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CONTINUITY ON GENERALISED TOPOLOGICAL SPACES VIA HEREDITARY CLASSES

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Abstract

A generalised topology is a collection of subsets of a given nonempty set containing the empty set and arbitrary unions of the elements in the collection. By using the concept of hereditary classes, a generalised topology can be extended to a new one, called a generalised topology via a hereditary class. We study continuity on generalised topological spaces via hereditary classes in various situations.

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1. Introduction

Topology has played an important role in many other branches of mathematics. Point-set topology deals with the class of open sets or of nearly open sets and their properties. In 1997, Császár [1] defined generalised open sets called γ-open sets covering every class of nearly open sets. He showed that arbitrary unions of γ-open sets are γ-open. In 2002, he used properties of γ-open sets to define a generalised topology [2]. The definition of a generalised topology ignores two of the requirements for a topology: that the whole set belongs to the topology and that a finite intersection of open sets is open. In 2007, Császár [4] gave an extension of generalised topological spaces by using hereditary classes (a collection $\mathcal{H}$ of subsets of the space such that every subset of elements in $\mathcal{H}$ belongs to $\mathcal{H}$). Hereditary classes were first introduced in 1990 by Hamlett and Janković [5] to generalise the set of ω-accumulation points on a set A (a point such that each of its neighbourhoods contains infinitely many elements of A) and the set of condensation points on a set A (a point such that each of its neighbourhoods contains uncountably many elements of A). In this paper, we study continuity on generalised topological spaces via hereditary classes.

The organisation of the paper is as follows. In Section 2 we explain generalised topological spaces and hereditary classes, based on [1–4, 6]. The main results of the paper are obtained in Section 3. The first theorem is a generalisation of the pasting lemma where we try to reduce some conditions by using hereditary classes.

Further theorems show that continuity between two generalised topological spaces can be preserved on generalised topological spaces via hereditary classes in various situations. By applying these theorems, we obtain results concerning open maps and hereditary classes on subspaces of generalised topologies via hereditary classes. Finally, we prove that in some conditions one can construct a hereditary class that makes a given function continuous on the generalised topological space via this hereditary class.

2. Preliminaries

Császár [1] introduced γ-open sets generalising open and nearly open sets.

DEFINITION 2.1. Let X be a nonempty set, and denote its power set by $\mathcal{P}(X)$. The function $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is called *monotonic* if $A \subset B$ implies $\gamma A \subset \gamma B$ for all $A, B \in \mathcal{P}(X)$. The set of all monotonic functions is denoted by $\Gamma(X)$.

DEFINITION 2.2. Let X be a set and γ a monotonic function. A set $A \subset X$ is called *γ-open* if $A \subset \gamma A$.

Császár [2] observed that the collection of γ-open sets has some properties similar to those of the classical open sets. That is, for each monotonic function γ, the empty set is γ-open and an arbitrary union of γ-open sets is γ-open. This motivates his definition of a generalised topological space.

DEFINITION 2.3. Let X be a nonempty set. A collection μ of subsets of X is a *generalised topology* on X if it satisfies the following conditions.

(1) The empty set is in μ.
(2) An arbitrary union of elements in μ is in μ.

The pair (X, μ) is called a *generalised topological space* and an element in μ is called a *μ-open* set. A set $A \subset X$ is called *μ-closed* if $X - A$ is μ-open. We observe that any topology is also a generalised topology.

DEFINITION 2.4. Let (X, μ) be a generalised topological space. For $A \subset X$, the *μ-interior* of A, denoted by $i_\mu A$, is the union of all μ-open subsets of A, and the *μ-closure* of A, denoted by $c_\mu A$, is the intersection of all μ-closed supersets of A. If there is no ambiguity, then $i_\mu A$ and $c_\mu A$ will be denoted by iA and cA, respectively.

DEFINITION 2.5. Let (X, μ) be a generalised topological space. Then μ is said to be a *quasi-topology* on X if it satisfies the property

$$A, B \in \mu \text{ implies } A \cap B \in \mu$$

and the pair (X, μ) is called a *quasi-topological space*. The *relative generalised topology* μ_A on A is defined by

$$\mu_A = \{M \cap A \mid M \in \mu\}$$

and the pair (A, μ_A) is called a *subspace* of (X, μ).

