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FINITE PROJECTIVE PLANES THAT ADMIT A STRONGLY IRREDUCIBLE COLLINEATION GROUP

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1. Introduction. This paper studies how coding theory and group theory can be used to produce information about a finite projective plane π and a collineation group G of π.

A new proof for Hering's bound on $|G|$ is given in 2.5. Using the idea of coding theory developed in [**9**], a relation between two rows of the incidence matrix of π with respect to a tactical decomposition is obtained in 2.1. This result yields, among other things, some techniques in calculating $|G|$, and generalizes a result of Roth [**16**], [see 2.4 and 2.5].

Hering [**7**] introduced the notion of strong irreducibility of G, that is, G does not leave invariant any point, line, triangle or proper subplane. He showed that if in addition G contains a non-trivial perspectivity, then there is a unique minimal normal subgroup of G. This subgroup is either non-abelian simple or isomorphic to the elementary abelian group $\mathbf{Z}_3 \times \mathbf{Z}_3$ of order 9. In Section 3, it is shown that if a minimal normal subgroup of a strongly irreducible collineation group is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$ and the order of the plane n is odd, then n is a square and is congruent to 1 modulo 3 (see 3.2).

A long standing question in the study of projective planes is whether a projective plane of prime order is Desarguesian. On one hand the answer is affirmative for projective planes of order 2, 3, 5 or 7 [**15**]. On the other hand little is known for projective planes of prime order larger than 7. Towards this we ask a more restricted question: Is a projective plane π of prime order n admitting a strongly irreducible group G Desarguesian? Theorem 3.3 gives some preliminary results on this matter. Sections 4 and 5 treat the special cases $n = 11$ and $n = 13$. It is shown that except for some explicit possibilities, π is Desarguesian (see 4.7 and 5.4). Also the structure of an arbitrary odd order collineation group of π is determined (see 4.5 and 5.3). By similar methods one can show (see 6.1) that if $n \leqq 37$ and G contains a non-trivial perspectivity, then, except for some specific cases, π is Desarguesian. The family of the 2 dimensional projective special linear group $L_2(q)$ seems to play an important role in studying the restricted question for arbitrary prime n. Some properties of $L_2(q)$ are presented in 3.4.

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2. Definitions and preliminaries. In this paper $\pi = (\mathscr{P}, \mathscr{L})$ will be a finite projective plane of order n and G will be a collineation group of π.

For $g \in G$ let $\mathscr{P}(g)$ (resp. $\mathscr{L}(g)$) be the set of fixed points (resp. lines) of G in $\mathscr{P}$ (resp. $\mathscr{L}$). Define

$$\mathrm{Fix}(g) = (\mathscr{P}(g), \mathscr{L}(g)) \quad \text{and} \quad \mathrm{Fix}(G) = \bigcap_{g \in G} \mathrm{Fix}(g).$$

For any $X \in \mathscr{P}$, $[X]$ denotes the set of all lines through X. If σ is a perspectivity of π, then $\mathscr{C}(\sigma)$ denotes the center of σ and $a(\sigma)$ denotes the axis of σ. For any subset S of G, let

$$IH(S) := \{\tau | \tau \text{ is an involutorial homology in } S\},$$

and

$$I(S) := \{\tau | \tau \text{ is an involution in } S\}.$$

We call a collineation g of π *regular* if

$$\mathrm{Fix}(g) = (\phi, \phi);$$

a *flag* if $\mathrm{Fix}(g)$ consists of a point and a line such that the point is on the line; an *anti-flag* if $\mathrm{Fix}(g)$ consists of a point and a line such that the point is not on the line; *planar* if $\mathrm{Fix}(g)$ is a subplane; a *generalized homology* if $\mathscr{P}(g) = \{P\} \cup x$ and $\mathscr{L}(g) = \{l\} \cup \{PQ | Q \in x\}$ for some line l, subset $x \leqq l$ and point P not on l; a *generalized elation* if $\mathscr{P}(g) \leqq l$ and $\mathscr{L}(g) \leqq [P]$ for some line $l \in \mathscr{L}(g)$ and $P \in \mathscr{P}(g)$. A generalized homology is of type $D(k)$ if it fixes exactly $k + 1$ points. A generalized homology of type D is called *triangular*. A *generalized perspectivity* is a generalized homology or a generalized elation. We also apply these terms to groups of collineations of π by considering $\mathrm{Fix}(G)$ instead of $\mathrm{Fix}(g)$.

For $L \leqq \mathscr{L}$ let

$$P(L) = \{x \cap y | x, y \in L\}.$$

For $J \leqq \mathscr{P}$ let

$$L(J) = \{AB | A, B \in J\}.$$

Let f_G be the least common multiple of the orders of the point-wise stabilizers in G of quadrangles in π. For any subgroup H of G, $N_G(H)$ denotes the normalizer of H in G, $Z(H)$ denotes the center of H, and $C_G(H)$ denotes the centralizer of H in G.

We say that G acts *strongly irreducibly* on π if G does not leave invariant any point, line, triangle, or proper subplane.

Other definitions in group theory can be found in [**6**, **12**]. Although the following proposition can be generalized to other incidence structures, the present form is good enough for the application in this paper.

2.1. PROPOSITION. *Let* $\mathscr{P}$ *be the disjoint union of* $\mathscr{P}_1, \ldots, \mathscr{P}_v$. *Suppose* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are two sets of lines in* $\mathscr{L}$ *such that for* $1 \leqq j \leqq v$, $|[P] \cap \mathscr{L}_1|$ *(resp.* $|[P] \cap \mathscr{L}_2|$ *) is a constant* $(\mathscr{L}_1\mathscr{P}_j)$ *(resp.* $(\mathscr{L}_2\mathscr{P}_j)$ *) for any point P of* $\mathscr{P}_j$. *Then*

$$\sum_{j=1}^{v} |\mathscr{P}_j| (\mathscr{L}_1\mathscr{P}_j)(\mathscr{L}_2\mathscr{P}_j) = |\mathscr{L}_1|\,|\mathscr{L}_2| + n|\mathscr{L}_1 \cap \mathscr{L}_2|.$$

Proof. Let M be the set of all functions from $\mathscr{P}$ to the integers. For f, $g \in M$, set

$$(f, g) = \sum_{X \in \mathscr{P}} f(X)g(X).$$

Let $l, h \in \mathscr{L}$. We identify a line in $\mathscr{L}$ with its characteristic function of the set of all points incident with it. For $i = 1, 2$, set

$$f_i = \sum_{f \in \mathscr{L}_i} f.$$

Thus for $1 \leqq j \leqq k$, and $X \in \mathscr{P}_j$, and $1 \leqq i \leqq 2$,

$$f_i(X) = (\mathscr{L}_i\mathscr{P}_j).$$

Hence

$$(f_1, f_2) = \sum_{j=1}^{v} |\mathscr{P}_j| (\mathscr{L}_1\mathscr{P}_j)(\mathscr{L}_2\mathscr{P}_j).$$

On the other hand

$$(f_1, f_2) = \sum_{f \in \mathscr{L}_1} \sum_{g \in \mathscr{L}_2} (f, g) = |\mathscr{L}_1|\,|\mathscr{L}_2| + n|\mathscr{L}_1 \cap \mathscr{L}_2|$$

as $(f, g) = 1$ when $f \neq g$ and $(f, g) = n + 1$ when $f = g$. This completes the proof of the proposition.

Let Δ be a tactical decomposition of π in the sense of [**2**], and let the point and line classes of Δ be numbered in an arbitrary but fixed way: $\mathscr{P}_1, \ldots, \mathscr{P}_v$ and $\mathscr{L}_1, \ldots, \mathscr{L}_l$. We define three integral matrices $B = (b_{ij})$, $C = (C_{ij})$ and $D = (d_{ij})$ by

$$b_{ij} = |\mathscr{L}_i|\,|\mathscr{L}_j|, \ c_{ij} = (\mathscr{P}_i\mathscr{L}_j), \ d_{ij} = (\mathscr{L}_i\mathscr{P}_j),$$

where $(\mathscr{P}_i\mathscr{L}_j)$ means the number of points of $\mathscr{P}_i$ on a line of $\mathscr{L}_j$, and where $(\mathscr{L}_i\mathscr{P}_j)$ is defined dually. By 2.1 we obtain the following:

$$D \operatorname{diag}(|\mathscr{P}_1|, \ldots, |\mathscr{P}_k|)D^t = nI_l + B,$$

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where I_l is the l by l identity matrix.

In this paper, the tactical decomposition formed by the point orbits and line orbits of G is used to yield information about G and π. For this tactical decomposition we call the square matrix D the G-incidence matrix of π and write $D(G)$ if we want to emphasize the dependence of the group G. For convenience, we call the row indexed by an orbit I of lines of G the I-row. For any two line orbits L, I of G let

$$[L|I] = \sum_{j=1}^{v} |\mathcal{P}_j|\,(L\mathcal{P}_j)(I\mathcal{P}_j).$$

Let π_s be a subplane of π. A *tangent* (resp. *exterior*) *line* of π_s is a line which is incident with exactly one (resp. no) point of π_s. Dually a *tangent* (resp. *exterior*) *point* of π_s is a point which is incident with exactly one (resp. no) line of π_s. For brevity we use t- for tangent and e- for exterior in the rest of this paper.

2.2. LEMMA. *A subplane of order m has* $(n - m)(m^2 + m + 1)$ *t-lines (resp. points), and* $(n - m)(n - m^2)$ *e-lines (resp. points). An e-line carries exactly* $m^2 + m + 1$ *t-points and* $n-(m + m^2)$ *e-points.*

Proof. This is clear from the definitions.

2.3. LEMMA. *Assume that* $\mathrm{Fix}(g) = \mathrm{Fix}(G)$ *is a subplane of order m for all* $g \neq 1$ *in G. Then the following conclusions hold.*

a) $|G|$ *divides* $n - m$.

b) *There are* $(m^2 + m + 1)(n - m)/|G|$ *orbits of t-lines (resp. points) and* $(n - m)(n - m^2)/|G|$ *orbits of e-lines (resp. points) of G, all of size* $|G|$. *Any other orbit of G has size* 1 *which consists of either a point or a line of* $\mathrm{Fix}(G)$.

c) *If L is an orbit of e-lines (resp. t-lines) of G, and J is an orbit of t-points (resp. e-points) of G, then* $(LJ) \leqq 1$. *If* $|G| = n - m$, *then* $(LJ) = 1$.

d) *Let L be an orbit of e-lines and J be an orbit of e-points of G. Then*

$$P(L) = \bigcup_{k=1}^{r} J_k$$

is a union of orbits of e-points and

$$L(J) = \bigcup_{t=1}^{s} L_t$$

is a union of orbits of e-lines. Also $(LJ) = (JL)$ *and*

$$\sum_{i=1}^{r} (LJ_k)((LJ_K) - 1) = |G| - 1.$$

Furthermore

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$$\sum_{k=1}^{r} (LJ_k) \leqq n - (m + m^2).$$

Proof. a) and b) follow from the fact that G acts fix-point-freely outside Fix(G).

c) Let L consist of e-lines and J consist of t-points. Then there exists a line h of Fix(G) such that h contains J. Let $l \in L$. The action of G on $l \cap h$ yields the desired result. The case that L consists of t-lines and J consists of e-points is proved similarly.

d) All conclusions, except the last, are consequences of a), b), c) and a simple counting incidence in $\mathscr{P}(L)$. The number of points of $\bigcup_{k=1}^{r} J_k$ on a line l of L is $\sum_{k=1}^{r} (LJ_k)$ as $(LJ_k) = (J_kL)$. Since J_k consists of e-points for $1 \leqq k \leqq r$, 2.2 implies that

$$\sum_{k=1}^{r} (LJ_k) \leqq n - (m + m^2).$$

2.4. Lemma. *Let* (H, Ω) *be a finite group space and let*

$$l = \text{l.c.m.}\{ |H_\alpha| : \alpha \in \Omega\}.$$

Then $|H| \mid |\Omega| l$.

