Canad. Math. Bull. Vol. **38** (4), 1995 pp. 469–480

CUT AND PASTE IN 2-DIMENSIONAL PROJECTIVE PLANES AND CIRCLE PLANES

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ABSTRACT. We describe two methods to combine sets of lines of different 2-dimensional projective planes into line sets of new 2-dimensional projective planes. Using these methods we describe several ways in which sets of circles of different 2-dimensional circle planes can be combined into circle sets of new 2-dimensional circle planes.

1. **Introduction.** Let us first recall the definitions of and some basic facts about 2-dimensional projective planes and 2-dimensional circle planes, *i.e.*, 2-dimensional Möbius planes, Laguerre planes and Minkowski planes. Starting with very general definitions of these incidence structures it has been shown that all of them can be represented in certain normal forms. We incorporate these normal forms in our definitions. For more details the reader is referred to [19], [24, 7.6], [3], [4, 3.10], [20, 4.4].

A 2-*dimensional projective plane* $\mathcal{P} = (P, \mathcal{L})$ is a projective plane whose point set P is the real projective plane (viewed as a 2-dimensional topological space) and whose lines are subsets of P homeomorphic to the unit circle $\mathbb{S}^1$. The affine plane one arrives at by deleting a line from $\mathcal{P}$ has a point set that is homeomorphic to $\mathbb{R}^2$. Its lines all separate the point set in two open components and are homeomorphic to $\mathbb{R}$. Every affine plane of this form is called a 2-*dimensional affine plane* and it can be shown that the projective extension of such a plane can be turned into a 2-dimensional projective plane in a unique way. The classical examples of 2-dimensional affine and projective planes are the affine and projective planes over the real numbers.

A 2-*dimensional Möbius plane* $\mathcal{M} = (\mathbb{S}^2, \mathcal{C})$ is an incidence structure with a point set and a circle set. Incidence is defined by inclusion. The point set is the 2-sphere $\mathbb{S}^2$, and the circles are simply closed continuous curves on $\mathbb{S}^2$. Furthermore, the incidence structure has to satisfy the following axioms:

(M1) Three distinct points are contained in a uniquely determined circle.

(M2) For two distinct points p, q and a circle c through p there exists a uniquely determined circle through q that *touches* c at p, *i.e.*, intersects c only in the point p, or coincides with c.

A 2-*dimensional Laguerre plane* $\mathcal{L} = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}, \|)$ is an incidence structure consisting of a point set, a circle set and an equivalence relation (parallelism) defined on the point set. Incidence is defined by inclusion. The point set is the cylinder $\mathbb{S}^1 \times \mathbb{R}$, the

This research was supported by a Feodor Lynen fellowship.
Received by the editors July 18, 1994; revised December 5, 1994.
AMS subject classification: Primary: 51H10, 51H15; secondary: 51B10, 51B15, 51B20.

https://doi.org/10.4153/CMB-1995-069-1 Published online by Cambridge University Press

circles are graphs of continuous functions $\mathbb{S}^1 \longrightarrow \mathbb{R}$, and the equivalence classes (parallel classes) of $\|$ are the verticals in $\mathbb{S}^1 \times \mathbb{R}$, *i.e.*, the sets $\{(a,y) \mid y \in \mathbb{R}\}, a \in \mathbb{S}^1$. Furthermore, the incidence structure has to satisfy the following axioms:

(A1) Three pairwise non-parallel points are contained in a uniquely determined circle.

(A2) For two non-parallel points p, q and a circle c through p there exists a uniquely determined circle through q that touches c at p, *i.e.*, intersects c only in the point p, or coincides with c.

A *2-dimensional Minkowski plane* $\mathcal{M} = (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}, \|_v, \|_h)$ is an incidence structure consisting of a point set, a circle set and two equivalence relations (parallelisms) defined on the point set. Incidence is defined by inclusion. The point set is the torus $\mathbb{S}^1 \times \mathbb{S}^1$, the circles are graphs of homeomorphisms $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$. The equivalence classes (parallel classes) of $\|_v$ are the verticals in $\mathbb{S}^1 \times \mathbb{S}^1$, *i.e.*, the sets $\{(a,y) \mid y \in \mathbb{S}^1\}, a \in \mathbb{S}^1$ and the equivalence classes of $\|_h$ are the horizontals in $\mathbb{S}^1 \times \mathbb{S}^1$, *i.e.*, the sets $\{(x,a) \mid x \in \mathbb{S}^1\}$, $a \in \mathbb{S}^1$. Furthermore, the incidence structure has to satisfy axioms A1 and A2 above (in the case of a Minkowski plane, non-parallel in A1 and A2 means non-parallel with respect to both $\|_v$ and $\|_h$).

The point sets of the three types of 2-dimensional circle planes, as described above, are metrizable 2-dimensional topological spaces. Circles are homeomorphic to the unit circle $\mathbb{S}^1$. When the circle sets are topologized by the Hausdorff metric with respect to a metric that induces the topology of the point set, then the planes are *topological* in the sense that the operations of joining three points, intersecting of circles, and touching are continuous with respect to the induced topologies on their respective domains of definition.

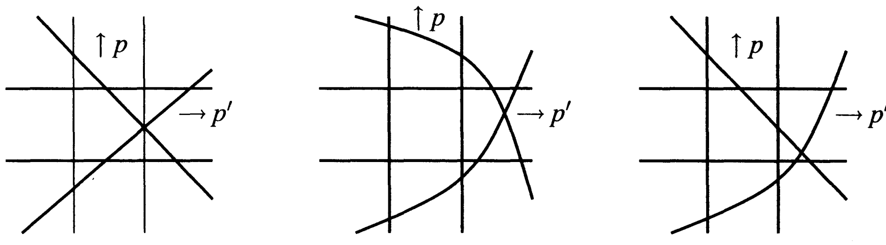
The classical 2-dimensional Möbius, Laguerre and Minkowski planes are obtained as the planes of non-trivial plane sections of an elliptic quadric, an elliptic cone with its vertex removed, and a ruled quadric, respectively, in the 3-dimensional projective space over the real numbers.

