Bull. Aust. Math. Soc. **83** (2011), 369–375
doi:10.1017/S0004972711002012

ON NONNILPOTENT SUBSETS IN GENERAL LINEAR GROUPS

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(Received 13 January 2010)

Dedicated to Professor B. Neshvadian-Bakhsh on his retirement.

Abstract

Let G be a group. A subset X of G is said to be nonnilpotent if for any two distinct elements x and y in X, $\langle x, y\rangle$ is a nonnilpotent subgroup of G. If, for any other nonnilpotent subset X' in G, $|X| \geq |X'|$, then X is said to be a maximal nonnilpotent subset and the cardinality of this subset is denoted by $\omega(\mathcal{N}_G)$. Using nilpotent nilpotentizers we find a lower bound for the cardinality of a maximal nonnilpotent subset of a finite group and apply this to the general linear group $\mathrm{GL}(n, q)$. For all prime powers q we determine the cardinality of a maximal nonnilpotent subset of the projective special linear group $\mathrm{PSL}(2, q)$, and we characterize all nonabelian finite simple groups G with $\omega(\mathcal{N}_G) \leq 57$.

2010 *Mathematics subject classification*: primary 20D60.
Keywords and phrases: general linear group, nonnilpotent set.

1. Introduction and results

Let $n > 0$ be an integer. Given a class of groups $\mathcal{X}$, we say that a group G satisfies the condition $(\mathcal{X}, n)$ whenever in every subset of G with $n + 1$ elements, there exist distinct elements x, y such that $\langle x, y\rangle$ is in $\mathcal{X}$. Let $\mathcal{N}$ be the class of nilpotent groups.

In 1994, finite groups satisfying the condition $(\mathcal{N}, n)$ were considered by Endimioni and in [4] he proves that every finite group G satisfying $(\mathcal{N}, n)$ is nilpotent if $n \leq 3$ and is soluble if $n \leq 20$; furthermore, these bounds cannot be improved. Tomkinson [7] proves that if G is a finitely generated soluble group which satisfies the condition $(\mathcal{N}, n)$, then $|G/Z^*(G)| \leq n^{n^4}$, where $Z^*(G)$ is the hypercentre of G. Also, for a finite insoluble group G, it has been proved that G satisfies the condition $(\mathcal{N}, 21)$ if and only if $G/Z^*(G) \cong A_5$ [2, Theorem 1.2].

A subset X of a group G is said to be a *nonnilpotent subset* if for any two distinct elements x and y in X, $\langle x, y\rangle$ is a nonnilpotent subgroup of G. If, for any other nonnilpotent subset X' in G, $|X| \geq |X'|$, then X is said to be a maximal nonnilpotent subset and the cardinality of this set is denoted by $\omega(\mathcal{N}_G)$. For convenience, if G is a nilpotent group we define $\omega(\mathcal{N}_G) = 1$.

It is clear that G satisfies the condition $(\mathcal{N}, n)$ if and only if $\omega(\mathcal{N}_G) \leq n$. Also, $\omega(\mathcal{N}_G) = n$ if and only if n is the smallest number such that G satisfies the condition $(\mathcal{N}, n)$. (We call n the smallest number for which the finite group G satisfies the condition $(\mathcal{N}, n)$ if G does not satisfy the condition $(\mathcal{N}, n-1)$.)

In this paper we introduce $\mathcal{N}n$-groups and give some of their properties. Using nilpotent nilpotentizers and Sylow p-subgroups, we obtain a lower bound for maximal nonnilpotent subsets of n-dimensional general linear groups and we determine the cardinality of a maximal nonnilpotent subset of $\mathrm{PSL}(2, q)$ (see Theorems 4.1 and 4.4).

We know that $\mathrm{PSL}(2, 5) \cong A_5$, the alternating group of degree five. It is the least (with respect to the order) nonabelian simple group and $\omega(\mathcal{N}_{A_5}) = 21$. Also, the cardinality of the maximal nonnilpotent subset of $\mathrm{PSL}(2, 7)$, the second least order nonabelian simple group, is 57. Here we give a characterization of finite nonabelian simple groups with $\omega(\mathcal{N}_G) \leq 57$ (see Theorem 4.5).