Proof. Let P be a Sylow p-subgroup of G, and consider the group space (P, Ω). Among the orbits of P, choose $\Delta = \alpha^P$ such that $|\Delta|$ is smallest. Since Ω is the union of all distinct orbits of P, $|\Delta|$ divides $|\Omega|$. Hence $|P|$ divides $|\Omega|\,|P_\alpha|$ and so

$$|P| \mid |\Omega| l.$$

Since this is true for any prime divisor p of $|H|$, $|H|$ divides $|\Omega| l$ as desired.

2.5. Theorem (Hering). *Set*

$f_G = \text{l.c.m.}\{ |G_{ABCD}| \mid \{A, B, C, D\}$ *ranges over the quadrangles in* $\mathscr{P}\}$.

Suppose $\{A, B, C\}$ *is a triangle. Then*

a) $|G| \mid n^3(n - 1)^2(n + 1)(n^2 + n + 1)f_G$.

b) $|G_A| \mid n^3(n - 1)^2(n + 1)f_G$.

c) $|G_{AB}| \mid n^2(n - 1)^2 f_G$.

d) $|G_{ABC}| \mid (n - 1)^2 f_G$.

Proof. This is an application of 2.4 to various group spaces (H, Ω).

Let Ω be the set of all ordered quadrangles in $\mathscr{P}$ in case a), the set of all ordered quadrangles with first vertex A fixed in case b), the set of all ordered quadrangles whose first two vertices A, B are fixed in case c) and the set of all ordered quadrangles whose first three vertices A, B, C are fixed in case d). Then

$$|\Omega| = n^3(n - 1)^2(n + 1)(n^2 + n + 1),\ n^3(n - 1)^2(n + 1),\ n^2(n - 1)^2,\ (n - 1)^2$$

respectively in a), b), c) and d). Correspondingly let H be the groups G, G_A, G_{AB}, G_{ABC} in a), b), c) and d) respectively. Observe that in all cases $|H_\alpha|$ divides f_G. Now 2.4 implies the desired result.

2.6. Proposition. *If* $n = 11$, *then* $f_G|3$. *If* $n = 13$, *then* $f_G = 1$.

Proof. Let $1 \neq H \leqq G$ such that Fix(H) is a subplane of order m. Let $l \in \mathscr{L}(H)$ and let L be an orbit of e-lines of H. The possible values for m are 2 and 3. From this it is easy to see that for all $1 \neq h \in H$, $\mathrm{Fix}(h) = \mathrm{Fix}(H)$. Hence H acts semi-regularly on the $n - m$ points of l not in $\mathscr{P}(H)$.

Case 1. $n = 11$. Thus $m = 2$ and $|H|\,|9$. Suppose $|H| = 9$. By 2.3.d we infer that $P(L)$ is the union of 2 orbits of e-points of H, J_1 and J_2, such that $(LJ_1) = 2$ and $(LJ_2) = 3$. Since each e-line carries exactly 5 e-points, $(LE) = 0$ for the other orbits of e-points of E of H. By 2.3.d again $L(J_1)$ is the union of 3 orbits of e-lines L_1 and L_2, where we may assume without loss of generality that $(L_1J_1) = 3$. Applying the above argument to L_1 in place of L we obtain $(L_1J_2) = 0$ or 2. Hence $[L|L_1] = 9.13$ or 9.19 by combining the fact that $(LX) = 0$ for $X \in \mathscr{P}(H)$ with 2.3.c. However this contradicts 2.1. Therefore $f_G|3$ as desired.

Case 2. $n = 13$. In this case we get $m = 2$ and $|H| = 11$. By 2.3.d we get that $P(L)$ is the union of 3 H-orbits of e-points J_1, J_2, J_3 such that

$$(LJ_1) = (LJ_2) = 2 \quad \text{and} \quad (LJ_3) = 3.$$

Consider $L(J_3)$. From 2.3.d we infer that

$$L(J_3) = L \cup L_1 \cup L_2,$$

where L_1 and L_2 are H-orbits of e-lines and

$$(L_1J_3) = (L_2J_3) = 2.$$

Since an e-line carries exactly 7-points, $(LE) = 0$ for the other orbits of e-points. Clearly $(LX) = 0$ for $X \in \mathscr{P}(H)$. Hence

$$[L|L_1] = 11(2(L_1J_1) + 2(L_1J_2) + 6 + 7)$$

by 2.3.c. By 2.1, we get

$$2(\,(L_1J_1) + (L_1J_2)\,) \equiv 11 \pmod{13}.$$

Hence

$$(L_1J_1) + (L_1J_2) \equiv 12 \pmod{13}.$$

By 2.3.d we have $0 \leqq (L_1J_1), (L_1J_2) \leqq 3$. Hence

$$(L_1J_1) + (L_1J_2) \leqq 9$$

and so cannot be congruent to 12 modulo 13. This contradiction implies $H = 1$ and $f_G = 1$ as desired. The proof of the lemma is complete.

In calculating f_G, it seems worthwhile to record the following result dealing with $n - (m^2 + m)$ being not too big, where m is the order of a proper subplane.

2.7. LEMMA. *Suppose* $\mathrm{Fix}(G) = \mathrm{Fix}(g)$ *for all* $g \neq 1$ *in* G *and* $\mathrm{Fix}(G)$ *is a proper subplane of order* m *such that* $n \neq m^2$. *Let* L *be a* G-*orbit of e-lines with*

$$P(L) = \bigcup_{k=1}^{r} J_k.$$

Set $d = n - (m^2 + m)$. *Then* $2 \leqq d$ *and for* $d \leqq 7$ *we have the following table. For a fixed* d *we list the possibilities for* r, *for each* r *we list the only possibilities for the* (LJ_k), $1 \leqq k \leqq r$, *and for a possible* $\{(LJ_k) \mid 1 \leqq k \leqq r\}$ *the corresponding* $|G|$ *is given. The last column gives the unique solution when* $|G| = n - m$. *Also* $|G|^*$ *means* $|G| > n - m$. *In particular,* $3 \leqq d$ *if* n *is odd.*

d	r	$\Gamma(L, J_1)$	$\Gamma(L, J_2)$	$\Gamma(L, J_3)$	$\lvert G \rvert$	n	When $\lvert G \rvert = n - m$
2	1	2	–	–	3*	even	–
3	1	2	–	–	3*	$n \equiv 0 \pmod 3$	–
		3	–	–	7	$n \equiv \pm 2 \pmod 7$	–
4	1	4	–	–	13	$n \equiv \pm 3 \pmod{13}$	$m = 3, n = 16$
	2	2	2	–	5*	$n = \pm 1 \pmod 5$	–
5	1	2	–	–	3*	$n \not\equiv 0 \pmod 3$	–
		3	–	–	7*	$n \equiv \pm 4 \pmod 7$	–
		5	–	–	21	$n \not\equiv 0 \pmod 3$ and $n \equiv \pm 4 \pmod 7$	$m = 4, n = 25$
	2	2	2	–	5*	$n \equiv 0 \pmod 5$	–
		2	3	–	9*	$n \equiv \pm 2 \pmod 9$	–
6	1	2	–	–	3*	$n \equiv 0 \pmod 3$	–
		3	–	–	7*	$n \equiv \pm 1 \pmod 7$	–
		5			21*	$n \equiv 0 \pmod 3$ and $n \equiv \pm 1 \pmod 7$	
		6	–	–	31	$n \equiv \pm 5 \pmod{31}$	$m = 5, n = 36$

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	2	2	2	–	5*	$n \equiv \pm 2 \pmod 5$	–
		2	3	–	9*	$n \equiv 0 \pmod 3$	–
		2	4	–	15	$n \equiv 0 \pmod 3$ and $n \equiv \pm 2 \pmod 5$	$m = 3, n = 18$
	3	2	2	2	7*	$n \equiv \pm 1 \pmod 7$	–
7	1	3	–	–	7*	$n \equiv 0 \pmod 7$	–
		7	–	–	43*	$n \equiv \pm 6 \pmod{43}$	–
	2	2	5	–	23	$n \equiv \pm 4 \pmod{23}$	$m = 4, n = 27$
	3	2	2	2	7*	$n \equiv 0 \pmod 7$	–
		2	2	3	11*	$n \equiv \pm 2 \pmod{13}$	–

Proof. By 2.3.d we get $3 \leqq d$. The inequality

$$\sum_{k=1}^{r} (LJ_r) \leqq d$$

provides possible possibilities for r and (LJ_k), $k = 1, \ldots, r$. The equality

$$\sum_{k=1}^{r} (LJ_i)((LJ_k) - 1) = |G| - 1$$

now yields the corresponding $|G|$. We now eliminate the cases not mentioned in 2.7.

Since $n \equiv m \pmod{|G|}$ by 2.3.a, $n = m^2 + m + d$ implies that

$$-d \equiv n^2 \pmod{|G|}.$$

In particular $-d \equiv n^2 \pmod q$ for any prime divisor q of $|G|$. This eliminates the possibilities not listed in the table by a direct calculation with the help of the quadratic reciprocity law. The only eliminated cases with $|G|$ not a prime are $d = 7$, $r = 1$, $(LJ_1) = 5$, $|G| = 21$ and $d = 7$, $r = 2$, $(LJ_1) = 2$, $(LJ_2) = 4$, $|G| = 21$. In both cases we use $q = 3$ and $\left(\frac{-7}{3}\right) = -1$.

If $|G| = n - m$, then $n = m^2 + m + d$ implies that $m^2 = |G| - d$. This enables us to put * on $|G|$ as shown in the table except in the following cases.

(1) $d = 3, r = 1, |G| = 7, m = 2$ and $n = 9$.

(2) $d = 5, r = 2, |G| = 9, m = 2$, and $n = 11$.

(3) $d = 7, r = 1, |G| = 43, m = 6$.

(4) $d = 7, r = 3, |G| = 11, m = 2$, and $n = 13$.

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Proposition 2.6 eliminates cases (2) and (4). Case (3) is eliminated by the fact that 6 cannot be the order of a projective plane [**15**].

Since $m^2 + m + 2$ is even, n is even when $d = 2$. The rest of the congruences for n come from $-d \equiv n^2 \pmod{q}$, where q is a prime divisor of $|G|$. The information in the last column comes from solving $m^2 = |G| - d$ and then using $n = m^2 + m + d$. The proof of the lemma is complete.

We record some known results in the following for the convenience of the reader.

2.8. THEOREM ([**7**]). *Suppose G does not leave invariant any point, line or triangle. Assume that G contains an abelian normal subgroup M. Then* $\mathrm{Fix}(M)$ *is a subplane or is* (ϕ, ϕ). *Furthermore each element in M is planar or triangular or regular.*

2.9. THEOREM ([**7**]). *Suppose G acts strongly irreducibly on π and let M be a minimal normal subgroup of G. If M is solvable, then one of the following holds.*

a) *Each element of M is regular or planar.*

b) $M \cong \mathbf{Z}_3 \times \mathbf{Z}_3$ *and* $C_G(M) = M$. *Either each subgroup of M is triangular, or M contains 2 triangular and 2 planar subgroups of order 3. G has even order.*

Furthermore, if G contains a non-trivial perspectivity, then there is a unique minimal normal subgroup of G. This subgroup is either non abelian simple or isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$.

2.10. THEOREM. ([**10**]). *Notation as in 2.9. If M is isomorphic to a simple Chevalley group of type A_2 or of rank 1, then one of the following holds:*

a) *π is Desarguesian.*

b) $M \simeq PSL(2, p^r)$, *where p is an odd prime and each non-trivial perspectivity of G is an involutory homology.*

c) $M \simeq PSU(3, q)$, *and either each non-trivial perspectivity of G is a homology or each non-trivial perspectivity of G is an involutory elation.*

Furthermore in the case $M \simeq PSL(3, q)$, *π is Desarguesian of order q except possibly when $q = 2$ and* $G \simeq PSL(3, 2) \simeq PSL(2, 7)$ *or* $G \simeq PGL(2, 7)$.

The following result due to Hering [**7**] is applied frequently in this paper.