Let $(P, \mathcal{C})$ be an incidence structure with a point set P and line (or circle) set $\mathcal{C}$. Let p be a point in P and let $\mathcal{S} \subset \mathcal{C}$. Then $\mathcal{S}_p$ denotes the set of lines (or circles) in $\mathcal{S}$ passing through p. Let $\mathcal{B} = (P, \mathcal{C})$ be an incidence structure with point set $P \in \{\mathbb{S}^2, \mathbb{S}^1 \times \mathbb{R}, \mathbb{S}^1 \times \mathbb{S}^1\}$ and circles and parallel classes as in the case of the 2-dimensional circle planes with the respective point set. So, for example, if $P = \mathbb{S}^2$, then all circles in $\mathcal{C}$ are simply closed continuous curves on $\mathbb{S}^2$. Associated with every point $p \in P$ is a *derived incidence structure* $\mathcal{B}_p$ whose point set consists of all points not parallel to p and whose line set consists of all circles in $\mathcal{C}_p$ that have been punctured at p and all parallel classes that do not contain p. It is known that the incidence structure $\mathcal{B}$ is a 2-dimensional circle plane if and only if the derived incidence structures at all its points are 2-dimensional affine planes (see, *e.g.*, [18]). So, for example, if $P = \mathbb{S}^2$ and $\mathcal{C}$ is a set of simply closed continuous curves on $\mathbb{S}^2$, then $\mathcal{B}$ is a 2-dimensional Möbius plane if and only if all incidence structures $\mathcal{B}_p, p \in \mathbb{S}^2$ are 2-dimensional affine planes.

The derived incidence structure of a 2-dimensional circle plane $\mathcal{B}$ at a point p is called the *derived (2-dimensional) affine plane at p*. In order to be able to describe circles that do not pass through p in this derived affine plane, let us recall that an *oval* in an affine or projective plane is a set of points that contains no three distinct collinear points. Moreover, for every point in the set there is a uniquely determined line that intersects the oval in just this point. A line is called an *exterior line*, *tangent* or *secant* of the oval if it has 0, 1, or 2 points in common with the oval, respectively. A *topological oval* in a 2-dimensional affine or projective plane is an oval that is homeomorphic to the unit circle $\mathbb{S}^1$. For more information about (topological) ovals the reader is referred to [2] and [7]. Now every circle $c \in \mathcal{C}$ that does not pass through p induces a topological oval in the projective extension of the derived affine plane at p, *i.e.*, there exists a uniquely determined topological oval in the projective extension whose affine part coincides with the circle c that has been punctured in the points of c that are parallel to p. This means that in the case of a 2-dimensional Möbius, Laguerre, or Minkowski plane the topological oval intersects the infinite line of the derived affine plane in 0, 1, or 2 points, respectively.

2. **Projective planes.** Let $\mathcal{P} = (P, \mathcal{L})$ be a 2-dimensional projective plane. Given two distinct points p, p' on a line l, the complement $l \setminus \{p, p'\}$ consists of two connected components which we label $l^+_{p,p'}$ and $l^-_{p,p'}$. Furthermore, we can partition $\mathcal{L}$ by $(\mathcal{L}_p \cup \mathcal{L}_{p'}) \cup \mathcal{L}^+ \cup \mathcal{L}^-$ where $\mathcal{L}^+$ and $\mathcal{L}^-$ denote the collection of all lines that intersect the line l through p and p' in $l^+_{p,p'}$ and $l^-_{p,p'}$, respectively.

PROPOSITION 1. *Let $\mathcal{P}_i = (P, \mathcal{L}_i)$, $i = 1, 2$ be two 2-dimensional projective planes, and suppose the line l is contained in both line sets. Suppose further that $\mathcal{L}_0 \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$ consists of two line pencils through the points $p, p' \in l$, i.e., $\mathcal{L}_0 = (\mathcal{L}_1)_p \cup (\mathcal{L}_1)_{p'} = (\mathcal{L}_2)_p \cup (\mathcal{L}_2)_{p'}$. Let $\mathcal{L} := \mathcal{L}_0 \cup \mathcal{L}_1^+ \cup \mathcal{L}_2^-$. Then $\mathcal{P} := (P, \mathcal{L})$ is a 2-dimensional projective plane (see Figure 1).*



$\mathcal{P}_1$ $\mathcal{P}_2$ $\mathcal{P}$

FIGURE 1

PROOF. We identify the point sets of the two 2-dimensional affine planes that correspond to the common line l with the real xy-plane. We identify in such a way that the common lines through p (p') correspond to the verticals (horizontals) in the xy-plane. So, both times l becomes the infinite line and p and p' become the infinite points of the verticals and horizontals, respectively (see Figure 1). Under these identifications the affine

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part of a line $k \in \mathcal{L}_1$ ($k \in \mathcal{L}_2$) that is not contained in the two line pencils through p and p' becomes the graph of a homeomorphism of $\mathbb{R}$. If this homeomorphism is strictly increasing, strictly decreasing, then k belongs to $\mathcal{L}_1^+$, $\mathcal{L}_1^-$, respectively ($\mathcal{L}_2^+$, $\mathcal{L}_2^-$, respectively).

