THE MAXIMUM NUMBER OF DISJOINT PERMUTATIONS CONTAINED IN A MATRIX OF ZEROS AND ONES

D. R. FULKERSON

1. Introduction. A well-known consequence of the König theorem on maximum matchings and minimum covers in bipartite graphs **(5)** or of the P. Hall theorem on systems of distinct representatives for sets **(4)** asserts that an n by n (0, 1)-matrix A having precisely p ones in each row and column can be written as a sum of p permutation matrices:

$$A = P_1 + P_2 + \ldots + P_p. \tag{1.1}$$

Our main objective is a generalization of (1.1) along the following lines. Let A be an arbitrary m by n (0, 1)-matrix. Call an m by n (0, 1)-matrix P a *permutation matrix* if $PP^T = I$, where P^T is the transpose of P and I is the identity matrix of order m. This definition implies $m \leqslant n$ and we shall assume throughout that this inequality holds. As in (1.1), we seek a decomposition

$$A = P_1 + P_2 + \ldots + P_p + R, \tag{1.2}$$

where each P_i, $i = 1, \ldots, p$, is a permutation matrix, R is a (0, 1)-matrix, and the integer p is maximal.

If p is maximal in (1.2), the remainder R, of course, contains no permutation matrix. The converse statement is false, however. For example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has the decompositions

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + R_1,$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + R_2,$$

and neither R_1 nor R_2 contains a permutation matrix.

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In Section 4 a formula for the maximum value of p in (1.2) is derived. Letting $\pi = \pi(A)$ denote this maximum value, it is shown that

$$\pi(A) = \min_{A'} \left[\frac{N_1(A')}{s(A')} \right]. \tag{1.3}$$

Here A' is an e by f minor of A, $N_1(A')$ denotes the number of ones in A', $s(A') = e + f - n$, brackets denote the biggest integer, and the minimum is understood to be taken over all minors A' of A with $s(A') > 0$. The constructive proof singles out a critical minor A' of A that yields the minimum in (1.3). In the example above, critical minors are the identity of order two in the lower right corner, the last column, or the last row.

Using the formula (1.3) and a result due to Haber **(3)**, the integer

$$\tilde{\pi} = \min_{A \in \mathfrak{A}} \pi(A) \tag{1.4}$$

can also be evaluated. In (1.4) $\mathfrak{A}$ is the class of all m by n (0, 1)-matrices that have the same row and column sums as A. The resulting formula for $\tilde{\pi}$ is given in Section 5.

2. Reformulation. The problem of determining $\pi(A)$ for an m by n (0, 1)-matrix $A = (a_{ij})$ may be reformulated in the following way. Determine an m by n (0, 1)-matrix $X = (x_{ij})$ satisfying the constraints

$$\sum_{j=1}^{n} x_{ij} = p, \qquad i = 1, \ldots, m, \tag{2.1}$$

$$\sum_{i=1}^{m} x_{ij} \leqslant p, \qquad j = 1, \ldots, n, \tag{2.2}$$

$$x_{ij} \leqslant a_{ij}, \tag{2.3}$$

and maximizing p.

To see this, first note that if (1.2) holds, then $X = P_1 + P_2 + \ldots + P_p$ satisfies (2.1), (2.2), (2.3). On the other hand, if X is an m by n (0, 1)-matrix with row sums equal to p and column sums at most p, then X is a sum of p permutation matrices. This assertion can be proved in various ways. For instance, a theorem of Mann and Ryser **(6)** concerning the existence of a system of distinct representatives that includes a prescribed set of elements implies that such a matrix X contains a permutation matrix P_1 having a 1 in each column of X of sum p. We may thus write $X = P_1 + R_1$, where R_1 has row sums $p - 1$ and column sums at most $p - 1$, and apply the theorem to R_1. Repeated applications produce the desired decomposition.

The construction described in the next section solves the maximum problem (2.1), (2.2), (2.3) by increasing the parameter p by one at each major cycle until the maximum value π is obtained. Thus, for each value of p encountered in the construction, there will be a decomposition (1.2) for A. The construction can be started with $p = 0$, $X = 0$, although this would, in general, be relatively inefficient from the computational standpoint.

3. A construction. We may suppose, in describing the construction, that $A = X + R$, where each row sum of the (0, 1)-matrix X is either p or $p + 1$, each column sum of X is at most $p + 1$, and R is a (0, 1)-matrix. The aim of a major cycle of the construction is to produce a decomposition $A = X' + R'$, with X' having precisely $p + 1$ 1's in each row and at most $p + 1$ 1's in each column. If successful, the process is repeated with the new decomposition and p replaced by $p + 1$. If unsuccessful, $\pi(A) = p$.

We distinguish the 1's of X in the matrix A and call these "marked 1's" of A; other 1's of A are "unmarked." A row or column of A that contains fewer than $p + 1$ marked 1's will be termed "short." Thus short rows have p marked 1's, but short columns may have fewer than p marked 1's.

The basic routine in the construction assigns "marks" to certain rows and columns of A using the iterative procedure (3.1), (3.2) below.