2.10 LEMMA. *Suppose that α, β are perspectivities. Then $\alpha\beta$ is a generalized perspectivity or trivial. In particular $\alpha\beta$ is not planar.*

2.11. THEOREM ([**13**]). *Let* $G \leqq \mathrm{Aut}(\mathscr{P}, \mathscr{L})$. *Assume that G satisfies the following conditions.*

a) *G contains involutory homologies with distinct centers and involutory homologies with distinct axes.*

b) *Each involution of G is a homology.*
c) $Z(G/O(G)) = 1$.
Then $G/O(G)$ is isomorphic to one of the following groups:
i) *a subgroup of* $P\Gamma L(2, q)$ *containing* $PSL(2, q)$, *q odd.*
ii) *a subgroup of* $P\Gamma L(3, q)$ *containing* $PSL(3, q)$, *q odd.*
iii) *a subgroup of* $P\Gamma U(3, q)$ *containing* $PSU(3, q)$, *q odd.*
iv) A_7
v) M_{11}
vi) $PSU(3, 4)$
vii) *a subgroup of* $P\Gamma L(2, 2^e)$, $\mathrm{Aut}(Sz(2^e))$, *or* $P\Gamma U(3, 2^e)$, $e \geqq 3$ *containing, respectively,* $PSL(2, 2^e)$, $Sz(2^e)$ *or* $PSU(3, 2^e)$. *In this case commuting involutions have the same center and the same axis.*

3. Some general results.

3.1. LEMMA. *Suppose G leaves invariant a subplane π' of order m and G does not contain any Baer involution. If $n \not\equiv m$* (mod 2), *then $|G|$ is odd.*

Proof. Since G does not contain any Baer involution, the kernel of the action of G on π' is of odd order.

Suppose $|G|$ is even. Let α be an involution and $\bar{\alpha}$ its action on π'. Clearly $\bar{\alpha}$ cannot be a Baer involution. Assume that $\bar{\alpha}$ is a homology. Then α fixes at least 3 lines incident with the center of $\bar{\alpha}$. This shows that α is also a homology. As α acts fix-point-freely on the points on a line through its center not in π', we get $n \equiv m$ (mod 2), a contradiction. Assume now that $\bar{\alpha}$ is an elation. This implies that α is an elation and so $2|n$ and $2|m$. Hence $n \equiv m$ (mod 2), a contradiction. This completes our proof.

3.2. THEOREM. *Suppose G acts strongly irreducibly on π. Let M be a minimal normal subgroup of G. Then the following holds.*
a) *If* $M \cong \mathbf{Z}_3 \times \mathbf{Z}_3$, *then n is a square,* $n \equiv 1$ (mod 3), *and* $\mathrm{Fix}(M) = (\phi, \phi)$.
b) *If G contains non trivial perspectivity and n is odd but not a square, then M is the unique minimal normal subgroup, M is non-abelian simple, and M acts strongly irreducibly on π.*

Proof. a) Since M is normal in G, and G is strongly irreducible we get $\mathrm{Fix}(M) = (\phi, \phi)$ by 2.8.

We now show $n \equiv 1$ (mod 3). If M contains a regular or triangular element, then

$$n^2 + n + 1 \equiv 0 \pmod 3$$

and so $n \equiv 1$ (mod 3) in this case. By 2.9 we may assume that each element in M is planar. Let $M = \langle g, h \rangle$. Since $\mathrm{Fix}(M) = (\phi, \phi)$, h acts fix-point-freely on $\mathrm{Fix}(g)$ which is a subplane of order m. This implies $m \equiv 1$ (mod 3). Let l be a line of $\mathrm{Fix}(g)$. Then g acts fix-point-freely on the points of l not in Fix(g). This implies that $3|n - m$. Hence

$n \equiv m \equiv 1 \pmod 3$.

Finally suppose n is not a square. Assume that there is an involution $\sigma \in C_G(M)$. Then σ is a perspectivity. Hence M fixes at least a point and a line, which contradicts Fix(M) $= (\phi, \phi)$. Therefore $|C_G(M|$ is odd. Since G induces an irreducible linear group on M, there is an involution α of G which inverts each element of M. As n is not a square this implies that no subgroup of order 3 of M is planar or regular. This contradicts 2.9. Therefore n is a square.

b) By a) and 2.9 we see that M is the unique minimal normal subgroup, M is non-abelian simple, and Fix(M) $= (\phi, \phi)$. Since M is non-abelian simple, M does not leave invariant any triangle. Suppose M leaves invariant a subplane π'. By 3.1 we see that π' has odd order. Since n is not a square, all involutions of M are homologies. Since π' has odd order, the center and axis of any involution of M belongs to π'. The substructure generated by the involutory centers and axes of M is inside π' and is G-invariant. Since G is strongly irreducible, $\pi' = \pi$ and M is strongly irreducible as desired.

3.3. THEOREM. *Suppose G is a collineation group of the projective plane π of prime order p. Then the following conclusions hold.*

a) *Any element of order p in G is either an elation or a flag collineation.*

b) *A Sylow p-subgroup of G is isomorphic to a subgroup of the non-abelian p-group of order p^3 with exponent p, except in the case $p = 2$.*

c) *If p^2 divides $|G|$, then G contains a non-trivial elation.*

d) *If p^3 divides $|G|$, then π is Desarguesian.*

e) *If G contains a non-trivial elation then π is Desarguesian or the subgroup generated by elations is a normal subgroup of order p. In particular π is Desarguesian if* Fix(G) $= (\phi, \phi)$.

Proof. a) This follows from the fact that there is no planar collineation of order p and a direct counting argument.

In proving b), c) and d) we may assume that G is a p-group. By a) we see from 2.8 that $|G|$ divides p^3. Suppose σ is an element of order p^2. Then $\langle\sigma\rangle$ acts transitively on the points outside the axis or the axis of the flag according to σ^p is an elation or a flag collineation. In either case a result of Hoffman [**2**, article 6 of p. 210] implies that $p = 2$.

b) We may assume that the exponent of G is p and p is odd. Suppose G is elementary abelian of order p^3. Then G contains a flag collineation τ. The action of G on the points of the fix-line of τ not equal to $\mathscr{P}(\tau)$ shows that G contains a subgroup of order p^2 which consists of elations with center $\mathscr{P}(\tau)$ and axis $\mathscr{L}(\tau)$. However this implies that p^2 divides $n = p$, a contradiction. The proof of b) is complete.

c) This is a consequence of a) and b) and a direct counting argument.

d) Using a) and b) and an easy counting argument we see that the center of G consists of elations with the same center and the same axis. The action of G on the points of this axis shows that G contains a subgroup of elations of order p^2 with the same axis. Therefore π is a translation plane of order p and so is Desarguesian.

e) Let σ and τ be two non-trivial elations such that $\langle\sigma\rangle \neq \langle\tau\rangle$. If $\mathscr{C}(\tau) = \mathscr{C}(\sigma)$ or $a(\sigma) = a(\tau)$ then π will be a translation plane and so is a Desarguesian plane. Hence we may assume that

$$\mathscr{C}(\tau) \neq \mathscr{C}(\sigma) \text{ and } a(\tau) \neq a(\sigma).$$

If $\mathscr{C}(\tau) \in a(\sigma)$ then the action of $\langle\sigma\rangle$ on $[\mathscr{C}(\tau)]$ shows that π is a Desarguesian plane again. Therefore we may assume that $\mathscr{C}(\tau) \notin a(\sigma)$ and similarly $\mathscr{C}(\sigma) \notin a(\sigma)$ and similarly $\mathscr{C}(\sigma) \in a(\sigma)$. This implies that $\langle\sigma, \tau\rangle$ induces a 2-transitive group H on the points of $\mathscr{C}(\sigma)\mathscr{C}(\tau) = l$.

Assume now that π is not Desarguesian. Then the Sylow p-subgroup of G has order at most p^2 by d) and $p > 7$. If $\langle\sigma\rangle$ is not normal in the stabilizer of $\mathscr{C}(\sigma)$ of H then π is a translation plane and so is Desarguesian. Let K be the kernel of $\langle\sigma, \tau\rangle$ on the points of l. Then K consists of homologies with center $a(\sigma) \cap a(\tau)$ and axis $\mathscr{C}(\sigma)\mathscr{C}(\tau)$, and K lies in the center of $\langle\sigma, \tau\rangle$. Suppose H is solvable and sharply 2-transitive. Then $H = O_2(H)\langle\sigma\rangle$, where $O_2(H)$ is elementary abelian of order $p + 1$. Since $O_2(H)$ acts regularly on l, there exists $h \in \langle\sigma, \tau\rangle$ such that $1 \neq h^2 \in K$. Since $\langle\sigma\rangle$ acts transitively on $O_2(H) - \{1\}$,

$$\langle h^x | x \in \langle\sigma\rangle \rangle / \langle h^2\rangle = O_2(H).$$

Thus $O_2(H)$ can be viewed as a vector space with a nondegenerate quadratic form and $\langle\sigma\rangle$ belongs to its orthogonal group. However from the order formula the only possible case is $p = 3$, a contradiction as $p \geqq 7$. Therefore $H \cong PSL(2, p)$ by [**3**, p. 78, Theorem 9.1.1]. So $\langle\sigma, \tau\rangle \cong SL(2, p)$, or $PSL(2, p)$. In both cases π is Desarguesian [**2**, article 13, p. 184; and article 15, p. 186]. Therefore all elations belong to a subgroup of order p and the proof is complete.

3.4. THEOREM. *Suppose* $G \simeq L_2(q)$, *and* $\text{Fix}(G) = (\phi, \phi)$, *and* G *contains an involutory perspectivity. Then the following conclusions hold.*

a) *Let* $i \neq j$ *be two involutions of* G. *Each of the following conditions will imply that* $\mathscr{C}(i) \neq \mathscr{C}(j)$ *and* $a(i) \neq a(j)$.

1) $ij \neq ji$;

2) $q = 5$;

3) $q \equiv \pm 1 \pmod 8$

and G *contains an involutory homology.*

b) *If* G *contains an elation, then the order of any elation of* G *is a power of* 2. *In particular* $n = 2$ *or* $n \equiv 0 \pmod 4$.

c) *If* $q = 7$ (*resp.* 9) *and* G *contains an elation, then there is a subplane of order* 2 (*resp.* 4) *which is invariant under* G.

Proof. Since G contains one conjugate class of involutions, all involutions of G are either all homologies or all elations. Let i, j be two involutions of G, and let $H = \langle i, j \rangle$. First we prove a.1, a.2.

a) Suppose $a(i) = a(j) = x$. Assume $ij \neq ji$. Suppose that a prime divisor p of the order of ij divides q. Then G_x contains a Sylow p-subgroup of G. Since G_x contains $C_G(\sigma)$ for any involution σ in H and $C_G(\sigma)$ contains a Sylow 2-subgroup of G, $G_x = G$. This contradicts $\mathrm{Fix}(G) = (\phi, \phi)$. Therefore $H \subseteq K \simeq D_{2s}$, where

$$s = (q \pm 1)/k \quad \text{and} \quad k = (q - 1, 2).$$

Let $1 \neq d \in K$ such that $d^s = 1$. Since d centralizes ij, $x^d = x$. Hence G_x contains $C_G(\sigma)$ for any involution in

$$L = \langle H^y | y \in \langle d \rangle \rangle.$$

Suppose $s > 6$. Then G_x contains at least 5 Sylow 2-subgroups of G. As $s > 6$, $G_x \not\simeq A_5$. Hence $G_x \simeq L_2(q')$ with $q'|q$ by Dickson's theorem. Since $q - 1 \leqq ks \leqq q' + 1$, $q \leqq q' + 2$. This forces $q = 4$ which contradicts $s > 6$. Therefore $s \leqq 6$ and $q \leqq 2s + 1 = 13$. Suppose $q = 3$. Since $ij \neq ji$, $s = 3$. Thus G_x contains at least 3 Sylow 2-subgroups of G. Hence $G = G_x$, a contradiction. Suppose $q = 5$. Then $s = 3$ or 5. Since G can be generated by two conjugates of H, G will leave invariant a point or a line, a contradiction.