2. Some properties of $\mathcal{N}n$-groups

Let G be a group and a be an element of G. Define

$$\mathrm{nil}_G(a) = \{b \in G : \langle a, b\rangle \text{ is nilpotent}\}$$

and call it the *nilpotentizer* of a in G. Also, for a nonempty subset X of G, define the nilpotentizer of X in G to be $\mathrm{nil}_G(X) = \bigcap_{x \in X} \mathrm{nil}_G(x)$. In particular, the set

$$\mathrm{nil}(G) = \{x \in G : \langle x, y\rangle \text{ is nilpotent for all } y \in G\}$$

is called the *nilpotentizer* of G.

We know that for any group G and arbitrary a in G, the subset $\mathrm{nil}_G(a)$ is not necessarily a subgroup of G. For example, in the symmetric group S_4, $|\mathrm{nil}_{S_4}((12)(34))| = 16$. We call a group G an n-group if $\mathrm{nil}_G(a)$ is a subgroup of G for every $a \in G$.

DEFINITION 2.1. A group G is said to be $\mathcal{N}n$-group if $\mathrm{nil}_G(a)$ is a nilpotent subgroup of G, where $a \in G \setminus \mathrm{nil}(G)$.

PROPOSITION 2.2. *The following are equivalent.*

(i) *G is an $\mathcal{N}n$-group.*
(ii) *If $\langle a, b\rangle$ is nilpotent, then $\mathrm{nil}_G(a) = \mathrm{nil}_G(b)$ whenever $a, b \in G \setminus \mathrm{nil}(G)$.*
(iii) *If $\langle a, c\rangle$ and $\langle a, c\rangle$ are nilpotent subgroups of G, then $\langle b, c\rangle$ is nilpotent whenever $a \in G \setminus \mathrm{nil}(G)$.*
(iv) *If A and B are subgroups of G and $\mathrm{nil}(G) < \mathrm{nil}_G(A) \leq \mathrm{nil}_G(B) < G$, then $\mathrm{nil}_G(A) = \mathrm{nil}_G(B)$.*

PROOF. (i) $\Rightarrow$ (ii). Suppose that $a, b \in G \setminus \mathrm{nil}(G)$ and $\langle a, b\rangle$ is a nilpotent subgroup of G. Then $a \in \mathrm{nil}_G(b)$. Let $x \in \mathrm{nil}_G(b)$. Since $\mathrm{nil}_G(b)$ is a nilpotent group, $\langle a, x\rangle$ is nilpotent subgroup of $\mathrm{nil}_G(b)$ and so $x \in \mathrm{nil}_G(a)$. Thus $\mathrm{nil}_G(b) \subseteq \mathrm{nil}_G(a)$. Similarly, $\mathrm{nil}_G(a) \subseteq \mathrm{nil}_G(b)$.

https://doi.org/10.1017/S0004972711002012 Published online by Cambridge University Press

(ii) $\Rightarrow$ (iii). If b or c is an element of $\mathrm{nil}(G)$, then $\langle b, c\rangle$ is nilpotent. If neither b nor c is an element of $\mathrm{nil}(G)$, then, by (ii), $\mathrm{nil}(a) = \mathrm{nil}(b)$ and $\mathrm{nil}(b) = \mathrm{nil}(c)$. Thus $\mathrm{nil}(b) = \mathrm{nil}(c)$ and so $\langle b, c\rangle$ is a nilpotent subgroup of G.

(iii) $\Rightarrow$ (iv). Suppose $\mathrm{nil}(G) < \mathrm{nil}_G(A) \leq \mathrm{nil}_G(B) < G$. Let $u \in A$, $v \in B \setminus \mathrm{nil}_G(A)$, $x \in \mathrm{nil}_G(A) \setminus \mathrm{nil}(G)$ and $y \in \mathrm{nil}_G(B) \setminus \mathrm{nil}_G(A)$. It follows that $\langle x, u\rangle$ and $\langle x, v\rangle$ are nilpotent subgroups. Hence, by (iii), $\langle u, v\rangle$ is nilpotent. Also, by assumption, $\langle y, v\rangle$ is nilpotent. Thus, by (iii), $\langle y, v\rangle$ is nilpotent. So $\langle u, y\rangle$ is a nilpotent. Consequently $y \in \mathrm{nil}_G(A)$, a contradiction.