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Let (X, τ) be a topological space and $A \subset X$. The *closure* $\overline{A}$ of A in X is the set of all points in X such that every neighbourhood of x meets A, that is,

$$x \in \overline{A} \Longleftrightarrow A \cap O \neq \emptyset \quad \text{for all } O \in \tau \text{ containing } x.$$

The condition $A \cap O \neq \emptyset$ can be rewritten as $A \cap O \notin \{\emptyset\}$. Following Hamlett and Janković [5], we can generalise the concept of the closure by replacing $\{\emptyset\}$ by a collection of subsets of X to obtain a new topology. Note that not every collection of subsets of X can be used. This leads to a concept of a hereditary class and an ideal.

Definition 2.6. Let X be a nonempty set. A collection $\mathcal{H}_X$ of subsets of X is said to be a *hereditary class* on X if for each $A, B \in \mathcal{P}(X)$,

$$A \subset B \text{ and } B \in \mathcal{H}_X \text{ imply } A \in \mathcal{H}_X.$$

If we add the property that

$$A, B \in \mathcal{H}_X \text{ implies } A \cup B \in \mathcal{H}_X \text{ for each } A, B \in \mathcal{P}(X),$$

then a hereditary class on X is said to be an *ideal*, usually denoted by $\mathcal{I}_X$.

Császár [4] used this concept to construct generalised topologies via hereditary classes. Throughout this paper, $(X, \mu, \mathcal{H})$ denotes a generalised topological space (X, μ) together with a hereditary class $\mathcal{H}$.

Definition 2.7. Let $(X, \mu, \mathcal{H})$ be a generalised topological space. For each $A \subset X$,

$$A^*_{\mathcal{H}} = \{x \in X \mid M \cap A \notin \mathcal{H} \text{ when } M \in \mu \text{ containing } x\}.$$

In particular, $A^*_{\{\emptyset\}} = cA$. If there is no ambiguity, then $A^*_{\mathcal{H}}$ will be denoted by A^*.

Example 2.8. Observe that $A^*_{\mathcal{H}}$ depends on the hereditary class on X. For example, let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then μ is a generalised topology on X. Let $A = \{a, c\}$ and consider the hereditary classes $\mathcal{H}_1 = \{\emptyset, \{a\}\}$ and $\mathcal{H}_2 = \{\emptyset, \{b\}\}$ on X. Then $A^*_{\mathcal{H}_1} = \{b, c\}$ and $A^*_{\mathcal{H}_2} = \{a, b, c\}$. Note that $A \cap \{a\} \in \mathcal{H}_1$ implies $a \notin A^*_{\mathcal{H}_1}$, and $a \in A^*_{\mathcal{H}_2}$ because $A \cap \{a\} \notin \mathcal{H}_2$ and $A \cap \{a, b, c\} \notin \mathcal{H}_2$.

Next, we describe some properties of A^*. In a topological space, $\overline{A}$ also satisfies these properties.

Proposition 2.9 [4]. *Let $(X, \mu, \mathcal{H})$ be a generalised topological space and $A, B \subset X$.*

(1) *$A \subset B$ implies $A^* \subset B^*$.*
(2) *$A^* \subset c_\mu A$.*
(3) *If $M \in \mu$ and $M \cap A \in \mathcal{H}$, then $M \cap A^* = \emptyset$.*
(4) *A^* is μ-closed.*
(5) *A is μ-closed implies $A^* \subset A$.*
(6) *$(A^*)^* \subset A^*$ when $A \subset X$.*
(7) *$X = X^*$ if and only if $\mu \cap \mathcal{H} = \{\emptyset\}$.*

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Proposition 2.9(7) leads to the following definition.

Definition 2.10. Let $(X, \mu, \mathcal{H})$ be a generalised topological space. The hereditary class $\mathcal{H}$ is said to be *μ-codense* if $\mathcal{H} \cap \mu = \{\emptyset\}$.