(3.1) Mark all short rows of A.

(3.2) Repeat the following two steps in order until there are either no newly marked rows or no newly marked columns of A. (The short rows of A are called "newly marked" after application of (3.1), and any other row or column marked in the immediately preceding application of (3.2*b*) or (3.2*a*) is called "newly marked.")

(*a*) For each newly marked row, mark each unmarked column containing an unmarked 1 in that row.

(*b*) For each newly marked column, mark each unmarked row containing a marked 1 in that column.

At the conclusion of the row and column marking process, we distinguish two cases.

CASE 1. A short column of A has been marked. In this case the marking procedure has located a sequence of $r \geqslant 1$ unmarked and $r - 1$ marked 1's having the form

$$a_{i_1 j_1},\quad a_{i_2 j_1},\quad a_{i_2 j_2},\quad \ldots,\quad a_{i_r j_{r-1}},\quad a_{i_r j_r} \tag{3.3}$$

with $a_{i_1 j_1}$ unmarked, $a_{i_2 j_1}$ marked, ..., $a_{i_r j_r}$ unmarked, and such that row i_1 and column j_r are short. We then interchange marked and unmarked 1's in (3.3). This yields a new decomposition $A = X^* + R^*$, with X^* containing one more 1 than X and having row sums p or $p + 1$, column sums at most $p + 1$. With reference to the new marking of 1's given by X^*, A has one less short row. The procedure (3.1), (3.2) is then repeated with the new set of marked 1's of A.

CASE 2. No short column of A has been marked. If A has no marked rows (and hence no short rows), replace p by $p+1$ and repeat (3.1), (3.2) with the new meaning for short row and column. If A has marked rows (and hence short rows), the construction ends.

It is clear that the process terminates. We shall show in the next section that the terminal value of p is $\pi(A)$.

4. A formula for $\pi(A)$. To establish the formula (1.3) for $\pi(A)$, we show first that if (1.2) holds, and if A' is a minor of A with $s(A') > 0$, then

$$p \leqslant [N_1(A')/s(A')]. \tag{4.1}$$

Thus, assume (1.2) and let A' be an e by f minor of A with $s(A') = e+f-n > 0$. Rearranging rows and columns of A, we may write

$$A = \begin{bmatrix} A' & * \\ * & A'' \end{bmatrix} = \begin{bmatrix} X' & * \\ * & X'' \end{bmatrix} + \begin{bmatrix} R' & * \\ * & R'' \end{bmatrix}. \tag{4.2}$$

It follows that

$$p(e+f-n) \leqslant N_1(X') - N_1(X'') \leqslant N_1(A'), \tag{4.3}$$

verifying (4.1).

To complete the proof, it suffices to show that there is a p corresponding to a decomposition (1.2), and an e by f minor A' of A with $s(A') > 0$, for which equality holds in (4.1). Let p be the terminal value in the construction of the preceding section, let A' be the e by f minor corresponding to the terminal sets of marked rows and unmarked columns, and let A'' be the $m-e$ by $n-f$ complementary minor. By (3.2a), A' contains no unmarked 1's; by (3.2b), A'' contains no marked 1's. Moreover, by (3.1), each unmarked row of A contains $p+1$ marked 1's; and, by the Case 2 hypothesis, each marked column of A contains $p+1$ marked 1's. Let t be the total number of marked 1's in A. The above remarks provide a count for t:

$$t = (p+1)(m-e) + (p+1)(n-f) + N_1(A'). \tag{4.4}$$

Since the construction has ended, there is at least one row of A having p marked 1's. Consequently,

$$t < (p+1)m. \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$(p+1)(e+f-n) > N_1(A'). \tag{4.6}$$

In particular, $s(A') = e+f-n > 0$.

Since A has a decomposition (1.2) for the terminal value p, we have, from (4.3) and (4.6),

$$p(e+f-n) \leqslant N_1(A') < (p+1)(e+f-n). \tag{4.7}$$

This shows that the terminal p is $\pi(A)$ and establishes the formula (1.3) for $\pi(A)$.

In case A is n by n with precisely p 1's in each row and column, the formula (1.3) reduces to $\pi(A) = p$. This may be checked as follows. Let $A' = A$ in the right side of (1.3) to get

$$N_1(A)/s(A) = np/n = p.$$

On the other hand, if such an A contained an e by f minor A' with $s(A') > 0$ and

$$[N_1(A')/s(A')] < p,$$

then

$$N_1(A') < p(e + f - n).$$

But writing

$$A = \begin{bmatrix} A' & * \\ * & A'' \end{bmatrix}$$

we see that

$$p(e + f - n) = N_1(A') - N_1(A''),$$

a contradiction. Hence, for such an A, the right side of (1.3) is p.