We verify that there is a uniquely determined connecting line in $\mathcal{L}$ for any pair of distinct points in P. If at least one of the two points is contained in l, then there is clearly a uniquely determined connecting line. If both points are contained in the xy-plane, the relative position of the points alone determines whether the connecting lines in $\mathcal{L}_1$ and $\mathcal{L}_2$ correspond to verticals, horizontals, or to the graphs of strictly increasing or strictly decreasing functions. This guarantees the uniqueness of the connecting line in this case. By [19, 2.5], $\mathcal{P}$ is a 2-dimensional projective plane. ■

We note that given any two 2-dimensional projective planes $\mathcal{P}_i = (P, \mathcal{L}_i)$, $i = 1, 2$ and points $p_i, p_i' \in P$, $p_i \neq p_i'$ with connecting lines l_i, there exists a homeomorphism of the common point set P that maps the pencils of lines through p_1 and p_1' to the pencils of lines through p_2 and p_2', respectively. (This follows from the familiar coordinatization of 2-dimensional projective planes; compare [19, Section 7]). This implies that the homeomorphism maps p_1 to p_2, p_1' to p_2' and l_1 to l_2. So the two projective planes can always be matched up such that the above proposition can be applied.

The general definition of 2-dimensional projective planes implies that the dual projective plane of such a plane can be turned into a 2-dimensional projective plane in a unique way (see [19]). So, it is possible to dualize Proposition 1, and we arrive at the following

COROLLARY. *Let $\mathcal{P}_i = (P, \mathcal{L}_i)$, $i = 1, 2$ be two 2-dimensional projective planes, and suppose that $(\mathcal{L}_1)_p = (\mathcal{L}_2)_p$ for some point $p \in P$. Let l, l' be two lines in this common pencil and let P', P'' be the two open components of $P \setminus (l \cup l')$. For every line $k \in \mathcal{L}_1 \setminus (\mathcal{L}_1)_p$ let k' be the line in $\mathcal{L}_2 \setminus (\mathcal{L}_2)_p$ that connects the two points $k \wedge l$ and $k \wedge l'$ and let $k_{\text{new}} := (k \cap P') \cup (k' \cap P'') \cup \{k \wedge l, k \wedge l'\}$. The set k_{new} is homeomorphic to the unit circle $\mathbb{S}^1$ and $\mathcal{P} := (P, \mathcal{L})$, with $\mathcal{L} := \{k_{\text{new}} \mid k \in \mathcal{L}_1 \setminus (\mathcal{L}_1)_p\} \cup (\mathcal{L}_1)_p$ is a 2-dimensional projective plane (see Figure 2).*

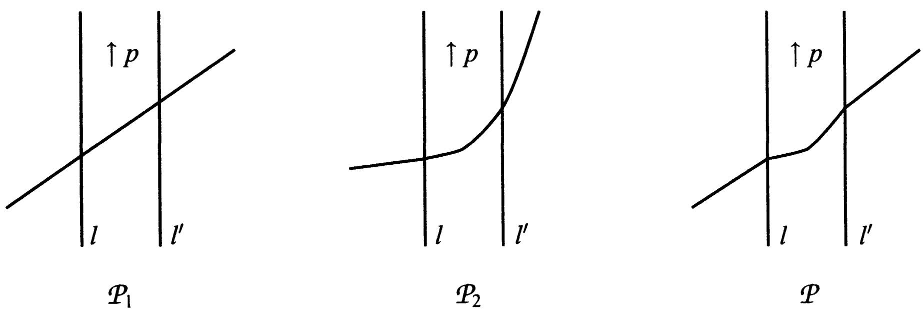

FIGURE 2

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The Moulton planes (see [13], [15], [16], [17], [19]) and semiclassical planes (see [21]) are examples of planes that can be constructed in this manner.

Let O be a topological oval in a 2-dimensional projective plane $\mathcal{P} = (P, \mathcal{L})$. Let $\mathcal{L}^i$ be the collection of all exterior lines, tangents and secants of O for $i = 0, 1, 2$, respectively. Let I (O) be the set of all points in P that are contained in precisely 0 (2) tangents of O. A point in I (O) is called an *inner* (*outer*) point of O. It can be shown that the sets I and O are the two open components of $P \setminus O$ [2, 3.7]. So, every point in P is either an inner point of O, a point on the oval O, or an outer point of O.

PROPOSITION 2. *Let $\mathcal{P}_i = (P, \mathcal{L}_i)$, $i = 1, 2$ be two 2-dimensional projective planes, and suppose that O is a topological oval in both planes and that all tangents of O are contained in both planes, i.e., $\mathcal{L}_1^1 = \mathcal{L}_2^1$. Let $\mathcal{L} := \mathcal{L}_1^0 \cup \mathcal{L}_1^1 \cup \mathcal{L}_2^2$. Then $\mathcal{P} := (P, \mathcal{L})$ is a 2-dimensional projective plane (see Figure 3).*

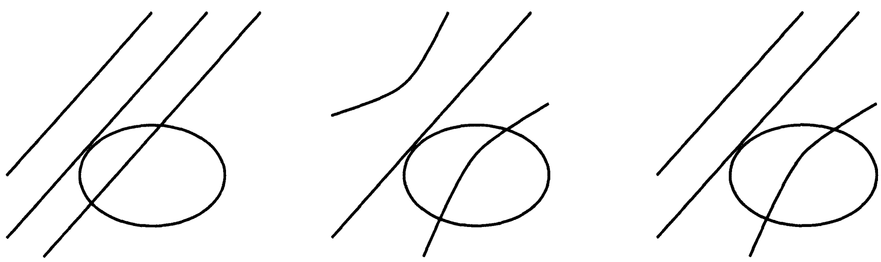

FIGURE 3

Note that the sets of inner points and the sets of outer points of O in both planes coincide, since the set of tangents of O is contained in the line sets of both planes.