In [6], Renukadevi and Vimladevi proved the following theorem and also gave a counterexample showing that the theorem is not true if $\mathcal{H}$ is a hereditary class but not an ideal.

Theorem 2.11 [6]. *Let $(X, \mu, \mathcal{I})$ be a quasi-topological space together with an ideal $\mathcal{I}$. For each $A, B \subset X$, $A^* \cup B^* = (A \cup B)^*$.*

In some situations, as in Example 2.8, A^* does not contain A, so A^* cannot be regarded as the closure of A. To generalise the concept of the closure of A, we need the following definition.

Definition 2.12. Let $(X, \mu, \mathcal{H})$ be a generalised topological space. For each $A \subset X$,

$$c^*_{\mu,\mathcal{H}}A = A \cup A^*.$$

If there is no ambiguity, then $c^*_{\mu,\mathcal{H}}A$ can be denoted by c^*A. Császár [4] proved that there is a generalised topological space μ^* such that c^*A is the intersection of all μ^*-closed supersets of A , that is, $M \in \mu^*$ if and only $c^*(X - M) = X - M$. This leads to a new generalisation of a topological space.

Definition 2.13. Let $(X, \mu, \mathcal{H})$ be a generalised topological space. Define a *generalised topology on X via a hereditary class $\mathcal{H}$* by

$$\mu^*_{\mathcal{H}} = \{M \subset X \mid c^*(X - M) = (X - M)\}.$$

An element in $\mu^*_{\mathcal{H}}$ is said to be $\mu^*_{\mathcal{H}}$-open. Again, if there is no ambiguity, then $\mu^*_{\mathcal{H}}$ will be denoted by μ^*.

Example 2.14. In the setting of Example 2.8, we have $\mu^*_{\mathcal{H}_1} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mu^*_{\mathcal{H}_2} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Remark 2.15. If $\mathcal{H} = \{\emptyset\}$, then $\mu^* = \mu$.

The following are some properties of the generalised topology μ^*.

Proposition 2.16 [4]. *Let $(X, \mu, \mathcal{H})$ be a generalised topological space. Then:*

(1) *F is μ^*-closed if and only if $F^* \subset F$;*
(2) $\mu \subset \mu^*$.

From Example 2.8, if a generalised topology contains a large number of elements, then it is complicated to calculate the generalised topology via the hereditary class. However, there is an easier way to determine a generalised topology via a hereditary class using the concept of a base for a generalised topology.

Definition 2.17. Let (X, μ) be a generalised topological space. The collection $\mathcal{B}$ is a *base* for μ if and only if every $M \in \mu$ is a union of elements of $\mathcal{B}$.

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THEOREM 2.18 [4]. *Let $(X, \mu, \mathcal{H})$ be a generalised topological space. The set*

$$\{M - H \mid M \in \mu \text{ and } H \in \mathcal{H}\}$$

constitutes a base for μ^*.

3. Main results

In this section we will prove the main results of this paper. The definition of a continuous function between generalised topological spaces was introduced by Császár [2].

DEFINITION 3.1. Let (X, μ) and (Y, ν) be generalised topological spaces. A function f from X to Y is *(μ, ν)-continuous* if $f^{-1}(N) \in \mu$, for each $N \in \nu$.

THEOREM 3.2 (Pasting lemma on quasi-topological spaces). *Let (X, μ) be a quasi-topological space and (Y, ν) a generalised topological space. Let $X = A \cup B$ where A and B are both μ-closed or μ-open. Let $f : X \to Y$ be a function such that $f|_A$ is (μ_A, ν)-continuous and $f|_B$ is (μ_B, ν)-continuous. Then f is (μ, ν)-continuous.*

PROOF. First, consider the case where A and B are both μ-open. Let $N \in \nu$. Then $f|_A^{-1}(N) \in \mu_A$ and $f|_B^{-1}(N) \in \mu_B$. That is, $f^{-1}(N) \cap A \in \mu_A$ and $f^{-1}(N) \cap B \in \mu_B$. Since A and B are μ-open and μ is a quasi-topology on X, $f^{-1}(N) \cap A$ and $f^{-1}(N) \cap B$ are in μ. So $f^{-1}(N) = f^{-1}(N) \cap X = f^{-1}(N) \cap (A \cup B) = (f^{-1}(N) \cap A) \cup (f^{-1}(N) \cap B) \in \mu$. Hence $f : X \to Y$ is (μ, ν)-continuous. If A and B are both μ-closed, we use a similar argument and the fact that f is continuous if the preimage of a closed set is closed. □

The following example shows that the above theorem is not true if (X, μ) is just a generalised topological space and not a quasi-topological space.

