbb{Z}$, and similarly for $f_{i,s}1_{\underline{k}}$ in the opposite direction with $s \in \mathbb{Z}$. Next, for $\psi^{\pm}_{i,\pm s_i^{\pm}}1_{\underline{k}}$ we define their action on $K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ as the follows

$$\begin{aligned}\psi^+_{i,s_i^+}1_{\underline{k}} :& K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)),\\ & x \mapsto x \otimes (-1)^{k_{i+1}-1}[\det(\mathcal{V}_{i+1}/\mathcal{V}_i)][\mathrm{Sym}^{s_i^+-k_{i+1}}(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})]\end{aligned} \tag{A.10}$$

where $s_i^+ \geq k_{i+1}$ and similarly

$$\begin{aligned}\psi^-_{i,-s_i^-}1_{\underline{k}} :& K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)),\\ & x \mapsto x \otimes (-1)^{k_i}[\det(\mathcal{V}_i/\mathcal{V}_{i-1})^{-1}][\mathrm{Sym}^{s_i^--k_i}(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})^{\vee}]\end{aligned} \tag{A.11}$$

where $s_i^- \geq k_i$.

Then note that all the results (except Proposition 5.10) we prove above to prove Theorem 5.6 do not assume any conditions for $r,\ s$ for $e_{i,r}1_{\underline{k}}$ and $f_{i,s}1_{\underline{k}}$. Thus, passing the works in the main text to the Grothendieck group and using Corollary 6.10, plus a little extra check, it is easy to verify the following theorem.

Theorem A.2. *There is an action of* $\dot{\mathbf{U}}'_{0,N}(L\mathfrak{sl}_n)$ *on* $\bigoplus_{\underline{k}} K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$.

Next, we expect that the two presentations in Definition 2.6 and Definition A.1 are equivalent like the two presentations of shifted quantum affine algebras in [22] (Theorem 5.5 in loc. cit.).

To relate this definition to Definition 2.6, we introduce another set of Cartan generators $\{h_{i,\pm r}\}^{r\geq 1}_{1\leq i\leq N-1}$ with the relations to $\{\psi^{\pm}_{i,\pm s_i^{\pm}}1_{\underline{k}}\}$ via the following

$$\begin{aligned}(\psi^+_{i,k_{i+1}}z^{-k_{i+1}})^{-1}\psi^+_{i,\underline{k}}(z) &= (1+h_{i,+}(z))1_{\underline{k}},\\ (\psi^-_{i,-k_i}z^{k_i})^{-1}\psi^-_{i,\underline{k}}(z) &= (1+h_{i,-}(z))1_{\underline{k}},\end{aligned}$$

where $h_{i,\pm}(z) = \sum_{r>0} h_{i,\pm r}z^{\mp r}$.

Remark A.3. From (A.10) and (A.11), it is easy to see that the action of $h_{i,\pm r}$ on the Grothendieck group $K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N))$ is given by

$$\begin{aligned}h_{i,r}1_{\underline{k}} &: K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)),\ x \mapsto x \otimes [\mathrm{Sym}^r(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})]\\ h_{i,-r}1_{\underline{k}} &: K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)) \to K(\mathrm{Fl}_{\underline{k}}(\mathbb{C}^N)),\ x \mapsto x \otimes [\mathrm{Sym}^r(\mathcal{V}_{i+1}/\mathcal{V}_{i-1})^{\vee}]\end{aligned}$$

where $r \geq 1$.

Then, we define inductively

$$\begin{aligned}e_{i,r}1_{\underline{k}} &:= \begin{cases} -\psi_i^+ e_{i,r-1}(\psi_i^+)^{-1}1_{\underline{k}} & \text{if } r>0\\ -(\psi_i^+)^{-1}e_{i,r+1}\psi_i^+1_{\underline{k}} & \text{if } r<-k_i,\end{cases}\\ f_{i,r}1_{\underline{k}} &:= \begin{cases} -(\psi_i^+)^{-1}f_{i,r-1}\psi_i^+1_{\underline{k}} & \text{if } r>k_{i+1}\\ -\psi_i^+ f_{i,r+1}(\psi_i^+)^{-1}1_{\underline{k}} & \text{if } r<0,\end{cases}\\ \psi^+_{i,r}1_{\underline{k}} &:= [e_{i,r-k_{i+1}}, f_{i,k_{i+1}}]1_{\underline{k}} \text{ for } r \geq k_{i+1}+1,\\ \psi^-_{i,r}1_{\underline{k}} &:= -[e_{i,r}, f_{i,0}]1_{\underline{k}} \text{ for } r \leq -k_i-1.\end{aligned}$$

We propose the following conjecture, which, roughly speaking, says that the two presentations defined by Definition A.1 and Definition 2.6 are equivalent.

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Conjecture A.4. *There is a* $\mathbb{C}$*-algebra isomorphism* $\dot{\mathbf{U}}_{0,N}(L\mathfrak{sl}_n) \to \dot{\mathbf{U}}'_{0,N}(L\mathfrak{sl}_n)$ *such that*

$$e_{i,r}1_{\underline{k}} \mapsto e_{i,r}1_{\underline{k}},\ f_{i,r}1_{\underline{k}} \mapsto f_{i,r}1_{\underline{k}},\ \psi_i^+1_{\underline{k}} \mapsto \psi_{i,k_{i+1}}^+1_{\underline{k}},\ \psi_i^-1_{\underline{k}} \mapsto \psi_{i,-k_i}^-1_{\underline{k}},$$

for $1 \le i \le n-1$.

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