PROOF. We verify that there is a uniquely determined connecting line in $\mathcal{L}$ for any pair of distinct points $p, q \in P$. If at least one of the points belongs to $I \cup O$, then a joining line must be a tangent or a secant. Hence there is exactly one line in $\mathcal{L}$ that connects both points. It is contained in $\mathcal{L}_2$. If both p and q belong to O, then we consider the two tangents to O (contained in both $\mathcal{L}_1$ and $\mathcal{L}_2$) that pass through p. The complement of the two lines has two connected components, P_s and P_e. One, P_s, contains all points of O except the two points of tangency. Joining a point of P_s with p yields a secant of O. Joining a point of P_e with p yields an exterior line of O. Hence, if q falls on one of the two tangents or in P_s, there is exactly one line that contains both points. It comes from $\mathcal{L}_2$. If q falls in P_e, there is also exactly one joining line. It comes from $\mathcal{L}_1$. This shows that two points are uniquely joined by a line in $\mathcal{L}$. By [19, 2.5], $\mathcal{P}$ is a 2-dimensional projective plane. ∎

We note that given any two 2-dimensional projective planes $\mathcal{P}_i = (P, \mathcal{L}_i)$, $i = 1, 2$ and two topological ovals O_i contained in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, there exists a homeomorphism of the common point set that maps O_1 to O_2 and all tangents of O_1 to tangents of O_2 (this is a direct consequence of [2, 3.10]). So the two projective planes can always be matched up such that the above theorem can be applied.

The set of tangents of a topological oval O in a 2-dimensional projective plane is itself a topological oval in the dual (2-dimensional) projective plane (see [2, 3.8]). Furthermore, the inner (outer) points of O turn into exterior lines (secants) of the dual oval, and vice versa. It is therefore possible to dualize Proposition 2, and we arrive at the following

COROLLARY. *Let $\mathcal{P}_i = (P, \mathcal{L}_i)$, $i = 1, 2$ be two 2-dimensional projective planes, and suppose that O is a topological oval in both planes and that all tangents of O are contained in both planes, i.e., $\mathcal{L}_1^1 = \mathcal{L}_2^1$. For every line $k \in \mathcal{L}_1^2$ let k' be the line in $\mathcal{L}_2^2$ that connects the two points of intersection of k with O. Let $k_{\text{new}} := (O \cap k) \cup (I \cap k') \cup (k \cap O)$. The set k_{new} is homeomorphic to the circle and $\mathcal{P} := (P, \mathcal{L})$ with $\mathcal{L} := \{k_{\text{new}} \mid k \in \mathcal{L}_1^2\} \cup \mathcal{L}_1^0 \cup \mathcal{L}_1^1$ is a 2-dimensional projective plane (see Figure 4).*

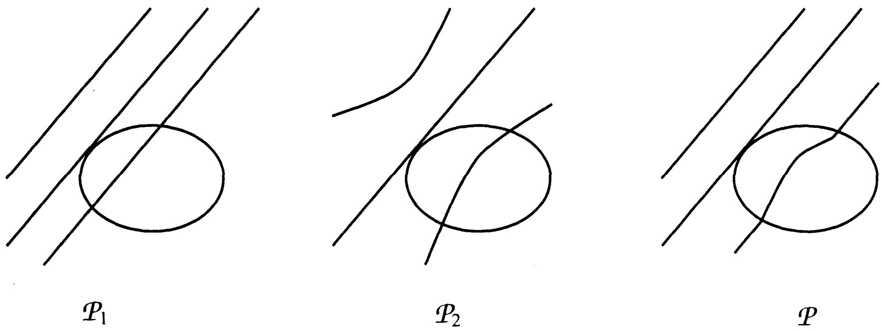

FIGURE 4

Hilbert's example of a non-desarguesian projective plane [10], [11] is an example for such a construction.

We remark that the results in this section are special cases of [23, 12].

3. **Minkowski planes.** Let $\mathcal{M} = (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}, \|_v, \|_h)$ be a 2-dimensional Minkowski plane as described in Section 1. Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the set of all circles in $\mathcal{C}$ that are graphs of orientation-preserving and orientation-reversing homeomorphisms of $\mathbb{S}^1$, respectively. Clearly $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$.

PROPOSITION 3. *Let $\mathcal{M}_i = (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}_i, \|_v, \|_h)$, $i = 1, 2$ be two 2-dimensional Minkowski planes and let $\mathcal{C} := \mathcal{C}_1^+ \cup \mathcal{C}_2^-$. Then $\mathcal{M} := (\mathbb{S}^1 \times \mathbb{S}^1, \mathcal{C}, \|_v, \|_h)$ is a 2-dimensional Minkowski plane (see Figure 5).*

In order to prove this result we need the following (well-known)

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LEMMA 1. *Let ϕ be a homeomorphism of $\mathbb{S}^1$. If ϕ is orientation-reversing, then ϕ fixes exactly two points. If ϕ fixes at most one or at least three points, then ϕ is orientation-preserving.*

PROOF. Let ϕ be an orientation-reversing homeomorphism. As a consequence of Brouwer's fixed-point theorem (see, *e.g.*, [14, p. 373, Ex. 7]), we know that ϕ has at least one fixed point. We identify $\mathbb{S}^1$ with $\mathbb{R}\cup\{\infty\}$ in a natural way such that ∞ gets identified with a fixed point. Then ϕ induces a strictly decreasing homeomorphism ϕ' of $\mathbb{R}$. Such a homeomorphism of $\mathbb{R}$ has precisely one fixed point because the function $\phi' - id_{\mathbb{R}}$ is a strictly decreasing homeomorphism. Hence ϕ has precisely two fixed points. The second part of this lemma follows immediately. ∎

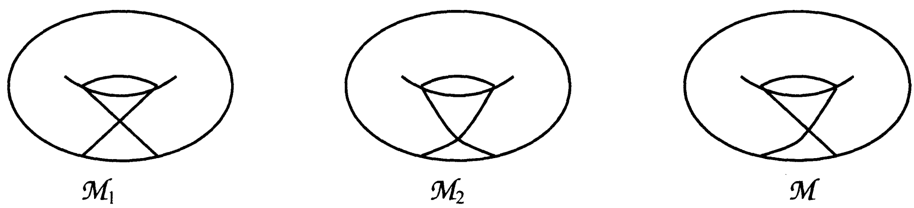

FIGURE 5

PROOF OF PROPOSITION 3. We prove that $\mathcal{M}$ is a 2-dimensional Minkowski plane.