Suppose q is even. Then $q \leqq s + 1 = 7$ which forces $q = 2$ or 4. If $q = 2$, then $G = H \subseteq G_x$, a contradiction. Since $L_2(4) \simeq L_2(5)$, $q \neq 4$. Therefore q is odd. If $s = 6$, then $q = 11$ or 13. If $s = 4$, then $q = 7$ or 9. In both cases G is generated by two conjugates of H which implies that G leaves invariant a point or a line, a contradiction. If $s = 5$, then $q = 9$ or 11. Since G_x contains at least 5 Sylow 2-subgroups, $G_x \simeq L_2(m)$ with $m \geqq 4$. Hence G_x is simple and so x is a common axis for G_x. A contradiction follows from the fact that G is generated by two conjugates of G_x. If $s = 3$, then $q = 7$ as $q \neq 5$. However this implies that $G = L \subseteq G_x$, a contradiction. Since $ij \neq ji$, the last contradiction shows that $a(i) \neq a(j)$.

In the case $q = 5$ we only need to treat the situation $ij = ji$, which is quite clear. A similar argument treats case $\mathscr{C}(i) \neq \mathscr{C}(j)$. The proof of a.1, a.2 is complete.

b) The last conclusion of this part follows from a result of Hughes [**11**, p. 268]. By way of contradiction, assume that there exists an elation $1 \neq \sigma$ of prime order r. If $r|q$, then a Sylow r-subgroup of G has a common axis or a common center. Since G is generated by 2 Sylow r-subgroups, G leaves invariant a point or a line, a contradiction. Hence r does not divide q. Hence there exist involutions α, β of G such that $\alpha\beta = \sigma \neq \beta\alpha$. Part a) implies that $\mathscr{C}(\alpha) \neq \mathscr{C}(\beta)$ and $a(\alpha) \neq a(\beta)$. Hence $\mathscr{P}(\sigma)$ is contained in

$$\{a(\alpha) \cap a(\beta)\} \cup (\mathscr{C}(\alpha)\mathscr{C}(\beta)).$$

Since σ is an elation,

$$a(\sigma) = \mathscr{C}(\alpha)\mathscr{C}(\beta).$$

However this implies that

$$\mathscr{C}(\beta) \in \mathscr{C}(\alpha)^{\langle\sigma\rangle} = \mathscr{C}(\alpha)$$

which contradicts $\mathscr{C}(\beta) \neq \mathscr{C}(\alpha)$. The proof of part b) is complete.

We now prove a.3. It suffices to treat the case $ij = ji$. Since G contains an involutorial homology, G does not contain any elation by b). Assume $a(i) = a(j) = x$. Let S be a Sylow 2-subgroup of G containing H. Then $S = \langle i, t\rangle \simeq D_8$, where $t^2 = 1$. By Andre's theorem, $\mathscr{C}(i) = \mathscr{C}(j) = \mathscr{C}(u)$ where u is the central involution of S. Suppose $\mathscr{C}(u) \neq \mathscr{C}(t)$. Then $a(t) \neq x$ by Andre's theorem again. This implies that the homology ut has axis $\mathscr{C}(t)\mathscr{C}(u)$. Since $ut = t^i$, ut should have axis $a(t)^i$. However $\mathscr{C}(i) = \mathscr{C}(u)$ is on $a(t)$. Hence ut has axis

$$a(t)^i = a(t) \neq \mathscr{C}(t)\mathscr{C}(u)$$

as t is a homology. This contradiction shows that $\mathscr{C}(u) = \mathscr{C}(t)$ and so $a(u) = a(t)$. Thus S has a common center and a common axis. Since G_x contains $C_G(\sigma)$ for $1 \neq \sigma \in S$, G_x contains at least 5 Sylow 2-subgroups of order 8. Hence $G_x = G$ by Dickson's theorem which contradicts $\text{Fix}(G) = (\phi, \phi)$. A similar argument treats the case $\mathscr{C}(i) = \mathscr{C}(j)$. The proof of a.3 is complete.

c) Let S be a Sylow 2-subgroup of G. Then $S = \langle\alpha, \beta\rangle \simeq D_8$, where $\alpha^2 = \beta^2 = 1$. Let u be the central involution of S. Suppose $a(\alpha) = a(u) = x$. Since $\text{Fix}(G) = (\phi, \phi)$ and G is generated by two conjugates of S, $a(\beta) \neq x$. Since $x^\beta = x$, $\mathscr{C}(\beta) \in x$. Since $a(\beta)^u = a(\beta)$, $\mathscr{C}(u) \in a(\beta)$. As all involutions are elations, this implies $\mathscr{C}(\beta) = \mathscr{C}(u)$. If $\mathscr{C}(\alpha) = \mathscr{C}(u)$, then $\mathscr{C}(u)$ will be a common center of S which contradicts $\text{Fix}(G) = (\phi, \phi)$. Hence $\mathscr{C}(\alpha) \neq \mathscr{C}(u)$. Since $\text{Fix}(G) = (\phi, \phi)$ and $G_{\mathscr{C}(u)}$ contains $N_G(\langle u, \beta\rangle)$ and G_x contains $N_G\langle\alpha, u\rangle$,

$$G_{\mathscr{C}(u)} \simeq G_x \simeq S_4.$$

As all involutions are conjugate, this shows that there are exactly 3 involutory centres on an involutory axis and exactly 3 involutory axes through an involutorial center. Suppose $q = 7$. Then

$$|\mathscr{C}(u)^G| = 168/24 = 7 = |x^G|.$$

It is easy to see that {involutory centers, involutory axes} forms a subplane of order 2 which is G invariant.

Assume $q = 9$. Then

$$|\mathscr{C}(u)^G| = |x^G| = 15.$$

Consider

$$\{l \cap m | l, m \in x^G\} = \bigcup_{s=1}^{v} O_s,$$

where O_s is an orbit of G for $s = 1, \dots, v$. This yields

$$\binom{15}{2} = \sum_{s=1}^{v} |O_s| \binom{k_s}{2},$$

where k_s is the number of lines in x^G through a point in O_s. Without loss of generality we may assume that $O_1 = \mathscr{C}(\alpha)^G$. Hence $k_1 = 3$. There is a subgroup N isomorphic to D_{10}. By b) all 5 involutions have different axes and different centers. The 5 axes meet at a point D and the 5 centers lie on a line d. Thus $D \notin O_1$ and $d \notin x^d$. Let $O_2 = D^G$. Then $k_2 \geqq 5$. Since $G \simeq A_6$, $|O_2| \geqq 6$. Hence

$$\binom{15}{2} \geqq 15 \binom{3}{2} + 6 \binom{5}{2} + \sum |O_s| \binom{k_s}{2},$$

which implies $v = 2$, $|O_2| = 6$, and $k_2 = 5$. Therefore

$$\{l \cup m | l, m \in x^G\} = O_1 \cup O_2.$$

Similarly

$$\{XY | X, Y \in O_1\} = x^G \cup d^G,$$

where $|d^G| = 6$ and there are exactly 5 involutory centers on a line of d^G.

We claim that $\{O_1 \cup O_2, x^G \cup d^G\}$ is a subplane of order 4. Since an involution has exactly 2 fixed points on O_2, the 4-transitivity of G on O_2 implies that any points of O_2 are joined by a line in x^G (and no 3 points of O_2 are collinear). Let $A \in O_2$. Then there are 5 lines in x^G passing through A. Each of these lines carry 3 points of O_1. This shows that any point in O_1 is joined to A by a line in x^G, and there exists a quadrangle of the substructure. Since

$$\{XY | X, Y \in O_1\} = x^G \cup d^G,$$

every two distinct points in $O_1 \cup O_2$ are joined by a line in $x^G \cup d^G$. Similarly every two lines in $x^G \cup d^G$ meet at a point in $O_1 \cup O_2$. It is now easy to see that this substructure is a subplane of order 4 invariant under G.

A similar argument treats the situation $\mathscr{C}(\alpha) = \mathscr{C}(\gamma)$. The proof of c) is complete.

4. Projective planes of order 11. In this section we assume that $n = 11$. Hence any proper subplane has order 2 if it exists. By 2.2 there are 63 t-lines (resp. t-points) and 63 e-lines (resp. e-points) of a proper subplane.

4.1. LEMMA. a) $|G|$ *divides* $2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$.

b) *If* $f_G = 3$, *then* G *has a normal* 3-*complement.*

Proof. a) follows from 2.6 and 2.8. In proving b) assume $f_G = 3$ and let $P \in Syl_3(G)$.

We claim that $P \leqq Z(N_G(P))$. Let H be a subgroup of order 3 of P such that Fix(H) is a subplane of order 2. If $N_G(P) \leqq N_G(H)$, then our claim follows from 3.1. Therefore we may assume that a distinct conjugate S of H lies in P. Since $f_G = 3$,

$$\text{Fix}(S) \neq \text{Fix}(H) \quad \text{and}$$

$$\text{Fix}(P) = \text{Fix}(S) \cap \text{Fix}(H) = \{A, l\} \quad \text{with } A \notin l.$$

Since P cannot act transitively on the 6 points of l not in $\mathcal{P}(H)$ or $\mathcal{P}(S)$, there is an orbit Ω of P of these points of size not bigger than 3. As S and H act semi-regularly on Ω we get $|\Omega| = 3$. Let K be the kernel of the action of P on Ω. Since K fixes A, Fix(K) is a subplane. Another similar argument shows that all 4 proper subgroups of P are planar. By 3.1 we have established our claim. The desired result now follows from Burnside's theorem [**6** or **12**].

4.2. LEMMA. a) *Any collineation of order* 3 *is either planar or anti-flag.*

b) *Any element of order* 5 *is a generalized homology of type* $D(2)$, $D(7)$ *or a homology.*

c) *Any element of order* 11 *is either an elation or a flag.*

d) *Any element of order* 7 *or* 9 *is regular.*

Proof. By 2.6 there is no planar collineation of prime order greater than 3. The lemma follows by inspection of the possible substructure of fixed points and lines.

4.3. LEMMA. *Suppose* $|G| = 21$ *and G leaves invariant a subplane* π_1. *Then any element of order* 3 *fixes at least one point outside this subplane.*

Proof. Suppose the lemma is false. Hence G acts semi-regularly outside π'. There are 7 orbits of points O_j, $j = 1, \dots, 7$ of G which we arrange in the following way: O_1 is the set of points of π', O_2, O_3, O_4 are orbits of t-points and O_5, O_6, O_7 are orbits of e-points. Thus $|O_1| = 7$ and $|O_j| = 21$ for $j \neq 1$.

We now consider the G-incidence matrix Γ introduced in Section 2. Let ϵ be a G-orbit of e-lines. Then we have

$$(1) \quad \sum_{j=2}^{4} (\epsilon O_j) = 7 \quad \text{and} \quad \sum_{j=5}^{7} (\epsilon O_j) = 5.$$

As $(\epsilon O_1) = 0$, 2.3(d) implies

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$$(2)\quad 20 = \sum_{j=2}^{7} (\epsilon O_j)((\epsilon O_j) - 1).$$

Thus $(\epsilon O_i) \leqq 5$ for $i \neq 1$ by (2). If $(\epsilon, O_i) = 5$ for some $i \neq 1$, then (O_i, ϵ) is a projective plane of order 4 which implies $4^2 + 4 \leqq 11$, a contradiction. Hence $(\epsilon O_i) \leqq 4$ for $i \neq 1$. By using (1), (2) and a direct calculation we get

$$(3)\quad \min\{ (\epsilon O_i), i = 2, 3, 4\} = 1.$$

From this we infer by (1) and (2) that the ϵ-row has 2 possible types: (0421311) or (0331320). We call these type I and type II respectively.