EXAMPLE 3.3. Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$ and define generalised topological spaces (X, μ) and (Y, ν) by $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\nu = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Note that μ is not a quasi-topology on X. Let $A = \{a, c\}$ and $B = \{b, c\}$. Then A and B are both μ-open. Define $f : X \to Y$ by

$$f(a) = 1, \quad f(b) = 2, \quad f(c) = 1.$$

Obviously, $f|_A$ is (μ_A, ν)-continuous. Since $\{b\}$ and $\{c\}$ are μ_B-open, $f|_B$ is (μ_B, ν)-continuous. However, f is not (μ, ν)-continuous since $\{b\}$ is not μ-open.

THEOREM 3.4 (Pasting lemma on quasi-topological spaces via codense ideals). *Let $(X, \mu, \mathcal{I})$ be a quasi-topological space with a μ-codense ideal $\mathcal{I}$ and (Y, ν) a generalised topological space. Let $X = A \cup B$ and let $f : X \to Y$ be a function such that $f|_{A^*}$ is (μ_{A^*}, ν)-continuous and $f|_{B^*}$ is (μ_{B^*}, ν)-continuous. Then f is (μ, ν)-continuous.*

PROOF. Let $N \in \nu$, so $Y - N$ is ν-closed. That is, $f^{-1}(Y - N) \cap A^* = f|_{A^*}^{-1}(Y - N)$ is μ_{A^*}-closed and $f^{-1}(Y - N) \cap B^* = f|_{B^*}^{-1}(Y - N)$ is μ_{B^*}-closed. Since A^* and B^* are μ-closed, $f^{-1}(Y - N) \cap A^*$ and $f^{-1}(Y - N) \cap B^*$ are μ-closed. Consider

$$f^{-1}(Y - N) \cap (A^* \cup B^*) = (f^{-1}(Y - N) \cap A^*) \cup (f^{-1}(Y - N) \cap B^*)$$

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which is μ-closed. Thus, $f^{-1}(Y - N) \cap (A^* \cup B^*) = f^{-1}(Y - N) \cap (A \cup B)^*$ by Theorem 2.11. Further, $f^{-1}(Y - N) \cap (A \cup B)^* = f^{-1}(Y - N) \cap (X)^* = f^{-1}(Y - N) \cap X$ by Proposition 2.9(7). So $f^{-1}(Y - N)$ is μ-closed and $f^{-1}(N) \in \mu$. Hence f is (μ, ν)-continuous. □

From Proposition 2.16, $\mu \subset \mu^*$. So we easily obtain the following theorem.

THEOREM 3.5. *Let (X, μ) and (Y, ν) be generalised topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that f is a (μ, ν)-continuous function from X to Y. Then f is (μ^*, ν)-continuous.*

In the theorem above, if we replace the generalised topological space (Y, ν) by the generalised topological space $(Y, \nu^*_{\mathcal{H}_Y})$ via some hereditary class $\mathcal{H}_Y$, it is natural to ask whether f is (μ^*, ν^*)-continuous or not. Consider the following example.

EXAMPLE 3.6. Take $X = [0, 1]$, $Y = [1, 2]$, μ the usual subspace topology on X and ν the discrete topology on Y, so that (X, μ) and (Y, ν) are topological spaces. Consider the (μ, ν)-continuous function $f : X \to Y$ defined by $f(x) = x + 1$. Let

$$\mathcal{H}_X = \{\emptyset\} \cup \{\{x\} \in \mathcal{P}(X) \mid x \in X\} \quad \text{and} \quad \mathcal{H}_Y = \{A \in \mathcal{P}(Y) \mid A \subset Y - \{1\}\}$$

be hereditary classes on X and Y, respectively. Then f is not $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-continuous because $\{1\} \in (\nu)^*_{\mathcal{H}_Y}$ but $\{0\} \notin \mu^*_{\mathcal{H}_X}$.