5. A formula for $\tilde{\pi}$. Let $\mathfrak{A}$ denote the class of all m by n (0, 1)-matrices having row sums

$$r_1 \geqslant r_2 \geqslant \ldots \geqslant r_m \geqslant 0 \tag{5.1}$$

and column sums

$$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n \geqslant 0. \tag{5.2}$$

The class $\mathfrak{A}$ can also be viewed as generated from an arbitrary A in $\mathfrak{A}$ by interchanges **(7)**. Here an *interchange* is a transformation on the elements of A that changes a minor of type (a) below into one of type (b), or vice versa, and leaves all other elements fixed:

$$(a)\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (b)\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this section we describe a formula for

$$\tilde{\pi} = \min_{A \in \mathfrak{A}} \pi(A), \tag{5.3}$$

which is similar to those for maximal term rank **(8)**, minimal term rank **(3)**, maximal and minimal trace **(9)**, and minimal width **(2)**. Note that the normalization assumptions (5.1) and (5.2) are no restriction in determining $\tilde{\pi}$.

The key additional result needed in evaluating $\tilde{\pi}$ has been furnished by

Haber in his study of minimal term rank **(3)**. To describe this result, we begin with the function

$$(5.4)\quad t(e,f) = ef + (r_{e+1} + \ldots + r_m) - (s_1 + \ldots + s_f), \qquad 0 \leqslant e \leqslant m,\ 0 \leqslant f \leqslant n.$$

This function, introduced in **(9)**, can also be viewed in the following manner. Let $A \in \mathfrak{A}$ be written

$$(5.5)\qquad A = \begin{bmatrix} A_1 & * \\ * & A_2 \end{bmatrix}$$

with A_1 of size e by f. Then $t(e,f)$ is the number of 0's in A_1 plus the number of 1's in A_2:

$$(5.6)\qquad t(e,f) = N_0(A_1) + N_1(A_2).$$

Next define, as in **(3)**,

$$(5.7)\qquad \psi(e,f) = \min_{\substack{e \leqslant i \leqslant m \\ f \leqslant j \leqslant n}} t_{ij},$$

$$(5.8)\qquad \phi(e,f) = \min_{\substack{0 \leqslant i \leqslant e \\ 0 \leqslant j \leqslant f \\ e \leqslant k \leqslant m \\ f \leqslant l \leqslant n}} (t_{il} + t_{kj} + (e-i)(f-j)).$$

Now let $H(e,f)$ denote the maximum number of 0's that any matrix in the normalized class $\mathfrak{A}$ can contain in its leading e by f minor. An ingenious argument in **(3)** shows that

$$(5.9)\qquad H(e,f) = \min(\psi(e,f), \phi(e,f)).$$

From (5.6), if $A \in \mathfrak{A}$ contains the maximum number of 0's possible in the leading e' by f' minor, then A contains the minimum number

$$(5.10)\qquad t(e',f') - H(e',f')$$

of 1's possible in the complementary $e = m - e'$ by $f = n - f'$ minor. Thus, provided it can be shown that the minimum number of 1's possible in any e by f minor of matrices in $\mathfrak{A}$ is achieved in the lower right e by f minor of some A in $\mathfrak{A}$, it would follow from (1.3) and (5.10) that

$$(5.11)\qquad \tilde{\pi} = \min_{e',f'} \left[\frac{t(e',f') - H(e',f')}{m - e' - f'} \right],$$

the minimum being taken over e', f' satisfying

$$(5.12)\qquad e' + f' < m.$$

The proviso above can be established by an interchange argument. Let e and f be fixed and consider an e by f minor A' of A in the normalized class $\mathfrak{A}$. Let i be a row of A that is not a row of A', and let i' be a row of A that is a row of A', with $i' < i$. By (5.1), interchanges confined to rows i' and i of A can be applied in such a way that the transformed matrix B has an e by f minor B' with the following properties: Columns of B' have the same index set as those of A'; rows of B' have the same index set as those of A', except that i has replaced i'; $N_1(B') \leqslant N_1(A')$. Repetition of this argument, first on rows, then on columns, shows that the e by f minor with the minimum number of 1's possible in the normalized class $\mathfrak{A}$ can be assumed to be in the lower right corner. Hence, (5.11) is valid.

6. Remarks and questions. The recursive procedure (3.1), (3.2), which was the main tool in establishing the formula (1.3) for $\pi(A)$, is a variation, suitable to the problem at hand, of the labelling method for constructing maximal flows in capacity-constrained networks **(1)**. Formula (1.3) can also be deduced from known theorems concerning network flows.

There is an analogue for matrices over non-negative reals. Specifically, suppose an m by n $A = (a_{ij})$, $a_{ij} \geqslant 0$, is given and we ask for a non-negative matrix $X = (x_{ij})$ satisfying (2.1), (2.2), (2.3) with p maximal. If we let $N_1(A')$ denote the sum of all entries of A', $s(A')$ the sum of the dimensions of A' minus n, then the maximum value of p is given by dropping brackets in (1.3).

If we return to (0, 1)-matrices, other related questions are suggested. For example, what is the minimum value of p in a decomposition (1.2), for a fixed matrix A, assuming that the remainder R contains no permutation matrix? Or what is the maximum value of $\pi(A)$ over the class generated by A? Both of these questions seem harder to answer than those considered here.

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