First assume that there exists an ϵ_1-row of type I. Without loss of generality we may assume that the ϵ_1-row is (0421311). Let the other 2 rows indexed by orbits of e-lines be ϵ_2, ϵ_3. Set

$$\Gamma(\epsilon_i, O_j) = \epsilon_{i,j} \quad \text{for } 1 \leqq i \leqq 3 \text{ and } 1 \leqq j \leqq 7.$$

Thus

$$\sum_{i=1}^{3} \epsilon_{i,2} = 7.$$

This together with (3) enables us to assume, by interchanging ϵ_3 and ϵ_2, if necessary, that $\epsilon_{2,2} = 2$ and $\epsilon_{3,2} = 1$. Hence the ϵ_2-row is of type I. Using

$$\sum_{k=1}^{3} \epsilon_{k,j} = 7 = \sum_{s=1}^{3} \epsilon_{i,s} \quad \text{for } 1 \leqq i \leqq 3, 2 \leqq j \leqq 4$$

we get

$$\epsilon_{2,3} = 1, \epsilon_{2,4} = 4, \epsilon_{3,3} \quad \text{and} \quad \epsilon_{3,4} = 2.$$

Since

$$\sum_{i=1}^{3} \epsilon_{i,5} = 5,$$

$\epsilon_{2,5} = 1 = \epsilon_{3,5}$. Let the 3 orbits of t-lines be τ_1, τ_2, τ_3. Set

$$(\tau_i O_j) = \tau_{i,j} \quad \text{for } 1 \leqq i \leqq 3 \text{ and } 1 \leqq j \leqq 7.$$

Consider $L(O_5)$. Since no e-point is incident with any line of π' by 2.1 we get that

$$20 = \sum_{i=1}^{3} (\epsilon_{i,5}(\epsilon_{i,5} - 1) + \tau_{i,5}(\tau_{i,5} - 1)).$$

This together with

$$\sum_{i=1}^{3} \tau_{i,5} = 7,\ \epsilon_{1,5} = 3, \text{ and } \epsilon_{2,5} = 1 = \epsilon_{3,5}$$

enables us to assume, by renaming τ_1, τ_2, τ_3 if necessary, that

$$\tau_{1,5} = 4,\ \tau_{2,5} = 2,\ \tau_{3,5} = 1.$$

Applying similar arguments to the columns of $(\tau_{i,j}|1 \leqq i \leqq 3, 5 \leqq j \leqq 7)$ in place of $(\epsilon_{i,j}|1 \leqq i \leqq 3, 2 \leqq j \leqq 4)$, we can assume that

$$\tau_{1,6} = 2,\ \tau_{2,6} = 1,\ \tau_{3,6} = 4,\ \tau_{1,7} = 1,\ \tau_{2,7} = 4 \text{ and } \tau_{3,7} = 2.$$

Consider $P(\tau_1)$. By 2.3(d) and the known values of $\tau_{1,j}$, we get

(4) $$6 = \sum_{j=2}^{4} \tau_{1,j}(\tau_{1,j} - 1).$$

Since each t-line carries exactly 4 t-points, we get

(5) $$4 = \sum_{j=2}^{4} \tau_{1,j}.$$

By 2.1 we get

$$4\tau_{1,2} + 2\tau_{1,3} + \tau_{1,4} \equiv 6 \pmod{11}.$$

However a direct calculation shows that the above equation has no non-negative integral solution subject to (4) and (5).

Therefore all ϵ-rows are of type II. Let ϵ_i, τ_i $1 \leqq i \leqq 3$ denote respectively the 3 orbits of e-lines and t-lines. Without loss of generality we may assume that ϵ_1-row is (0331320). Set

$$(\epsilon_i O_j) = \epsilon_{i,j} \text{ and } (\tau_i O_j) = \tau_{i,j} \quad \text{for } 1 \leqq i \leqq 3 \text{ and } 1 \leqq j \leqq 7.$$

Thus

$$\{\epsilon_{i,j}|2 \leqq j \leqq 4\} = \{3, 3, 1\} \text{ for } 1 \leqq i \leqq 3.$$

Since

$$\sum_{i=1}^{3} \epsilon_{i,2} = 7$$

we get

$$\{\epsilon_{i,2}|1 \leqq i \leqq 3\} = \{3, 3, 1\}.$$

Consider $L(O_2)$. Since a t-point is incident with exactly one line of the subplane, 2.1 and

$$\{\epsilon_{i,2} | 1 \leqq i \leqq 3\} = \{3, 3, 1\}$$

imply that

$$\sum_{i=1}^{3} \tau_{i,2}(\tau_{i,2} - 1) = 8.$$

As a t-point is incident with exactly 4 t-lines, we obtain

$$\sum_{i=1}^{3} \tau_{i,2} = 4.$$

However no non-negative integral solution exists subject to the last 2 equations. This contradiction completes the proof of our lemma.

4.4. Lemma. *If* $|G| = 21$, *then* G *is not abelian and* G *does not leave invariant any proper subplane.*

Proof. Let H be a Sylow 3-subgroup of G. If G is abelian, then $\text{Fix}(H)$ is invariant under G. However a contradiction is reached by 4.2.a, 4.2.d and 4.3. Therefore G is not abelian.

Suppose G leaves invariant a proper subplane Ω. Since H has to fix some point outside Ω by 4.3, $\text{Fix}(H)$ is a subplane of order 2 by 4.2.a. Since G is not abelian, $\text{Fix}(H)^G$ consists of 7 disjoint subplanes and each one of the subplanes intersects Ω in exactly one point and one line such that the point is not on the line. Let

$$(P, l) = \Omega \cap \text{Fix}(H).$$

Since G acts semi-regularly on the points not in any one of the subplanes, these points form 4 G-orbits of size 21. There are 3 lines t_1, t_2, t_3 of $\mathscr{L}(H)$ in $[P]$. For $1 \leqq j \leqq 3$ let $O_j = (l \cap t_j)^G$ and O_{j+5} be the G-orbit of points containing the fixed point of H on t_j not equal to P or $l \cap t_j$. All these orbits have size 7.

There are 5 G-orbits of t-points: O_1, O_2, O_3, O_4, O_5 where $|O_4| = |O_5| = 21$ and 5 G-orbits of e-points: O_6, O_7, O_8, O_9, O_{10}, where $|O_9| = |O_{10}| = 21$.

Let the columns of the G-incidence matrix Γ of π be indexed by $\Omega, O_1, \ldots, O_{10}$. Let $\tau_i = t_i^G$ for $1 \leqq i \leqq 3$, and set

$$\tau_{i,j} = (\tau_i O_j) \quad \text{for } 1 \leqq i \leqq 3,\ 1 \leqq j \leqq 10.$$

Let $\tau \in \{\tau_1, \tau_2, \tau_3\}$ and O a G-orbit of points such that $|O| = 21$. If $(\tau O) \geqq 2$, then each line in τ will carry at least 6 points of O. However, this contradicts $|O| = 21$ and $|\tau| = 7$. Therefore $(\epsilon O) \in \{0, 1\}$. Let $m \in \tau$ such that $m^H = m$. Set $(\tau O_j) = m_j$ for $1 \leqq j \leqq 10$. Since m carries at least 1 point in $\bigcup_{j=1}^{3} O_j$ and 3 points in $O_4 \cup O_5$, we get from

$$\sum_{j=1}^{3} m_j + 3(m_4 + m_5) = 4$$

that

(1) $m_1 + m_2 + m_3 = 1 = m_4 + m_5$ and $m_j \in \{0, 1\}$ for $1 \leqq j \leqq 5$.

Consider $P(\tau)$. Since $(\tau\Omega) = 1$ and $(\tau O) \leqq 1$ for any $|O| = 21$ we get from 2.1 that there exists exactly one G-orbit of e-points E such that $(\tau E) > 1$. In fact $|E| = 7$ and $(\tau E) = 3$. From

$$\sum_{j=6}^{8} m_j + 3(m_9 + m_{10}) = 7 \text{ and } m_9, m_{10} \in \{0, 1\}$$

we infer that

(2) $m_9 + m_{10} = 1$ and $\{m_6, m_7, m_8\} = \{0, 1, 3\}$.

Since $m^H = m$ and H fixes only 1 point of E, the 3 points of E on m form an H-orbit. Therefore the unique orbit O_k, $6 \leqq k \leqq 8$, such that $(\tau O_k) = 1$ can be characterized as the G-orbit among O_6, O_7, O_8 with the property that m carries a point of $\mathscr{P}(H) \cap O_k$. This implies that we may assume that

$$\tau_{1,6} = \tau_{2,7} = \tau_{3,8} = 1.$$

Applying (1), (2) to τ_1, we may assume, by interchanging τ_2, τ_3 and O_7, O_8, if necessary, that the τ_1-row is (11001013010). By definition $\tau_{2,2} \neq 0$. Hence (1) implies

$$\tau_{2,1} = 0 = \tau_{2,3}.$$

Since $\tau_{2,7} = 1$,

$$[\tau_1|\tau_2] = 7 + 21\tau_{2,4} + 7\tau_{2,6} + 7 \cdot 3 + 21\tau_{2,9}.$$

By 2.1 we get

$$3 \equiv 3\tau_{2,4} + \tau_{2,6} + 3\tau_{2,9} \pmod{11}.$$

Since the right hand side of this equation is less than 9 by (1) and (2), we obtain

(3) $3 = 3\tau_{2,4} + \tau_{2,6} + 3\tau_{2,9}$.

From $\tau_{3,3} \neq 0$ and $\tau_{3,8} = 1$ we get, similarly, that

$$6 \equiv 3\tau_{3,4} + \tau_{3,6} + 3\tau_{3,7} + 3\tau_{3,9} \pmod{11}$$

from $[\tau_1|\tau_3]$. By (2) and $\tau_{3,8} = 1$ we obtain

$$\tau_{3,6} + \tau_{3,7} = 3.$$

The last equation modulo 11 now reads

$$3 \equiv 3\tau_{3,4} + 2\tau_{3,7} + 3\tau_{3,9} \pmod{11}.$$

Since the right hand side of the equation is less than 12 we obtain

(4) $3 = 3\tau_{3,4} + 2\tau_{3,7} + 3\tau_{3,9}$.

Suppose $\tau_{2,4} = 1$. This implies by (3), (2), and (1), that the τ_2-row is (10101001301). Hence

$$[\tau_2|\tau_3] = 7 + 21\tau_{3,4} + 7\tau_{3,7} + 21 + 21\tau_{3,10}.$$

By 2.1 we get

$$3 \equiv 3\tau_{3,4} + \tau_{3,7} + 3\tau_{3,10} \pmod{11}.$$

As before the right hand side of this equation is less than 9. Hence we get

(5) $3 = 3\tau_{3,4} + \tau_{3,7} + 3\tau_{3,10}$.

From equations (4) and (5) we deduce that

$$\tau_{3,7} = 3(\tau_{3,10} - \tau_{3,9}).$$

Since $\tau_{3,8} = 1$, $\tau_{3,7} \in \{0, 3\}$ by (2). From

$$\tau_{3,10} + \tau_{3,9} = 1 \text{ and } \tau_{3,10}, \tau_{3,9} \in \{0, 1\}$$

we now get $\tau_{3,7} = 3$. However (4) has no non-negative integral solution. Therefore $\tau_{2,4} \neq 1$, and so $\tau_{2,4} = 0$ and $\tau_{2,5} = 1$ by (1).

Assume $\tau_{2,8} = 3$. Then (2) implies that the τ_2-row is (10100101310). Hence

$$[\tau_2|\tau_3] = 7 + 21\tau_{3,5} + 7\tau_{3,7} + 21 + 21\tau_{3,9}.$$

By 2.1.c we get

$$3 \equiv 3\tau_{3,5} + \tau_{3,7} + \tau_{3,9} \pmod{11}.$$

As the right hand side is less than 12 we get

(6) $3 = 3\tau_{3,5} + \tau_{3,7} + \tau_{3,9}$.