This implies that not every hereditary class on Y makes f a (μ^*, ν^*)-continuous function. So we can ask what conditions give rise to the (μ^*, ν^*)-continuity of a (μ, ν)-continuous function. Such conditions are described in the following theorems.

THEOREM 3.7. *Let (X, μ) and $(Y, \nu, \mathcal{H}_Y)$ be generalised topological spaces. If f is a (μ, ν)-continuous injection from X into Y, then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(A) \mid A \in \mathcal{H}_Y\},$$

the function f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-continuous.

PROOF. It is clear that $\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\}$ is a hereditary class on X. To see that f is (μ^*, ν^*)-continuous, take $G \in \nu^*$. From Theorem 2.18, ν^* has a base of the form $\{N - H \mid \text{for all } N \in \nu \text{ and } H \in \mathcal{H}_Y\}$. So

$$f^{-1}(G) = f^{-1} \bigcup_{\alpha \in \Lambda} (N_\alpha - H_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(N_\alpha - H_\alpha) = \bigcup_{\alpha \in \Lambda} (f^{-1}(N_\alpha) - f^{-1}(H_\alpha)).$$

Here, $f^{-1}(N_\alpha) \in \mu$ and $f^{-1}(H_\alpha) \in \mathcal{H}_X$ for each $\alpha \in \Lambda$. This implies that $f^{-1}(G) \in \mu^*$. Hence f is (μ^*, ν^*)-continuous. □

EXAMPLE 3.8. Consider the previous example. Define hereditary classes on X and Y by $\mathcal{H}_X = \{\emptyset, \{1\}\}$ and $\mathcal{H}_Y = \{\emptyset, \{2\}\}$, respectively. Then $\nu^*_{\mathcal{H}_Y} = \{\emptyset, [1, 2), [1, 2]\}$. For each $N \in \nu^*_{\mathcal{H}_Y}$, we can easily check that $f^{-1}(N) \in \mu^*_{\mathcal{H}_X}$. Hence f is (μ^*, ν^*)-continuous.

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The following example shows that the injective property is a necessary condition in the construction of the hereditary classes in Theorem 3.7.

Example 3.9. Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, $\mu = \{\emptyset, \{a, c\}, \{a, b, c\}\}$ and $\nu = \{\emptyset, \{1, 2\}\}$. Define $g : X \to Y$ by

$$g(a) = 1, \quad g(b) = 2, \quad g(c) = 2.$$

It is easy to check that g is (μ, ν)-continuous. Let $\mathcal{H}_Y = \{\emptyset, \{2\}\}$ be a hereditary class on Y. By the construction in Theorem 3.7, $\mathcal{H}_X = \{\emptyset, \{b, c\}\}$. However, $\mathcal{H}_X$ is not a hereditary class on X.

Similarly, given a hereditary class on X, we can construct a hereditary class on Y such that f is (μ^*, ν^*)-continuous.

Theorem 3.10. *Let $(X, \mu, \mathcal{H}_X)$ and (Y, ν) be generalised topological spaces. If f is a (μ, ν)-continuous bijection from X onto Y, then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(A) \mid A \in \mathcal{H}_X\},$$

the function f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-continuous.

Proof. Clearly $\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\}$ is a hereditary class on Y. We will prove that f is (μ^*, ν^*)-continuous. Let $G \in \nu^*$. We can write $G = \bigcup_{\alpha\in\Lambda}(N_\alpha - H_\alpha)$ when $N_\alpha \in \nu$ and $H_\alpha \in \mathcal{H}_Y$. Since $H_\alpha \in \mathcal{H}_Y$, there is a $K_\alpha \in \mathcal{H}_X$ such that $f(K_\alpha) = H_\alpha$. Therefore,

$$\begin{aligned} f^{-1}(G) &= f^{-1}\Big(\bigcup_{\alpha\in\Lambda}(N_\alpha - H_\alpha)\Big) = \bigcup_{\alpha\in\Lambda} f^{-1}(N_\alpha - H_\alpha) \\ &= \bigcup_{\alpha\in\Lambda} f^{-1}(N_\alpha - f(K_\alpha)) = \bigcup_{\alpha\in\Lambda}(f^{-1}(N_\alpha) - K_\alpha). \end{aligned}$$