We check that axiom A1 is satisfied. Let f and g be the homeomorphisms $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ whose graphs are the uniquely determined circles in $\mathcal{C}_1$ and $\mathcal{C}_2$ that contain three given pairwise non-parallel points in $\mathbb{S}^1 \times \mathbb{S}^1$. Then fg^{-1} has at least three fixed points. By Lemma 1, we know that fg^{-1} is orientation-preserving, *i.e.*, f and g are either both orientation-preserving or both orientation-reversing. This guarantees the existence and uniqueness of the connecting circle in $\mathcal{C}$.

We verify that axiom A2 is satisfied. Let c and c^* be two circles in $\mathcal{C}_1$ ($\mathcal{C}_2$) that touch in a point and let f and g be the corresponding homeomorphisms of $\mathbb{S}^1$, respectively. Then $f^{-1}g$ has precisely one fixed point if $c \neq c^*$ or infinitely many fixed points if $c = c^*$. By Lemma 1, f and g are either both orientation-preserving or both orientation-reversing, *i.e.*, either both c and c^* are contained in $\mathcal{C}_1^+$ ($\mathcal{C}_2^+$) or both are contained in $\mathcal{C}_1^-$ ($\mathcal{C}_2^-$). This implies that, given a circle $c \in \mathcal{C}$, a point p and one further point q that is not parallel to p, there is at least one circle $c^* \in \mathcal{C}$ through q that touches c in p. If $c \in \mathcal{C}_1^+$ ($c \in \mathcal{C}_2^-$), then the circle we have in mind is the only such circle in $\mathcal{C}_1^+$ ($\mathcal{C}_2^-$). As a consequence of Lemma 1, all circles in $\mathcal{C}_2^-$ ($\mathcal{C}_1^+$) intersect c in exactly two points, *i.e.*, none of these circles touch c. This means that c^* is uniquely determined.

Since all circles in $\mathcal{C}$ are graphs of homeomorphisms $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$, we conclude that $\mathcal{M}$ is a 2-dimensional Minkowski plane. ∎

Special cases of the construction described in Proposition 3 can be found, *e.g.*, in [1], [9], [20], [22].

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4. **Laguerre planes.** We represent 2-dimensional Laguerre planes as in Section 1.

PROPOSITION 4. *Let $\mathcal{L}_i = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}_i, \|)$, $i = 1,2$ be two 2-dimensional Laguerre planes. Suppose that $\mathcal{C}_0 \subseteq \mathcal{C}_1 \cap \mathcal{C}_2$ is a non-empty closed subset of the topological space $\mathcal{C}_i$ and that it satisfies the following conditions:*

(1) there exist non-empty open subsets $\mathcal{C}_i^+$ and $\mathcal{C}_i^-$ of $\mathcal{C}_i$ such that $\mathcal{C}_i$ is the disjoint union of $\mathcal{C}_i^+$, $\mathcal{C}_i^-$ and $\mathcal{C}_0$;

(2) for all $p \in \mathbb{S}^1 \times \mathbb{R}$ either the collection $(\mathcal{C}_0)_p$ of all circles in $\mathcal{C}_0$ that pass through p is empty, or $(\mathcal{C}_0)_p = (\mathcal{C}_1)_p = (\mathcal{C}_2)_p$, or there exists a second point q such that all elements of $(\mathcal{C}_i^+)_p$ $\big((\mathcal{C}_i^-)_p\big)$ intersect the vertical through q above (below) q.

Let $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1^+ \cup \mathcal{C}_2^-$. Then $\mathcal{L} := (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}, \|)$ is a 2-dimensional Laguerre plane.

PROOF. We show that each derived incidence structure $\mathcal{L}_p$, $p \in \mathbb{S}^1 \times \mathbb{R}$ is a 2-dimensional affine plane. Then $\mathcal{L}$ is a 2-dimensional Laguerre plane (see the remark at the end of Section 1).

Suppose that $(\mathcal{C}_0)_p$ is empty. As a consequence of (1) we know that either

(a) $(\mathcal{C}_1)_p = (\mathcal{C}_1^+)_p$ and $(\mathcal{C}_2)_p = (\mathcal{C}_2^+)_p$; or

(b) $(\mathcal{C}_1)_p = (\mathcal{C}_1^-)_p$ and $(\mathcal{C}_2)_p = (\mathcal{C}_2^-)_p$; or

(c) $(\mathcal{C}_1)_p = (\mathcal{C}_1^+)_p$ and $(\mathcal{C}_2)_p = (\mathcal{C}_2^-)_p$; or

(d) $(\mathcal{C}_1)_p = (\mathcal{C}_1^-)_p$ and $(\mathcal{C}_2)_p = (\mathcal{C}_2^+)_p$.