Add (4) to (6). Using $\tau_{3,4} + \tau_{3,5} = 1$ we get

$$3 = 3\tau_{3,7} + 4\tau_{3,9}.$$

However $\tau_{3,7} \in \{0, 3\}$ and $\tau_{3,9} \in \{0, 1\}$ imply the last equation has no solution. Therefore $\tau_{2,8} \neq 3$. By (2) we get that $\tau_{2,8} = 0$ and $\tau_{2,6} = 3$, and the τ_2-row is (10100131001) by (3). Hence

$$[\tau_2|\tau_3] = 7 + 21\tau_{3,5} + 21\tau_{3,6} + 7\tau_{3,7} + 21\tau_{3,10}.$$

By 2.1 we get

$$6 \equiv 3\tau_{3,5} + 3\tau_{3,6} + \tau_{3,7} + 3\tau_{3,10} \pmod{11}.$$

Since $\tau_{3,6} + \tau_{3,7} = 3$ as $\tau_{3,8} = 1$, we get

$$3 \equiv 3\tau_{3,5} + 2\tau_{3,6} + 3\tau_{3,10} \pmod{11}.$$

Since the right hand side is less than 12 we get

(7) $3 = 3\tau_{3,5} + 2\tau_{3,6} + 3\tau_{3,10}$.

Using $\tau_{3,4} + \tau_{3,5} = 1$ and $\tau_{3,9} + \tau_{3,10} = 1$ and $\tau_{3,6} + \tau_{3,7} = 3$ we obtain, by adding (4) to (7), the final contradiction $6 = 3 + 6 + 3$. This completes the proof of the lemma.

4.5. THEOREM. *Suppose that G is a collineation group of odd order of the projective plane π of order 11. Then G is isomorphic to a subgroup of one of the following groups.*

a) *A group of order 9. If $|G| = 9$, then $f_G = 3$.*

b) *A semi-direct product of the non-abelian group of order 11^3, exponent 11 by an elementary abelian group of 25.*

1) *If $11^2 \mid |G|$, then G contains an elation. If $11^3 \mid |G|$, then π is Desarguesian.*

2) *Suppose $|G| = 25$. Then the number of subgroups of type $D(2)$ is equal to the number of subgroups of homologies, and $\mathrm{Fix}(G)$ is a triangle.*

3) *If G is cyclic of order 55, then G consists of perspectivities.*

c) *A semi-direct product of the elementary abelian group of order 25 with a group of order 3 acting irreducibly on the former, and $|G| = 3.25$. Any 3-element is anti-flag. There are 3 subgroups of type $D(2)$ and 3 subgroups of homologies in the Sylow 5-subgroup.*

d) *A semi-direct product of a cyclic group of order 7.19 by a group of order 3 such that the latter induces a fix-point-free automorphism group of the former. If $|G| = 21$ or $|G| = 21.19$, then any 3-element of G is anti-flag.*

e) *A semi-direct product of an elementary group of order 11^2 by a cyclic group of order 15 such that the latter acts on the first as linear group. If $33 \mid |G|$, then π is Desarguesian. If $|G| = 15$, then there is a homology of order 5. If $|G| \neq 3$, then any 3-element is anti-flag.*

Groups in b) *to* e) *are odd order subgroups of $PGL(3, 11)$.*

Proof. Let S be a Sylow 5-subgroup of G. Then $|S| \mid 25$ by 4.1. If S is cyclic of order 25, then 4.2.b implies that S acts semi-regularly on 10 points of a line. This is impossible. Hence S has exponent 5.

Suppose $|S| = 25$. From 4.2.b we see that S fixes the vertices and sides of a triangle. Let Ω_1, Ω_2, Ω_3 be the set of the points on the 3 sides of this triangle not equal to the vertices. Since $|\Omega_1| = |\Omega_2| = |\Omega_3| = 10$, S is not transitive on these sets. Let Ω be an S-orbit in one of these sets. Then $|\Omega| \leqq 5$. Hence the kernel K of the action of S on Ω is not trivial. Since a 5-element is not planar by 4.2.b, K acts faithfully on the other two sets among Ω_1, Ω_2, Ω_3. This shows that Fix(S) is a triangle and

$$\Omega_i = \Omega_{i1} \cup \Omega_{i2},$$

where $|\Omega_{i1}| = |\Omega_{i2}| = 5$ for $i = 1, 2, 3$. Let K_{ij} be the kernel of action of S on Ω_{ij} for $i = 1, 2, 3$ and $j = 1, 2$. As a 5-element is not planar, K_{ij} acts faithfully on Ω_{kl} for $k \neq i$. Clearly $K_{i1} = K_{i2}$ if S has a subgroup of homologies of order 5. If two subgroups of homologies have the same axis, then they have the same center. However this implies that these subgroups induce the same permutation group on one of the Ω_{ij} which will then yield a non-trivial homology fixing a point in Ω_{ij} not in the axis or equal to the center. Therefore different subgroups of homologies have different axes and different centers. Assume that there is triangular subgroup. Then it cannot be any of the 6 kernels. Since there are 6 proper subgroups of S, two of these kernels coincide, which implies that there exists a subgroup of homologies. The same argument shows that the number of triangular subgroups is not bigger than the number of subgroups of homologies. Assume that a subgroup of homologies exists. Then there are at most 4 more possibilities for the kernels. Hence we have at least one subgroup which acts faithfully on Ω_1, Ω_2, Ω_3. This implies that a triangular subgroup exists. By using the same argument, it is shown that the number of subgroups of homologies is not bigger than the number of triangular subgroups. Hence these two numbers are equal.

From 4.1, 4.2 and the fact that $|G|$ is odd we see that G has a normal 3-component G_1, G_1 has a normal 5-complement G_2, and G_2 has a normal 11-complement G_3. By 3.3.b we see that $G_3 \leqq Z(G_2)$.

Suppose $11 \mid |G|$. By 4.2 we obtain $G_3 = 1$. Assume $3 \mid |G|$. Since a 3-element cannot centralize an element of order 11 by 4.1 and 4.2, we have $11^2 \mid |G|$. If $11^3 \mid |G|$, then π is Desarguesian by 3.3. However this implies that $3 \nmid |G|$. Hence only $11^2 \mid |G|$, and $9 \nmid |G|$. Thus an element of order 3 acts irreducibly on the elementary abelian 11-subgroup V. By 3.3.c, V contains at least 2 distinct subgroups of elations which implies that π is Desarguesian. As no subgroup of $GL(2, 11)$ has order 3.25, we are in case e). Assume $3 \nmid |G|$. By 3.3 and 4.2.b and 4.2.c we see that case b) holds.

Suppose $11 \nmid |G|$. If $G_2 \neq 1$, then $G_2 = G_3$. By 4.2 we have $5 \nmid |G|$. Hence 4.1, 4.2, and 4.4 imply that G is isomorphic to a subgroup described in case d). Assume $|G| = 3.7.19$. Since any two different subgroups of order 3 have their fixed points intersecting trivially, any 3-element cannot be planar. Assume that $|G| = 21$ and a 3-element σ is planar. Hence there are 11 G-orbits of points: A_j, $1 \leqq j \leqq 11$ such that $|A_j| = 7$ for $1 \leqq j \leqq 7$ and $|A_j| = 21$ for $8 \leqq j \leqq 11$. Also 7 G-orbits of lines L_i, $1 \leqq i \leqq 7$, correspond to the 7 lines of the Fix(σ). Clearly $|L_i| = 7$ for $1 \leqq i \leqq 7$. By 4.4 we see that

$$(L_i A_j) \neq 3 \quad \text{for } 1 \leqq i, j \leqq 7.$$

Since each point of Fix(σ) lies on exactly 3 lines of Fix(σ) and each line of Fix(σ) carries exactly 3 points of Fix(σ), we see that the submatrix (L_iA_j), $1 \leqq i \leqq j \leqq 7$ is the incidence matrix of a projective plane of order 2. Consider $P(L_i)$ for $1 \leqq i \leqq 7$. By 2.1 we get

$$\{ (L_iA_j) \,|\, 1 \leqq i \leqq 7, 8 \leqq j \leqq 11\} = \{2, 1, 0, 0\}.$$

Therefore 2.1 implies that for $i \neq k$,

$$[L_i|L_k] = 7(1 + 3 \sum_{j=8}^{11} (L_iA_j)(L_kA_j)) \equiv 7^2 \pmod{11}.$$

Hence

$$\sum_{j=8}^{11} (L_iA_j)(L_kA_j) \equiv 2 \pmod{11}.$$

Since the left hand side of this equation is not bigger than 5, we have

$$\sum_{j=8}^{11} (L_iA_j)(L_kA_j) = 2.$$

A contradiction is obtained by looking at

$$((L_iA_j), 1 \leqq i \leqq 7, 8 \leqq j \leqq 11).$$

Therefore any 3-element is anti-flag if $|G| = 21$.

Assume now $11 \nmid |G|$ and $G_2 = 1$. By 4.2 G has order 3^f5^h where $1 \leqq f$, $h \leqq 2$. Since $f_G|3$, we get $|G| = 9$ if $f = 2$ by 4.2. Suppose H is a cyclic subgroup of order 15. Then 4.2 implies that the 5-elements of H are homologies and the 3-elements of H are anti-flag. Assume $|G| = 3.25$ and a 3-element normalizes a subgroup of order 5. Then G is the direct product of its Sylow 3-subgroup and Sylow 5-subgroup. Hence 3-elements are anti-flag. Since there is a subgroup of homologies of G, there is a triangular subgroup of the Sylow 5-subgroup which centralizes the anti-flag 3-elements. This is impossible. Therefore if $|G| = 3.25$, then N, a group of order 3, acts irreducibly on its Sylow 5-subgroup. The structure of the Sylow 5-subgroup now shows that there are 3 triangular subgroups and 3 subgroups of homologies of the Sylow 5-subgroup S of G. Hence Fix(S) is a triangle, and N permutes the 3 vertices and 3 sides of Fix(S). Therefore

$$\text{Fix}(N) \cap \text{Fix}(S) = (\phi, \phi).$$

There are exactly 4 S-orbits of points not in $\mathcal{P}(S)$. All these orbits are of size 25 and S acts regularly on each of these orbits. Suppose Fix(N) is a subplane. Then one of these orbits O contains at least two points of Fix(N). Since N permutes these 4 orbits, O is invariant under G. Since

$$|O \cap \mathscr{P}(N)| \geqq 2 \quad \text{and} \quad |O| = 25,$$

at least two subgroups of order 3 fix a common point in O which contradicts the fact that S acts regularly on O. Therefore N is anti-flag as desired. The proof of the theorem is complete.

4.6. COROLLARY. *If* $|G| = 21$, *then* π *contains an arc of size* 7, *which is the* G-*orbit of the fixed point of an element of order* 3.

Proof. Let A be the G-orbit of the fixed point of an element σ of order 3. Then $|A| = 7$. Let l be any line. Since $|l^G| = 7$ or 21, $|l \cap A| \leqq 3$. If $|l^G| = 7$, then $|l \cap A| \leqq 2$ by 4.4. Suppose $|l^G| = 21$ and $|l \cap A| = 3$. Let $P \in l \cap A$ be a fixed point of σ. Then $[P]$ contains, $l, l^\sigma, l^{\sigma^2}$. Let $h \in l^G$ and $h \notin \{l, l^\sigma, l^{\sigma^2}\}$. Then $h \cap l, h \cap l^\sigma, h \cap l^{\sigma^2}$ must be 3 distinct points of A, as otherwise $|A| > 7$. However there are at most 4 possibilities for such an h which contradicts $|l^G| = 21$. Therefore $|l \cap A| \leqq 2$ and A is an arc.

4.7. THEOREM. *Suppose* G *does not leave invariant any point, line or triangle. If* G *leaves invariant a subplane, then* $|G| \mid 9$ *or* $|G| = 7$. *If* G *does not leave invariant any subplane, then* π *is Desarguesian or one of the following holds.*

a) *G contains a unique minimal normal subgroup M such that* $M \cong A_5$ *or* $L_2(7)$.

b) *G is isomorphic to a subgroup of the semi-direct product of the cyclic group of order* 7.19 *by a group of order* 3 *such that the group of order* 3 *induces a fix-point-free automorphism group of the former and either* 7 *or* 19 *divides* $|G|$.