This expresses $f^{-1}(G)$ in terms of basis elements for μ^*, so $f^{-1}(G) \in \mu^*$. □

From Theorems 3.7 and 3.10, given any hereditary class on either X or Y under some assumption on the function f, we can always find a hereditary class which preserves the continuity of f. It is easy to prove the following corollary for the composition function.

Corollary 3.11. *Let (X, μ), (Y, ν) and (Z, ω) be generalised topological spaces. Let f be a (μ, ν)-continuous bijective function from X to Y and g a (ν, ω)-continuous bijective function from Y to Z.*

(1) *If $\mathcal{H}_X$ is a hereditary class on X, we can construct the hereditary classes $\mathcal{H}_Y$ and $\mathcal{H}_Z$ such that $g \circ f$ is (μ^*, ω^*)-continuous.*
(2) *If $\mathcal{H}_Z$ is a hereditary class on Z, we can construct the hereditary classes $\mathcal{H}_X$ and $\mathcal{H}_Y$ such that $g \circ f$ is (μ^*, ω^*)-continuous.*
(3) *If $\mathcal{H}_Y$ is a hereditary class on Y, we can construct the hereditary classes $\mathcal{H}_X$ and $\mathcal{H}_Z$ such that $g \circ f$ is (μ^*, ω^*)-continuous.*

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In a generalised topological space (X, μ), we can define an open map and a homeomorphism in the same way as in a topological space.

DEFINITION 3.12. Let (X, μ) and (Y, ν) be generalised topological spaces. A function $f : X \to Y$ is *(μ, ν)-open* if $f(M) \in \nu$ for each $M \in \mu$.

A function $f : X \to Y$ is a *(μ, ν)-homeomorphism* if f is a (μ, ν)-continuous bijection and f^{-1} is (ν, μ)-continuous.

THEOREM 3.13. *Let (X, μ) and (Y, ν) be generalised topological spaces and f a bijection from X onto Y. Then f is (μ, ν)-open if and only if f^{-1} is (ν, μ)-continuous.*

PROOF. Let $M \in \mu$. Since f is (μ, ν)-open, $f(M) \in \nu$ and $(f^{-1})^{-1}(M) \in \nu$. Hence f^{-1} is (ν, μ)-continuous. Conversely, assume that f^{-1} is (ν, μ)-continuous. Let $M \in \mu$. Since f^{-1} is (ν, μ)-continuous, $(f^{-1})^{-1}(M) \in \nu$ and so $f(M) = (f^{-1})^{-1}(M) \in \nu$. Hence f is (μ, ν)-open. □

COROLLARY 3.14. *Let (X, μ) and (Y, ν) be generalised topological spaces and f a bijection from X onto Y. Then f is a (μ, ν)-homeomorphism if and only if f is (μ, ν)-continuous and (μ, ν)-open.*

COROLLARY 3.15. *Let $(X, \mu, \mathcal{H}_X)$ and (Y, ν) be generalised topological spaces. If f is a (μ, ν)-open bijection from X to Y, then there is a hereditary class $\mathcal{H}_Y$ on Y such that f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-open.*

PROOF. Apply Theorems 3.7 and 3.13. □

COROLLARY 3.16. *Let $(X, \mu,)$ and $(Y, \nu, \mathcal{H}_Y)$ be generalised topological spaces. If f is a (μ, ν)-open bijection from X to Y, then there is a hereditary class $\mathcal{H}_X$ on X such that f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-open.*

PROOF. Apply Theorems 3.10 and 3.13. □

EXAMPLE 3.17. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, $\mu = \{\emptyset, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ and $\nu = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Define $f : X \to Y$ by

$$f(a) = 1, \quad f(b) = 3, \quad f(c) = 2.$$

Then f is a (μ, ν)-open bijection. For the hereditary class $\mathcal{H}_X = \{\emptyset, \{b\}\}$ on X, we have $\mu^*_{\mathcal{H}_X} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. By Theorems 3.7 and 3.13, we can define $\mathcal{H}_Y = \{\emptyset, \{3\}\}$ and then $\nu^*_{\mathcal{H}_Y} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Therefore, f^{-1} is $(\nu^*_{\mathcal{H}_Y}, \mu^*_{\mathcal{H}_X})$-continuous. By using Theorem 3.13 again, f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-open.