(iv) $\Rightarrow$ (i). Let $x \in G \setminus \mathrm{nil}_G(a)$, $y, z \in \mathrm{nil}_G(x)$ and $\langle y, z\rangle$ be nilpotent. Then $\mathrm{nil}(G) < \mathrm{nil}(\langle x, y\rangle) < \mathrm{nil}(x) < G$, a contradiction. □

LEMMA 2.3. *Let G be a finite group and N be a maximal nonnilpotent subset of G. Then $G = \bigcup_{x\in N} \mathrm{nil}_G(x)$,*

PROOF. If $y \in G \setminus \bigcup_{x\in N} \mathrm{nil}_G(x)$, then for all $x \in N$, $\langle x, y\rangle$ is not nilpotent. Hence $N \cup \{y\}$ is a nonnilpotent subset of size $|N| + 1$, a contradiction. □

3. Nonnilpotent subsets in finite groups

In this section we provide some conditions on a finite group G which extend every nonnilpotent subset to a maximal nonnilpotent subset. Also, by using Sylow p-subgroups, we give a nonnilpotent subset consisting of p-elements in arbitrary finite groups.

LEMMA 3.1. *Let G be a finite group. Then:*

(i) *for any subgroup H of G, $\omega(\mathcal{N}_H) \leq \omega(\mathcal{N}_G)$;*
(ii) *for any normal subgroup N of G, $\omega(\mathcal{N}_{G/N}) \leq \omega(\mathcal{N}_G)$.*

PROOF. (i) This is straightforward.

(ii) Let $\{a_1N, \ldots, a_kN\}$ be a nonnilpotent subset of G/N. Then, for any two distinct elements i, j in $\{1, 2, \ldots, k\}$, the subgroup

$$\langle a_iN, a_jN\rangle = \langle a_i, a_j\rangle N/N \cong \langle a_i, a_j\rangle/\langle a_i, a_j\rangle \cap N$$

is nonnilpotent. Thus $\{a_1, \ldots, a_k\}$ is a nonnilpotent subset of G. □

LEMMA 3.2. *Let G be a group and let the subgroups $A_1, A_2, \ldots, A_n$ of G form a partition of G. If $\mathrm{nil}_G(g) \leq A_i$, for all $g \in A_i \setminus \mathrm{nil}(G)$, then:*

(i) $\omega(\mathcal{N}_G) = \sum_{i=1}^{n} \omega(\mathcal{N}_{A_i})$;
(ii) *if A_i is nilpotent for all $i \in \{1, \ldots, n\}$, then every nonnilpotent subset of G can be extended to a maximal nonnilpotent subset of G.*

PROOF. (i) Let $N_i = \{a_{i1}, \ldots, a_{it_i}\}$ be a nonnilpotent subset of A_i. We show that $N = \bigcup_{i=1}^{n} N_i$ is a nonnilpotent subset of G. Suppose that there exist a and b in N such that $\langle a, b\rangle$ is a nilpotent group. So there exist $i \neq j$ such that $a \in A_i$ and $b \in A_j$.

Thus $a \in \mathrm{nil}_G(b) \le A_j$ and so $a \in A_i \cap A_j$, which is not possible. It follows that $\sum_{i=1}^{n} \omega(\mathcal{N}_{A_i}) \le \omega(\mathcal{N}_G)$. Now let X be a maximal nonnilpotent subset of G. Hence

$$X = X \cap G = X \cap \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} (X \cap A_i).$$

Since $X \cap A_i$, for $i = 1, \ldots, n$, is a nonnilpotent subset of A_i, $|X \cap A_i| \le \omega(\mathcal{N}_{A_i})$. So $|X| = \omega(\mathcal{N}_G) \le \sum_{i=1}^{n} \omega(\mathcal{N}_{A_i})$. Therefore $\omega(\mathcal{N}_G) = \sum_{i=1}^{n} \omega(\mathcal{N}_{A_i})$.