Proof. Suppose G leaves invariant a subplane. The order of this subplane must be 2. By 3.1 we get that $|G|$ is odd. By 4.4 and 4.5 and 2.6 we obtain $|G| \mid 9$ or $|G| = 7$ in this case. Assume now that G does not leave invariant any subplane. So G acts strongly irreducibly on π. If $|G|$ is odd, then the result follows from 4.5. Suppose $|G|$ is even. Then G contains an involutory homology. By 2.12, 2.9, 2.10 and 3.1 and 4.1 we see that π is Desarguesian except possibly that G contains a unique minimal normal subgroup M isomorphic to A_5 or $L_2(7)$ as desired.

5. Projective plane of order 13.

In this section we assume that $n = 13$.

5.1. LEMMA. $|G| \mid 2^5 \cdot 3^3 \cdot 7 \cdot 13^3 \cdot 61$.

Proof. This is clear from 2.6 and 2.5.

5.2. LEMMA. a) *Any collineation of order* 3 *is regular or a generalized homology of type* $D(k)$, $k = 2, 5, 8, 11$ *or* 14.

b) *Any collineation of order* 7 *is an anti-flag.*

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c) *Any collineation of order* 13 *is either an elation or a flag.*
d) *Any collineation of order* 61 *is regular.*

Proof. This is a consequence of $n = 13$ and $f_G = 1$ by 2.4.

5.3. THEOREM. *Suppose* $|G|$ *is odd. Then* G *is isomorphic to a subgroup of one of the following groups.*

a) *A semi-direct product of the non-abelian group of order* 13^3, *exponent* 13 *by an elementary abelian group of order* 9.

1) *If* $13^2 \mid |G|$, *then* G *contains an elation. If* $13^3 \mid |G|$, *then* π *is Desarguesian.*

2) *If* G *is cyclic of order* 39, *then* G *consists of perspectivities.*

b) *A semi-direct product of the elementary abelian group of order* 11^2 *by a group* $H = \langle \sigma, \tau, \gamma \rangle$, *where* $\langle \sigma, \tau \rangle$ *is a normal cyclic subgroup of order* 21 *and* $\gamma^3 = 1$ *and* $\langle \sigma, \gamma \rangle$ *is non-abelian of order* 21.

1) *If* $77 \mid |G|$, *then* π *is Desarguesian.*

2) *If* G *is cyclic of order* 21, *then* 3-*elements are homologies.*

3) *If* G *is non-abelian of order* 21 *then* 3-*elements are triangular.*

c) *A semi-direct product of a cyclic regular group of order* 183 *by a triangular group of order* 3 *such that the latter induces a fix-point-free automorphism group on the group of order* 61 *of the former.*

d) *A non-abelian group of order* 27, *exponent* 3. *If* $|G| = 27$, *then* $Z(G)$ *is triangular.*

Groups in a), b), c), d) *are subgroups of* $PGL(3, 13)$.

Proof. Let S be a Sylow 3-subgroup of G. Then $|S| \mid 27$. Suppose S has an element of σ of order 9. Since $9 \nmid 183$, σ is not regular. Hence σ^3 is a generalized homology. Since σ acts semi-regularly on points outside $\mathscr{P}(\sigma^3)$ and $9 \nmid 144$, σ^3 cannot be triangular. Therefore $\mathscr{P}(\sigma^3)$ consists of a point P and some points of a line l with $P \notin l$. Hence σ acts semi-regularly on the points outside l not equal to P. This implies that $9 \mid 168$, a contradiction. Therefore S has exponent 3. Assume that $|S| = 27$ and S is abelian. Then S is elementary. Since $27 \nmid 183$, S has a non-regular element τ. If τ is triangular, then the action of S on $\mathrm{Fix}(\tau)$ implies that S has a normal subgroup N of order 9 which fixes each vertex of $\mathrm{Fix}(\tau)$. On one side of $\mathrm{Fix}(\tau)$, N acts on the 12 points not equal to the vertices. Hence N contains a non-trivial element which fixes at least 3 points of these 12 points. Therefore we may assume, without loss of generality, that S contains a generalized homology of type $D(k)$ with $k > 2$, which we call τ again. Then there exists a unique line l in $\mathscr{L}(\tau)$ such that

$$|l \cap \mathscr{P}(\tau)| > 2.$$

Hence l is invariant under S which implies that there is a subgroup R of order 9 fixing at least 5 points on l. This implies that each element of R is a generalized homology of type $D(r)$, $r > 2$. Hence R acts semi-regularly on points outside l not equal to the unique fixed point of τ. Therefore $9|183 - 15$, a contradiction. So S is not abelian. Every subgroup of order 9 is elementary abelian. From 3.10 of [7] we see that the number of not regular subgroups in a subgroup of order 9 is 1 or 4. Since $|S| = 27$, this implies that $Z(S)$ is not regular. Assume that $Z(S)$ is not triangular. Then $\mathscr{P}(Z(S))$ consists of a point P and more than 2 points on a line l with $P \notin l$. Hence S leaves l invariant, which implies that S has a subgroup R of order 9 fixing at least 5 points on l. This shows $9|183 - 15$ as before, which is a contradiction. Therefore $Z(S)$ is triangular as desired.

Let H by a Sylow 7-subgroup of G. Suppose $|G| = 3^a \cdot 7$. Then $H \trianglelefteq G$. Since H is a flag, $a \leqq 2$. Assume that $|G| = 21$ and G is not abelian. As G leaves invariant the fixed line d of H, each Sylow 3-subgroup fixes at least 2 points on d. Since G is not abelian, these fixed point sets are mutually disjoint. This implies that each Sylow 3-subgroup fixes exactly 2 points on d and so is triangular. Suppose $|G| = 21$ and G is abelian. Then the Sylow 3-subgroup fixes pointwise all points on d. Hence 3-elements are homologies. Suppose $|G| = 3^2 \cdot 7$. Then there is a cyclic subgroup of order 21 and its 3-elements are homologies. Assume that G is abelian. Then all 3-elements are homologies. Since G has no elation, homologies from different subgroups of S have different centers and different axes. However this implies that the centers are the vertices of a triangle, which shows that at least one subgroup is triangular. This contradiction implies that G is not abelian. Hence $G = \langle \sigma, \tau, \gamma \rangle$ where $\sigma^7 = \tau^3 = \gamma^3 = 1$, and $\sigma\tau = \tau\sigma$ and $\tau\gamma = \gamma\tau$ and $\langle \sigma, \gamma \rangle$ is a non-abelian group of order 21. Therefore γ is triangular.

Let T be a Sylow 13-subgroup of G. Since $|G|$ is odd and $f_G = 1$, Burnside's theorem implies that G has a normal 3-complement G_1, that G_1 has a normal 7-complement G_2, and that G_2 has a normal 61 complement G_3. Thus $G_2 = T \times G_3$ by 3.4.

Suppose $1 \neq T$. Then $G_2 = T$ by 5.2. If $|T| = 13^3$, then π is Desarguesian by 3.4. If $|T| = 13$, then $7 \nmid |G|$ by 5.2. Assume that G is cyclic of order 39. Since a 13-element is either an elation or a flag, a 3-element is not regular. Since a 13-element acts on the fix-point-line structure of 3-element, 3-element must be a homology by 5.2. Therefore G consists of perspectivities in this case. If $|T| = 13$ and $|S| = 27$, then $Z(S)$ will centralize T. However $Z(S)$ is triangular. Therefore $|S| \mid 3^2$ when $|T| = 13$. Assume now $|T| = 13^2$. If $7 \mid |G|$, then a Sylow 7-subgroup acts irreducibly on T which implies that T contains more than one subgroup of elations. Hence π is Desarguesian in this case.

Suppose now $13 \nmid |G|$. Assume that $61 \mid |G|$. Then $7 \nmid |G|$ by 5.2. Since a Sylow 61-subgroup K is regular, every 3-element of $C_G(K)$ is regular. Hence $9 \nmid |C_G(K)|$. Since $9 \nmid |\mathrm{Aut}(K)|$, $|G| \mid 3^2 \cdot 61$ in this case. If $|G| = 3^2 \cdot 61$, then

$$|C_G(K)| = 3 \cdot 61.$$

As there are exactly 3 K-orbits of points of π, each of size 61, some subgroup of order 3 must fix each of these orbits. Let this subgroup be T_1. Then T_1 fixes at least one point from each of these 3 orbits. Hence T_1 is not regular. In particular $T_1 \nleq C_G(K)$. Since the Sylow 3-subgroup of $C_G(K)$ is regular and acts on $\mathrm{Fix}(T_1)$, T_1 must be triangular as desired. The proof of the theorem is complete.

5.4. THEOREM. *If G does not leave invariant any point, line or triangle, then π is Desarguesian or G is isomorphic to a subgroup of a semi-direct product of a cyclic regular group of order* 183 *by a triangular group of order* 3 *such that the latter induces a fix-point-free automorphisms group on the cyclic group of order* 61 *of the former and* $61 \mid |G|$.

Proof. Suppose G leaves invariant a subplane of order m. If $m = 2$, then $|G|$ is odd by 3.1. Assume $m = 3$. Since $f_G = 1$, G acts faithfully on this subplane. On this subplane the 13-element is regular and the 3-element is either an elation or a flag. By 5.2 we see that a 13-element cannot be an elation of π and so must be a flag. Also a 3-element cannot be an elation of the subplane. This implies $9 \nmid |G|$. If $13 \mid |G|$, then the Sylow 13-subgroup is normal. Since a 13-element is regular on this subplane and an involution induces homology on this subplane, $|G|$ is odd. Therefore $|G|$ is odd in any case. As G does not leave invariant any point, line, triangle, the case $m = 2$ cannot occur. If $m = 3$ and $13 \mid |G|$, then a Sylow 13-subgroup is normal and G leaves invariant the line fixed by this subgroup. If $m = 3$ and $13 \nmid |G|$, then $|G|$ is a 3-group which leaves invariant a line of the subplane. Therefore we conclude that G does not leave invariant any subplane and so G is strongly irreducible. If $|G|$ is odd, then 5.3 implies the desired result. Suppose $|G|$ is even. Then all involutions in G are homologies, and G contains a unique minimal normal non-abelian subgroup M, which acts strongly irreducibly on π. If π is not Desarguesian, then $M \cong L_2(7)$, $L_2(27)$ or $PSU(3, 3)$ by 2.12 and 5.2. To eliminate these possibilities, we present the following arguments, some of which are taken from **[17]**.

1) $M \cong L_2(7)$. It is easy to see that

$$|a(i)^M| = |\mathscr{C}(i)^M| = 21,$$

where i is an involution of M. Let $M_7 \in Syl_7(M)$. Then

$$N(7): = N_M(M_7) = M_3M_7 \quad \text{where } M_3 \in Syl_3(M).$$

By 5.2 $\mathrm{Fix}(M_7) = (\{P\}, \{l\})$, where $P \notin l$. Since $N(7)$ is a maximal subgroup in M, $M_P = N(7)$. Hence $|P^M| = 8$. This implies that $l \cap P^M = \emptyset$ as $|l^M| > 1$. Next $N(3): = N_M(M_3)$ is a dihedral group of order 6. By 5.2,

$$\text{Fix}(N(3)) = (\{T\}, \{t\}), \quad \text{where } T \notin t.$$

Also M_3 normalizes another $K \in Syl_7(M)$ which is conjugated to M_7 by an involution in $N(3)$. Let

$$\text{Fix}(K) = (\{P_1\}, \{l_1\}).$$

Thus

$$(l \cap l_1)^{N(3)} = l \cap l_1.$$

Hence $T = l \cap l_1$, and in particular $7 \nmid |M_T|$ as $l \cap P^M = \emptyset$. Therefore $|T^M| > 7$, which forces $M_T = N(3)$. So $|T^M| = 28$. Let $Q = l_1 \cap t$. Then $7 \nmid |M_Q|$, and $|Q^K| = 7$. Since $Q^{M_3} = Q$, M_3 leaves Q^K invariant as M_3 normalizes K. Since M_3 does not centralize K, M_3 fixes exactly one point in Q^K, namely Q. Therefore $T \notin Q^K$ as $Q \neq T$ is fixed by M_3. Assume

$$|[Q] \cap a(i)^M| > 0,$$

where i is an involution of M. Then

$$|[Q] \cap a(i)^M| \geqq 3$$

as $Q^{M_3} = Q$. Since $|M_{l_1}| = 21$, $l_1 = TQ$ is not an axis of an involution of M. Hence $\cup_{k \in K}[Q^k]$ contains at least 21 axes of involutions of M. Since

$$T \notin Q^K \quad \text{and} \quad |[T] \cap a(i)^M| \geqq 3,$$

there are at least 24 axes of involutions of M. This contradicts

$$|a(i)^M| = |\mathscr{C}(i)^M| = 21.$$

Therefore

$$[Q] \cap a(i)^M = \emptyset$$

and so $|M_Q|$ is odd. Hence $M_Q = M_3$ and $|Q^{M_3}| = 56$. As i commutes with exactly 4 other involutions and normalizes exactly 4 Sylow 3-subgroup of M, we have

$$|a(i) \cap \mathscr{C}(i)^M| = 4 = |a(i) \cap T^M|.$$

From

$$|[\mathscr{C}(i)] \cap a(i)^M| = 4 \quad \text{and} \quad |[T] \cap a(i)^M| = 3,$$

we see that the 20 axes of involutions of M are different from $a(i)$ intersect $a(i)$ at the above 8 points. Let R be a point of $a(i)$ different from these 8 points. Then

$$[R] \cap a(i)^M = a(i).$$

Hence $|M_R|$ is prime to 21 and so $|M_R| = 2$. Therefore $|R^M| = 86$. However,

$$|\mathscr{C}(i)^M \cup T^M \cup Q^M \cup P^M \cup R^M| = 197 \gneqq 183 = |\mathscr{P}|,$$

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which is a contradiction. Hence $M \not\cong L_2(7)$.