We need to exclude cases c) and d). Let us assume that $(\mathcal{C}_1)_p = (\mathcal{C}_1^+)_p$. For a point $q \in P$ that is not parallel to p we cannot have $(\mathcal{C}_0)_q = (\mathcal{C}_1)_q = (\mathcal{C}_2)_q$, since in this case the set $(\mathcal{C}_0)_q \subset \mathcal{C}_0$ also includes circles through p. This is not the case. If $(\mathcal{C}_0)_q$ is the empty set for all points $q \in P$ that are not parallel to p, then $\mathcal{C}_0$ is empty too. This is not the case. Condition (2) asserts that there is only one more possibility. There exists a point $p' \in P$ that is not parallel to p and one further point q' such that all elements of $(\mathcal{C}_i^+)_{p'}$ $\big((\mathcal{C}_i^-)_{p'}\big)$ intersect the vertical through q' above (below) q'. Notice that q' cannot be parallel to p'. We show that q' is parallel to p. If this were not the case, then the pencil of circles in $\mathcal{C}_1$ through p' and p would contain circles that intersect the parallel class of q' above q' as well as circles that intersect this parallel class below q'. This is not possible since, by assumption, all circles in this pencil are contained in $\mathcal{C}_1^+$ and all these circles are supposed to intersect the parallel class of q' above q'. So, q' is parallel to p. Remember that we started off with the assumption that $(\mathcal{C}_1)_p = (\mathcal{C}_1^+)_p$. Since the set $(\mathcal{C}_1)_p$ includes circles that pass through p', we know that p is situated above q'. As a consequence of this all circles in $\mathcal{C}_2$ that contain both p and p' are contained in $\mathcal{C}_2^+$. All these circles are contained in $(\mathcal{C}_2)_p$. Hence $(\mathcal{C}_2)_p = (\mathcal{C}_2^+)_p$. This excludes case c). We exclude case d) in a similar manner. We just showed that if $(\mathcal{C}_0)_p$ is empty, then $\mathcal{L}_p$ is a 2-dimensional affine plane.

If $(\mathcal{C}_0)_p = (\mathcal{C}_1)_p = (\mathcal{C}_2)_p$, then $\mathcal{L}_p = (\mathcal{L}_1)_p = (\mathcal{L}_2)_p$ is a 2-dimensional affine plane. Suppose that $(\mathcal{C}_0)_p$ is non-empty and let q be a point as in (2). Then the pencil of circles through p and q is contained in $(\mathcal{C}_0)_p$. The projective extensions of the derived (2-dimensional) affine planes $(\mathcal{L}_1)_p$ and $(\mathcal{L}_2)_p$ have the pencils of lines through q and the ideal point of the verticals in common. Using the two projective extensions together

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with these distinguished two common pencils of lines as input we construct a third 2-dimensional projective plane as in Proposition 1 and remove the line that connects q and the ideal point of the verticals from this plane. Now $\mathcal{L}_p$ is the resulting (2-dimensional) affine plane. ∎

EXAMPLES. We give two examples of subsets $\mathcal{C}_0$ of the circle space of a 2-dimensional Laguerre plane $\mathcal{L} = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}, \|)$ as in Proposition 4(1). This will allow us (below) to construct new Laguerre planes.

a) For each $p \in \mathbb{S}^1 \times \mathbb{R}$ let $\mathcal{C}_p$ be the set of all circles that pass through p and let $\mathcal{C}_{p^+}$ and $\mathcal{C}_{p^-}$ be the collection of all circles that intersect the parallel class of p above or below p, respectively. Clearly, $\mathcal{C}_0 := \mathcal{C}_p$ satisfies (1) in Proposition 4 and the two connected components of $\mathcal{C} \setminus \mathcal{C}_0$ are $\mathcal{C}_{p^+}$ and $\mathcal{C}_{p^-}$.

b) Suppose that $\mathcal{L}$ admits a fixed-point-free involutory automorphism γ and that there exists a circle c_0 with $c_0 \cap \gamma(c_0) = \emptyset$. We remark that γ, viewed as a bijection of the point set of $\mathcal{L}$, is a homeomorphism since, for every point p, it induces an isomorphism of the derived (2-dimensional) affine planes $\mathcal{L}_p$ and $\mathcal{L}_{\gamma(p)}$, which is continuous (see [19, 3.5]). Let $\mathcal{C}_0$ be the collection of all circles fixed by γ. Then $\mathcal{C}_0$ is a closed subset of $\mathcal{C}$ that is homeomorphic to $\mathbb{R}^2$ (see [6, 3.6]). For each $c \in \mathcal{C} \setminus \mathcal{C}_0$, c and $\gamma(c)$ are disjoint according to [6, 3.4]. Thus $f(x) > (\gamma f)(x)$ or $f(x) < (\gamma f)(x)$ for all $x \in \mathbb{S}^1$ where f and γf are continuous functions $\mathbb{S}^1 \longrightarrow \mathbb{R}$ whose graphs are c and $\gamma(c)$, respectively. We use the notation $c > \gamma(c)$ or $c < \gamma(c)$, respectively, to denote this fact. With this notation let $\mathcal{C}^+ := \{c \in \mathcal{C} \mid c > \gamma(c)\}$ and $\mathcal{C}^- := \{c \in \mathcal{C} \mid c < \gamma(c)\}$. Clearly, $\mathcal{C}_0$ satisfies (1) in Proposition 4 and the two connected components of $\mathcal{C} \setminus \mathcal{C}_0$ are $\mathcal{C}^+$ and $\mathcal{C}^-$.

PROPOSITION 5. *Let $\mathcal{L} = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}, \|)$ and $\mathcal{L}^* = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}^*, \|)$ be two 2-dimensional Laguerre planes.*

(1) Let $p_\infty \in \mathbb{S}^1 \times \mathbb{R}$. Suppose $\mathcal{C}_0 = \mathcal{C}_{p_\infty} = \mathcal{C}^*_{p_\infty}$ and define $\mathcal{C}_1^\pm := \mathcal{C}_{p_\infty^\pm}$, $\mathcal{C}_2^\pm := \mathcal{C}^*_{p_\infty^\pm}$; or

(2) Suppose both Laguerre planes admit the fixed-point-free involutory automorphism γ and there exist circles $c_0 \in \mathcal{C}$, $c_0^* \in \mathcal{C}^*$ with $c_0 \cap \gamma(c_0) = \emptyset$, $c_0^* \cap \gamma(c_0^*) = \emptyset$. Suppose further that the collections of all circles in $\mathcal{C}_1$ and $\mathcal{C}_2$ fixed by γ coincide and call this common set of circles $\mathcal{C}_0$. Finally, define $\mathcal{C}_i^+ := \{c \in \mathcal{C}_i \mid c > \gamma(c)\}$ and $\mathcal{C}_i^- := \{c \in \mathcal{C}_i \mid c < \gamma(c)\}$.