THEOREM 3.18. *Let $(X, \mu, \mathcal{H}_X)$ and (Y, ν) be generalised topological spaces. If f is a (μ, ν)-homeomorphism from X to Y, then there is a hereditary class $\mathcal{H}_Y$ on Y that makes f a (μ^*, ν^*)-homeomorphism.*

PROOF. Let $f : X \to Y$ be a (μ, ν)-homeomorphism so that f is a (μ, ν)-open and (μ, ν)-continuous bijection. By applying Theorem 3.10, we obtain a hereditary class $\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\}$ on Y such that f is (μ^*, ν^*)-continuous. It is easy to show that f is (μ^*, ν^*)-open. Thus f is a (μ^*, ν^*)-homeomorphism. □

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Corollary 3.19. *Let (X, μ) and $(Y, \nu, \mathcal{H}_Y)$ be generalised topological spaces. If f is a (μ, ν)-homeomorphism from X to Y, then there is a hereditary class $\mathcal{H}_X$ on X that makes f a (μ^*, ν^*)-homeomorphism.*

Proof. Consider f^{-1} and apply Theorem 3.18. □

We next consider the behaviour of continuous functions on subspaces of generalised topological spaces.

Definition 3.20. Let X be a nonempty set, $\mathcal{H}$ a hereditary class on X and $A \subset X$. The *relative hereditary class* $\mathcal{H}_A$ on A is

$$\mathcal{H}_A = \{H \cap A \mid \text{for all } H \in \mathcal{H}\}.$$

Remark 3.21. We will show that $\mathcal{H}_A$ is a hereditary class on A. Let $C \in \mathcal{H}_A$ and $D \subset C$. Since $C \in \mathcal{H}_A$, there exists $H \in \mathcal{H}$ such that $C = H \cap A$. Then $D \subset H \cap A \subset H$ and so $D \in \mathcal{H}$. Hence $D = D \cap A \in \mathcal{H}_A$.

Theorem 3.22. *Let $(X, \mu, \mathcal{H})$ be a generalised topological space and A a subset of X. For the relative hereditary class $\mathcal{H}_A$ on A, we have $(\mu_A)^* = (\mu^*)_A$.*

Proof. Let $V \in (\mu^*)_A$. There exists $G \in \mu^*$ such that $V = G \cap A$. Since $G \in \mu^*$, $G = \bigcup_{\alpha\in\Lambda}(M_\alpha - H_\alpha)$ with $M_\alpha \in \mu$ and $H_\alpha \in \mathcal{H}$. Then

$$\begin{aligned} V &= \Big(\bigcup_{\alpha\in\Lambda}(M_\alpha - H_\alpha)\Big) \cap A = \Big(\bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c)\Big) \cap A = \bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c \cap A) \\ &= \Big(\bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c \cap A)\Big) \cup \emptyset = \Big(\bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c \cap A)\Big) \cup \Big(\bigcup_{\alpha\in\Lambda}(M_\alpha \cap A^c \cap A)\Big) \\ &= \bigcup_{\alpha\in\Lambda}((M_\alpha \cap (H_\alpha)^c \cap A) \cup (M_\alpha \cap A^c \cap A)) = \bigcup_{\alpha\in\Lambda}((M_\alpha \cap A) \cap ((H_\alpha)^c \cup A^c)) \\ &= \bigcup_{\alpha\in\Lambda}((M_\alpha \cap A) \cap (H_\alpha \cap A)^c) = \bigcup_{\alpha\in\Lambda}((M_\alpha \cap A) - (H_\alpha \cap A)). \end{aligned}$$