2) $M \cong L_2(27)$. Let $K \in Syl_{13}(M)$, and $N = N_M(K)$. By 5.2, $\mathscr{P}(N) \neq \emptyset$. Let $P \in \mathscr{P}(N)$. Then N is a subgroup of M_P of order 26. Since

$$|M:N| \equiv 1 \equiv |M_P:N| \pmod{13},$$

$|M:M_P| \equiv 1 \pmod{13}$. Clearly

$$|M| = 2^2 \cdot 3^3 \cdot 4 \cdot 13.$$

If $3 \mid |M_P|$, then $3^3 \mid |M_P|$ as

$$|M_P:N| \equiv 1 \pmod{13}.$$

This forces $M_P = M$, a contradiction. Therefore $3^3 \mid |M:M_P|$. If $3^3 = |M:M_P|$, then

$$|M_P| = 4 \cdot 7 \cdot 13.$$

This implies that a Sylow 7-subgroup is normal in M_P, which forces K to be centralized by a 7-element, a contradiction. Hence

$$3^3 < |M:M_P| \,\big|\, 2 \cdot 3^3 \cdot 7.$$

Since $|M:M_P| \equiv 1 \pmod{13}$,

$$|M:M_P| = 2 \cdot 3^3 \cdot 7 > 183 = |\mathscr{P}|.$$

This contradiction shows that $M \not\cong L_2(27)$.

3) $M \cong PSU(3, 3)$. Let α be a 7-element of M. By 5.2,

$$\text{Fix}(\alpha) = (\{P\}, \{l\}).$$

Thus $N = N_M(\langle \alpha \rangle)$ is a subgroup of order 21 of M_P. Since $|M| = 2^5 \cdot 3^3 \cdot 7$,

$$|M_P:N| = 2^a 3^b = x \quad \text{and} \quad |M:M_P| = 2^{5-a} 3^{2-b} = y,$$

where $0 \leqq a \leqq 5$ and $0 \leqq b \leqq 2$. As M does not fix any point, $M \neq M_P$, and so $a < 5$. Clearly, $x \equiv 1 \pmod 7$ and $y \equiv 1 \pmod 7$. If $b = 2$, then $2^a 3^2 \equiv 1 \pmod 7$ implies $a = 2$ as $a < 5$. Hence $y = 8$ which implies that M is isomorphic to a subgroup of A_8. This in turn forces $3^3 | 8!$, a contradiction. If $b = 1$, then $x \equiv 1 \pmod 7$ implies $2^a \equiv 5 \pmod 7$, which has no solution for $0 \leqq a \leqq 5$. Hence $b = 0$. If $a = 0$, then $M_P = N$ and so

$$|P^M| = 2^5 \cdot 3^2 > |\mathscr{P}|,$$

a contradiction. From $y \equiv 1 \pmod 7$, we now get

$$y = 2^2 \cdot 3^2, \; x = 2^3, \quad \text{and} \quad |M_P| = 2^3 \cdot 3 \cdot 7.$$

As every proper subgroup of M is solvable, M_P has a non-trivial elementary abelian normal q-subgroup Q for some prime q. Since $x = 8$,

$q \neq 7$. If $q = 3$, then a 7-element of M_P will centralize a 3-element of M_P which is impossible. Hence $q = 2$. Since no 7-element of M_P centralizes a 2-element of M_P, $|K| = 8$. However, the Sylow 2-subgroup of M is isomorphic to $\mathbf{Z}_4 \wr \mathbf{Z}_2$, which does not contain any elementary abelian subgroup of order 8. This contradiction yields the fact that $M \not\cong PSU(3, 3)$, and the proof is now complete.

6. Remarks.

6.1. *Suppose n is a prime not bigger than* 37 *and G acts strongly irreducibly on π and contains a non-trivial perspectivity. Let M be the unique minimal normal subgroup of G. Then π is Desarguesian or one of the following holds.*

a) $n = 11$ *and* $M \cong L_2(5)$ *or* $L_2(7)$.
b) $n = 19$ *and* $M \cong L_2(5)$ *or* $L_2(9)$.
c) $n = 23$ *and* $M \cong L_2(7)$.
d) $n = 29$ *and* $M \cong L_2(5)$ *or* $L_2(7)$.
e) $n = 31$ *and* $M \cong L_2(5)$ *or* $L_2(7)^*$ *or* $L_2(9)$.
f) $n = 37$ *and* $M \cong L_2(7)$.

All situations from a) *to* f) *occur in the Desarguesian plane of the corresponding order except in the case marked by* *.

Proof. By 3.2 we see that M is non-abelian simple. For $n \leqq 7$, it is known that π is Desarguesian [**15**]. By 4.7 and 5.4 we may assume $17 \leqq n$. The case $n = 17$ is dealt with in [**5**]. Using 2.5, 2.9, 2.10, 2.12, a direct calculation as in [**17**] leads to the conclusion of the theorem except the following cases:

1) $n = 29$ and $M \cong PSU(3, 3)$ or $PSU(3, 5)$ or $L_2(13)$.

Since $29 \equiv -1 \pmod 3$, $PSU(3, 3)$ is eliminated by Lemma 3.3 of [**4**]. If $M \cong PSU(3, 5)$, then $25 | f_M$. However a 5-element not in the center of a Sylow 5-subgroup is inverted by an involution. This implies that $25 \nmid f_M$. Therefore $PSU(3, 5)$ is also eliminated. Since $29 \equiv -1 \pmod 3$, $L_2(13)$ is eliminated by Lemma 3.5 of [**4**].

2) $n = 37$ and $M \cong L_2(17)$.

In this case we have $7 | f_M$. However since $17 - 1 = 16$, an involution inverts the Sylow 7-subgroup which implies $7 \nmid f_M$. Therefore $L_2(17)$ is also eliminated.

3) $n = 31$ or 37 and $M \cong PSU(3, 3)$.

Let S be a Sylow 3-subgroup of M. Then $N_M(S) = T \cdot S$ where $T = \langle y \rangle$ is a cyclic group of order 8 and S is an extra special 3-group of order 27. Let $1 \neq \sigma \in Z(S)$.

Suppose σ is planar. An easy calculation shows that Fix(σ) has order 4. Since y^4 is an involution centralizing σ, we obtain a contradiction by 3.1. Therefore σ is not planar. As every 3-element of $S \setminus Z(S)$ is inverted by an involution, this shows that no 3-element is planar. Since σ centralizes the homology y^4, σ is not regular. Since $n \equiv 1 \pmod 3$, 3-element are generalized homologies.

Assume next that σ is triangular. The action of $\langle y \rangle$ on Fix(σ) implies that

$$\text{Fix}(\sigma) \leqq \text{Fix}(y^2).$$

Since y^2 permutes the 4 subgroups of order 3 of $S/Z(S)$ cyclicly, and y^4 inverts each of these subgroups,

$$\text{Fix}(\sigma) \leqq \text{Fix}(\langle y^2, S \rangle).$$

Since y^4 is a homology, one side l of Fix(σ) is the axis of y^4. Hence $N_M(S) \leqq M_l$. Clearly $C_M(y^4) \leqq M_l$. Therefore

$$M = \langle N_M(S), C_M(y^4) \rangle \leqq M_l,$$

which contradicts the fact that M does not fix any line. Thus σ cannot be triangular. Therefore σ is a generalized homology of type $D(k)$ with $k \geqq 5$, and there is an unique line u such that

$$|\mathscr{P}(\sigma) \cap u| > 2.$$

Hence u is left invariant by $N_M(S)$. By definition,

$$\mathscr{P}(\sigma) = \{P\} \cup \{\mathscr{P}(\sigma) \cap u\} \quad \text{with } P \notin u.$$

If $P = \mathscr{C}(y^4)$, the center of y^4, then $P^N M^{(S)} = P$ implies that all 9 involutions in $N_M(S)$ have P as common center. Since the axis of a homology does not go through its center and

$$\mathscr{L}(\sigma) \leqq [P] \cup \{u\},$$

u is the common axis for all involutions in $N_M(S)$. Therefore

$$M = \langle N_M(S), C_M(y^4) \rangle$$

is contained in M_u, a contradiction. Hence $\mathscr{C}(y^4) \in u$. Now y acts fix-point-freely on the points of l not equal to $\mathscr{C}(y^4)$ or $a(y^4) \cap l$. This implies that $8 | n - 1 = 30$ or 36, a contradiction.

By [**1**] all situations from a) to f) do occur in the Desarguesian plane of the corresponding order except the one marked by *.

REFERENCES

1. D. M. Bloom, *The subgroups of PSL*(3, *q*) *for odd q*, Trans AMS *127* (1967), 150-178.

2. P. Dembowski, *Finite geometries* (Springer, 1972).

3. W. Feit, *The current situation in the theory of finite simple groups*, Acte congré intern. math. *1* (1970) 55-93.
4. A. Goncalves and D. F. Barboza, *The projective plane of order* 47 *and* 53, to appear.
5. A. Goncalves and R. Pomareda, *Planos projectivos de orden* 17, Notas de la Soc. Mat. de Chile *1* (1981).
6. D. Gorenstein, *Finite groups* (Harper and Row, 1968).
7. C. Hering, *On the structure of finite collineation groups of projective planes*, Abh. Math. Sem. Univ. Hamburg *49* (1975), 155-182.
8. ——— *Finite collineation groups of projective planes containing non-trivial perspectivities*, to appear.
9. ——— *On codes and projective design*, Kyoto Univ. Math. Research Inst. Sem. *344* (1979), 26-60.
10. C. Hering and M. Walker, *Perspectivities in irreducible collineation groups of projective planes I*, Math. Z. *155* (1977), 95-101; II, J. of Statistic Planning Inference *3* (1979), 151-177.
11. D. Hughes and F. Piper, *Projective planes* (Springer, 1973).
12. B. Huppert, *Endliche Gruppen I* (Springer, 1967).
13. W. Kantor, *On unitary polarities of finite projective planes*, Can. J. Math. *23* (1971), 1060-1077.
14. ——— *On the structure of collineation groups of finite projective planes*, Proc. London Math. Soc. *32* (1976), 385-402.
15. G. Pickert, *Projektive Ebenen* (Springer, 1975).
16. R. Roth, *Collineation groups of finite projective planes*, Math. Z. *83* (1964), 409-421.
17. A. M. Serra Filho, *Planos projectivos de ordem prima*, Doctoral Thesis, Univ. of Brasília (1981).

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