(ii) Let $a_i \in A_i \setminus \mathrm{nil}(G)$, for $i \in \{1, \ldots, n\}$. Since A_i is nilpotent, $\langle a_i, x\rangle$ is a nilpotent group, for all $x \in A_i$. It follows that $A_i \subseteq \mathrm{nil}_G(a_i)$, for $i \in \{1, \ldots, n\}$. Hence, by assumption, $A_i = \mathrm{nil}_G(a_i)$, for $i \in \{1, \ldots, n\}$. Thus $\mathrm{nil}(a_i)$ is a nilpotent subgroup, for $i \in \{1, \ldots, n\}$ and $G = \bigcup_{i=1}^{n} \mathrm{nil}(a_i)$. Let X be a nonnilpotent subset of G. Then, for each $1 \le i \le n$, $|X \cap \mathrm{nil}_G(a_i)| \le 1$, as each $\mathrm{nil}_G(a_i)$ is nilpotent. Let $I = \{i \in \{1, \ldots, n\} : X \cap \mathrm{nil}_G(a_i) = \emptyset\}$. For each $k \in I$, choose an element $b_k \in \mathrm{nil}_G(a_k) \setminus \mathrm{nil}(G)$. Thus $X \cup \{b_k : k \in I\}$ is the maximal nonnilpotent subset of G. □

We denote the number of Sylow p-subgroups of a finite group G by $\nu_p(G)$.

LEMMA 3.3. *Suppose that G is a finite group and p is a prime number dividing $|G|$. Let $P = P_1, P_2, \ldots, P_{\nu_p(G)}$ be the Sylow p-subgroups of G. If $P \setminus \bigcup_{i=2}^{\nu_p(G)} P_i \neq \emptyset$, then $\nu_p(G) \le \omega(\mathcal{N}_G)$.*

PROOF. Let $a \in P \setminus \bigcup_{g \in G, P^g \neq P} P^g$. So P is the unique Sylow p-subgroup containing a. For each i, choose $x_i \in G$ such that $P^{x_i} = P_i$. Then it is easy to see that $a^{x_i} \in P_i \setminus (P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_{\nu_p(G)})$. Set $X = \{a^{x_1}, a^{x_2}, \ldots, a^{x_{\nu_p(G)}}\}$. We show that X is a nonnilpotent subset. Suppose to the contrary that $\langle a_i^x, a_j^y\rangle$ is a nilpotent subgroup of G. It follows that $\langle a_i^x, a_j^y\rangle$ is p-subgroup, and so there exists a Sylow p-subgroup P^{x_i} of G such that $\langle a_i^x, a_j^y\rangle \subseteq P^t$. This is a contradiction. Thus $\nu_p(G) \le \omega(\mathcal{N}_G)$. □

As a consequence of Lemma 3.3, we have the following result that was proved by Endimioni in [4, Lemma 3, p. 1246].

COROLLARY 3.4. *Let G be a finite group with $\omega(\mathcal{N}_G) = n$ and p be a prime number dividing $|G|$. If $P_1, \ldots, P_k$ are all Sylow p-subgroups of G such that $P_i \cap P_j = 1$, where $1 \le i \neq j \le k$, then $\nu_p(G) \le n$.*

4. Main results

Now, using the above results, we are ready to state the main results of this paper. For the convenience of the discussion, we define

$$\psi(q^n) = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)/(q - 1)^n.$$

THEOREM 4.1. *We have $\omega(\mathcal{N}_{\mathrm{GL}(n,q)}) \ge \psi(q^n)$.*

https://doi.org/10.1017/S0004972711002012 Published online by Cambridge University Press

PROOF. Let P be the subgroup of $\mathrm{GL}(n, q)$ of upper triangular matrices. By [5, Satz 7.1], P is a Sylow p-subgroup of $\mathrm{GL}(n, q)$. Set