Then $\mathcal{C}_0$, $\mathcal{C}_1^\pm$ and $\mathcal{C}_2^\pm$ satisfy the hypotheses of Proposition 4 in both cases. Hence $\mathcal{L} := (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}_0 \cup \mathcal{C}_1^+ \cup \mathcal{C}_2^-, \|)$ is a 2-dimensional Laguerre plane (see Figure 6 for (1)).

PROOF. We verify that the hypotheses of Proposition 4 are satisfied. Then the resulting incidence structure is a 2-dimensional Laguerre plane by that proposition. Condition (1) is satisfied in each case as seen in the above examples.

1) To verify Proposition 4(2) let $p \in \mathbb{S}^1 \times \mathbb{R}$. If $p = p_\infty$, then $(\mathcal{C}_0)_p = (\mathcal{C}_1)_p = (\mathcal{C}_2)_p$. If $p \neq p_\infty$, but p is parallel to p_∞, then $(\mathcal{C}_0)_p$ is empty. Finally, if p is not parallel to p_∞, then $(\mathcal{C}_0)_p$ is the pencil of circles through p and p_∞ and condition 4(2) is clearly satisfied (choose p_∞ as the point q).

2) To verify Proposition 4(2) let $p \in \mathbb{S}^1 \times \mathbb{R}$. By [6, 3.3], each circle passing through p and $\gamma(p)$ belongs to $\mathcal{C}_0$. Conversely, every circle through p that is fixed by γ must pass through $\gamma(p)$. Hence $(\mathcal{C}_0)_p$ is the pencil of circles through p and $\gamma(p)$. Let $c \in (\mathcal{C}_i^+)_p$ ($c \in (\mathcal{C}_i^-)_p$). Since $c > \gamma(c)$ ($c < \gamma(c)$) and $\gamma(p) \in \gamma(c)$, it is clear that c intersects the parallel class of $\gamma(p)$ above (below) $\gamma(p)$. ■

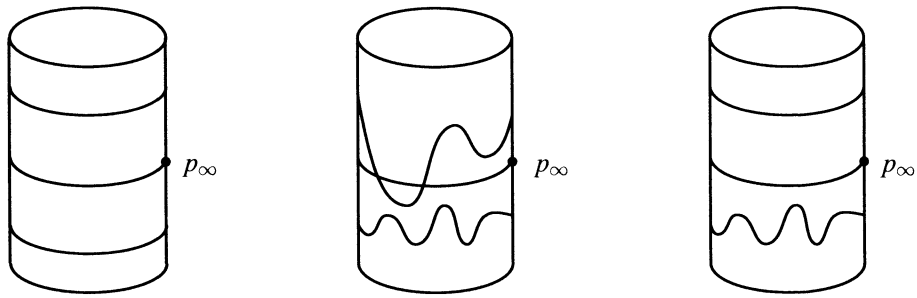

FIGURE 6

REMARKS. 1) Special cases of the construction described in Proposition 5(1) can be found, *e.g.*, in [1], [8], [12].

2) In [5] and [6] Groh characterized the 2-dimensional Laguerre planes that admit fixed-point-free involutory collineations γ such that there exist circles c with $c \cap \gamma(c) = \emptyset$. These planes can all be constructed from 2-dimensional Möbius planes.

In the following we describe one more method to combine sets of circles of different 2-dimensional Laguerre planes into circle sets of new 2-dimensional Laguerre planes. Let $\mathcal{L} = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}, \|)$ be a 2-dimensional Laguerre plane and let $c_0 \in \mathcal{C}$. Consider the collection $\mathcal{C}^1$ of all circles that touch c_0. The circle c_0 separates $\mathbb{S}^1 \times \mathbb{R}$ into two connected components C^+ and C^-. We define $\mathcal{C}^\pm$ to be the collection of all circles that are completely contained in $C^\pm$. Finally, let $\mathcal{C}^2$ be the set of all circles that intersect c_0 in precisely two points. Obviously, $\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^+ \cup \mathcal{C}^-$ is a partition of the circle set.

PROPOSITION 6. *Let $\mathcal{L}_i = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}_i, \|)$, $i = 1, 2, 3$ be three 2-dimensional Laguerre planes. Suppose that $\mathcal{C}_1^1 = \mathcal{C}_2^1 = \mathcal{C}_3^1$ for some common circle c_0. Let $\mathcal{C} := \mathcal{C}_1^1 \cup \mathcal{C}_1^2 \cup \mathcal{C}_2^+ \cup \mathcal{C}_3^-$. Then $\mathcal{L} := (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}, \|)$ is a 2-dimensional Laguerre plane.*

PROOF. As in Proposition 4 it suffices to show that each derived incidence structure $\mathcal{L}_p$, $p \in \mathbb{S}^1 \times \mathbb{R}$ is a 2-dimensional affine plane. Then $\mathcal{L}$ is a 2-dimensional Laguerre plane.