Since $M_\alpha \cap A \in \mu_A$ and $H_\alpha \cap A \in \mathcal{H}_A$, we find $V \in (\mu_A)^*$, so $(\mu^*)_A \subset (\mu_A)^*$. Conversely, let $W \in (\mu_A)^*$. Then $W = \bigcup_{\beta\in\Gamma}(U_\beta - K_\beta)$ with $U_\beta \in \mu_A$ and $K_\beta \in \mathcal{H}_A$. Since $U_\beta \in \mu_A$ and $K_\beta \in \mathcal{H}_A$, there exist $M_\beta \in \mu$ and $H_\beta \in \mathcal{H}$ such that $U_\beta = M_\beta \cap A$ and $K_\beta = H_\beta \cap A$. Consequently, $W = \bigcup_{\beta\in\Gamma}((M_\beta \cap A) - (H_\beta \cap A))$. Similarly,

$$W = \bigcup_{\beta\in\Gamma}(M_\beta \cap A - H_\beta \cap A) = \Big(\bigcup_{\beta\in\Gamma}(M_\beta - H_\beta)\Big) \cap A.$$

Since $\bigcup_{\beta\in\Gamma}(M_\beta - H_\beta)$ is an element in μ^*, we have $W \in (\mu^*)_A$ and so $(\mu_A)^* \subset (\mu^*)_A$. Therefore $(\mu_A)^* = (\mu^*)_A$. □

Corollary 3.23. *Let $(X, \mu, \mathcal{H})$ and (Y, ν) be generalised topological spaces and A a subset of X. For the relative hereditary class $\mathcal{H}_A$ on A and $f : A \to Y$, the function f is $((\mu_A)^*, \nu)$-continuous if and only if it is $((\mu^*)|_A, \nu)$-continuous.*

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So far, we have various situations where (μ, ν)-continuity implies (μ^*, ν^*)-continuity. The following examples investigate whether a function can be (μ^*, ν)-continuous without being (μ, ν)-continuous.

Example 3.24. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Define the generalised topologies $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, X\}$ on X and $\nu = \{\emptyset, \{1\}\}$ on Y. Let $f : (X, \mu) \to (Y, \nu)$ be defined by

$$f(a) = 1, \quad f(b) = 3, \quad f(c) = 2, \quad f(d) = 1,$$

so that f is not (μ, ν)-continuous. Choose the hereditary class $\mathcal{H}_X = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ on X. Then

$$\mu^* = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$$

and we can observe that f is (μ^*, ν)-continuous.

Example 3.25. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Define the generalised topologies $\mu = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}\}$ on X and $\nu = \{\emptyset, \{1\}, \{1, 2\}\}$ on Y. Let $f : (X, \mu) \to (Y, \nu)$ be defined by

$$f(a) = 1, \quad f(b) = 3, \quad f(c) = 2, \quad f(d) = 1,$$

so that f is not (μ, ν)-continuous. Moreover, f is not (μ^*, ν)-continuous for any hereditary class on X because μ^* has a base of the form $\{M - H \mid M \in \mu, H \in \mathcal{H}\}$ and so any open set in $\mu^*_{\mathcal{H}}$ cannot contain the element c.

Theorem 3.26. *Let (X, μ) and (Y, ν) be generalised topological spaces and $f : X \to Y$. If $X \in \mu$, then there is always a hereditary class on X such that f is (μ^*, ν)-continuous.*

Proof. Define $\mathcal{H}_f = \{A \subset X - f^{-1}(V) \mid \text{for all } V \in \nu\}$. We claim that $\mathcal{H}_f$ is a hereditary class on X. Let $C \in \mathcal{H}_f$ and $D \subset C$. Since $C \in \mathcal{H}_f$, there exists $V' \in \nu$ such that $C \subset X - f^{-1}(V')$. This implies $D \subset X - f^{-1}(V')$ and so $D \in \mathcal{H}_f$.

Now suppose that $X \in \mu$ and let $N \in \nu$. Then $X - (X - f^{-1}(N))$ is an element in a base for μ^*. Therefore $f^{-1}(N) \in \mu^*$. □

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