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We show that $A \in P \setminus \bigcup P^K$, for all $K \in \mathrm{GL}(n, q) \setminus N_{\mathrm{GL}(n,q)}(P)$. Suppose that there exists $K \in \mathrm{GL}(n, q) \setminus N_{\mathrm{GL}(n,q)}(P)$ such that $A \in P^K$. So $KA = CK$, where $C \in P$. Let

$$C = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 1 & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1n} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$B = A - I$ and $D = C - I$, where I is the identity matrix. It follows that $K(I + B) = (I + D)K$ and so $KB = DK$. An easy computation shows that $K \in N_{\mathrm{GL}(n,q)}(P)$, which is impossible. We know that $\nu_p(\mathrm{GL}(n, q)) = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)/(q - 1)^n$, so by Lemma 3.3, the proof is complete. □

LEMMA 4.2. *We have* $\omega(\mathcal{N}_{\mathrm{SL}(n,q)}) \geq \psi(q^n)$.

PROOF. The Sylow p-subgroup P in the proof of Theorem 4.1 is also a Sylow p-subgroup of $\mathrm{SL}(n, p)$, and we know that

$$\nu_p(\mathrm{SL}(n, q) = (q^n - 1)(q^{n-1} - 1) \cdots (q^2 - 1)/(q - 1)^{n-1}.$$

So an argument similar to that of Theorem 4.1 completes the proof. □

As an application of Corollary 3.4, we have the following lemma.

LEMMA 4.3. *Let p be a prime number. Then $\omega(\mathcal{N}_{S_p}) \geq (p - 2)!$, where S_p is the symmetric group of degree p.*

PROOF. We know that $\nu_p(S_p) = (p - 2)!$ and the size of Sylow p-subgroups of S_p is p. Clearly, if P and Q are Sylow p-subgroups of S_p, then $P \cap Q = 1$. Now by Corollary 3.4 the proof is complete. □

THEOREM 4.4. *Let q be a p-power (p prime). Then*

$$\omega(\mathcal{N}_{\mathrm{PSL}(2,q)}) = \begin{cases} 4 & \text{if } q = 2 \\ 10 & \text{if } q = 3 \\ 21 & \text{if } q = 4, 5 \\ q^2 + q + 1 & \text{if } q > 5. \end{cases}$$

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PROOF. Suppose that $G = \mathrm{PSL}(2, q)$, where q is a power of a prime p and $k = \gcd(q-1, 2)$. By [5, Satz 6.14, p. 183], $\mathrm{PSL}(2, 2) \cong S_3$, $\mathrm{PSL}(2, 3) \cong A_4$ and $\mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5) \cong A_5$. So, by [4, Proposition 1, Lemma 3], in the case $q = 2, 3, 4, 5$ the computation of $\omega(\mathcal{N}_G)$ is straightforward. So we may assume that $q > 5$. By [5, Satz 8.2, p. 191; Satz 8.2, 8.3, p. 192; Satz 8.5, p. 193]:

(1) a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroups of G is $q + 1$;
(2) G contains a cyclic subgroup A of order $t = (q-1)/k$;
(3) G contains a cyclic subgroup B of order $s = (q+1)/k$;
(4) the set $\{P^x, A^x, B^x : x \in G\}$ is a partition of G.

Let

$$X = \{p_i, a_j, b_k : p_i \in P^{x_i}, a_j \in A^{x_j}, b_k \in B^{x_k} \text{ and } x_l \in G,\ p_i^2, a_j^2, b_k^2 \neq 1\}.$$

Now, suppose that $a \in X$. It follows, by [1, Proposition 3.21], that $C_G(a) = P^x$ or A^x or B^x for some $x \in G$. Hence $C_G(a)$ is abelian. Suppose that $a, b \in X$ such that $\langle a, b\rangle$ is a nilpotent group. Since $Z(\langle a, b\rangle) \subseteq C_G(a) \cap C_G(b)$, we have $Z(\langle a, b\rangle) = 1$. This is a contradiction. It follows from Lemma 3.2 that X is a maximal nonnilpotent subset of G and

$$\omega(\mathcal{N}_G) = \sum_{x \in G} \omega(\mathcal{N}_{P^x}) + \sum_{x \in G} \omega(\mathcal{N}_{A^x}) + \sum_{x \in G} \omega(\mathcal{N}_{B^x}).$$