Let $p \in \mathbb{S}^1 \times \mathbb{R}$. If $p \in c_0$, then obviously $\mathcal{C}_p = (\mathcal{C}_1)_p$. Hence $\mathcal{L}_p = (\mathcal{L}_1)_p$ is a 2-dimensional affine plane. If $p \notin c_0$, say $p \in C^+$, then the projective extensions of the derived (2-dimensional) affine planes $(\mathcal{L}_1)_p$ and $(\mathcal{L}_2)_p$ have a topological oval O induced by c_0 in common. More precisely, O is obtained from c_0 by removing the point of c_0 parallel to p and adding the infinite point of lines that come from parallel classes of points in $\mathbb{S}^1 \times \mathbb{R}$. Furthermore, both projective planes have all tangents to O in common and $\mathcal{L}_p$ is obtained from $(\mathcal{L}_1)_p$ and $(\mathcal{L}_2)_p$ by replacing the lines in $(\mathcal{L}_1)_p$ that correspond to exterior lines of O by the lines in $(\mathcal{L}_2)_p$ that correspond to exterior lines of O. As a consequence of Proposition 2 this yields a 2-dimensional affine plane. If $p \in C^-$, $\mathcal{L}_p$ is obtained from $(\mathcal{L}_1)_p$ and $(\mathcal{L}_3)_p$ in a similar fashion. This shows that each derived incidence structure $\mathcal{L}_p$ is a 2-dimensional affine plane. ∎

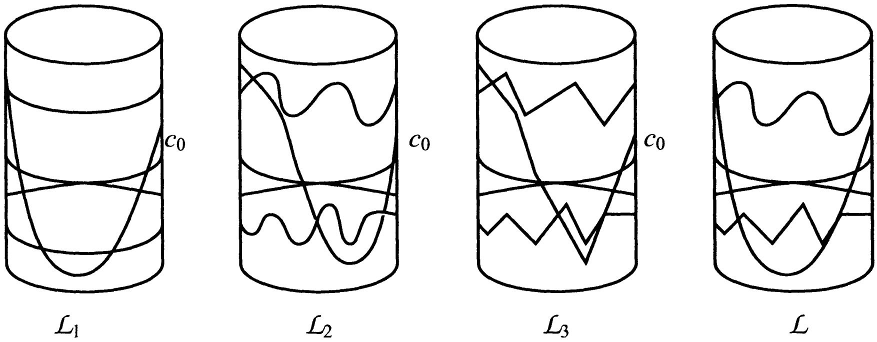

FIGURE 7

5. **Möbius planes.** Let $\mathcal{M} = (\mathbb{S}^2, \mathcal{C})$ be a 2-dimensional Möbius plane as described in Section 1. We fix one circle c_0 and consider the collection $\mathcal{C}^1$ of all circles that touch c_0. The circle c_0 separates $\mathbb{S}^2$ into two connected components C^+ and C^-. We define $\mathcal{C}^\pm$ to be the collection of all circles that are completely contained in $C^\pm$. Finally, let $\mathcal{C}^2$ be the set of all circles that intersect c_0 in precisely two points. Obviously, $\mathcal{C}^1 \cup \mathcal{C}^2 \cup \mathcal{C}^+ \cup \mathcal{C}^-$ is a partition of the circle set (similar to the one considered in Proposition 6).

PROPOSITION 7. *Let $\mathcal{M}_i = (\mathbb{S}^2, \mathcal{C}_i)$, $i = 1,2,3$ be three 2-dimensional Möbius planes. Suppose that $\mathcal{C}_1^1 = \mathcal{C}_2^1 = \mathcal{C}_3^1$ for some common circle c_0. Let $\mathcal{C} := \mathcal{C}_1^1 \cup \mathcal{C}_1^2 \cup \mathcal{C}_2^+ \cup \mathcal{C}_3^-$. Then $\mathcal{M} := (\mathbb{S}^2, \mathcal{C})$ is a 2-dimensional Möbius plane.*

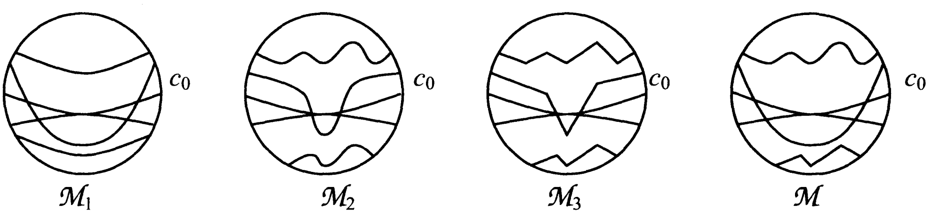

FIGURE 8

PROOF. We show that each derived incidence structure $\mathcal{M}_p$, $p \in \mathbb{S}^2$ is a 2-dimensional affine plane. Then $\mathcal{M}$ is a 2-dimensional Möbius plane (see the remark at the end of Section 1).

https://doi.org/10.4153/CMB-1995-069-1 Published online by Cambridge University Press

Let $p \in \mathbb{S}^2$. If $p \in c_0$, then obviously $\mathcal{C}_p = (\mathcal{C}_1)_p$. Hence $\mathcal{M}_p = (\mathcal{M}_1)_p$ is a 2-dimensional affine plane. If $p \notin c_0$, say $p \in C^+$, then $(\mathcal{M}_1)_p$ and $(\mathcal{M}_2)_p$ are two 2-dimensional affine planes that have a topological oval O induced by c_0 in common and whose line sets have all tangents to O in common. $\mathcal{M}_p$ is obtained from $(\mathcal{M}_1)_p$ and $(\mathcal{M}_2)_p$ by replacing the circles in $(\mathcal{M}_1)_p$ that correspond to exterior lines of O by the circles in $(\mathcal{M}_2)_p$ that correspond to exterior lines of O. As a consequence of Proposition 2 this yields a 2-dimensional affine plane. If $p \in C^-$, $\mathcal{M}_p$ is obtained from $(\mathcal{M}_1)_p$ and $(\mathcal{M}_3)_p$ in a similar fashion. This shows that each derived incidence structure $\mathcal{M}_p$ is a 2-dimensional affine plane. ∎

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