Therefore

$$\omega(\mathcal{N}_G) = (q+1) + \frac{(q+1)(q-1)q/k}{2(q-1)/k} + \frac{(q+1)(q-1)q/k}{2(q+1)/k} = q^2 + q + 1.$$

This concludes the proof. □

THEOREM 4.5. *Let G be a finite nonabelian simple group. Then $\omega(\mathcal{N}_G) \leq 57$ if and only if $G \cong A_5$ or $G \cong \mathrm{PSL}(2, 7)$.*

PROOF. Since $\mathrm{PSL}(2, 5)$ and $\mathrm{PSL}(2, 7)$ are nonabelian simple groups and since $\omega(\mathcal{N}_{\mathrm{PSL}(2,5)}) = 21$ and $\omega(\mathcal{N}_{\mathrm{PGL}(2,7)}) \leq \omega(\mathcal{N}_{\mathrm{GL}(2,7)}) = 57$, it suffices to show that these are the only nonabelian simple groups with $\omega(\mathcal{N}_G) \leq 57$. Suppose that the result is false, and let G be a minimal counterexample. Thus every proper nonabelian simple section of G is isomorphic to A_5 or $\mathrm{PSL}(2, 7)$. By [3, Proposition 2], G is isomorphic to one of the following groups:

- $\mathrm{PSL}(2, 2^m)$, $m = 4$ or m is a prime;
- $\mathrm{PSL}(2, 3^p)$, $\mathrm{PSL}(2, 5^p)$, $\mathrm{PSL}(2, 7^p)$, p a prime;
- $\mathrm{PSL}(2, p)$, p a prime greater than 11;
- $\mathrm{PSL}(3, 3)$, $\mathrm{PSL}(3, 5)$, $\mathrm{PSL}(3, 7)$;
- $\mathrm{PSU}(3, 3)$, $\mathrm{PSU}(3, 4)$, $\mathrm{PSU}(3, 7)$ (the projective special unitary groups of degree three over the finite fields of orders 3, 4 and 7, respectively); or
- $\mathrm{Sz}(2^p)$, p an odd prime.

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For every prime number p and every integer $n \geq 2$, by Theorem 4.4, $\omega(\mathcal{N}_{\mathrm{PSL}(2,p^n)}) = p^{2n} + p^n + 1$. Thus since $\mathrm{PSL}(2, 2^2) \cong A_5$, among the projective special linear groups, we only need to investigate PSL(3, 3), PSL(3, 5) and PSL(3, 7).

Now PSL(3, 3), PSL(3, 5) and PSL(3, 7) have orders $2^4 \times 3^3 \times 13$, $2^5 \times 3 \times 5^3 \times 31$ and $2^5 \times 3 \times 7^3 \times 19$, respectively. So by Corollary 3.4, $\nu_{13}(\mathrm{PSL}(3, 3)) = 144 > 57$, $\nu_{31}(\mathrm{PSL}(3, 5)) = 4000 > 57$ and $\nu_{19}(\mathrm{PSL}(3, 7)) = 32\,928 > 57$.

Also, PSU(3, 3), PSU(3, 4) and PSU(3, 7) have orders $2^5 \times 3^3 \times 7$, $2^6 \times 3 \times 5^2 \times 13$ and $2^7 \times 3 \times 7^3 \times 43$, respectively. So by Corollary 3.4, $\nu_7(\mathrm{PSU}(3, 3)) = 288 > 57$, $\nu_{13}(\mathrm{PSU}(3, 4)) = 1600 > 57$ and $\nu_{43}(\mathrm{PSU}(3, 7)) = 1 + 43k$, for some $k > 0$. Since 44 does not divide $|\mathrm{PSU}(3, 7)|$ we have $\nu_{13}(\mathrm{PSU}(3, 7)) > 56$.

Finally, $\mathrm{Sz}(2^m)$ has order $2^{2m}(2^m - 1)(2^{2m} + 1)$ and $\nu_2(\mathrm{Sz}(2^m)) = 2^{2m} + 1 \geq 65$ (see [6, Ch. XI, Theorem 3.10]). This completes the proof. □

Acknowledgement

The author is grateful to the referee for his/her helpful comments.

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https://doi.org/10.1017/S0004972711002012 Published online by Cambridge University Press