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THE TRANSFER OF THE KRULL DIMENSION AND THE GABRIEL DIMENSION TO SUBIDEALIZERS

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Let M be a right ideal of the ring T with identity. A unital subring R of T which contains M as a two-sided ideal is called a *subidealizer*; the largest such subring is the idealizer $\mathbf{I}(M)$ of M in T. M is said to be *generative* if $TM = T$. In this case M is idempotent, and it follows from the dual basis lemma that T is finitely generated projective as a right R-module (see [**7**, Lemma 2.1]); we make frequent use of these two facts in this paper.

In recent years, techniques involving subidealizers and idealizers have been employed successfully in order to provide solutions to a number of ring theoretical questions. While most of the results involve only the right-hand side, left-handed properties have been studied as well (see [**2**] and [**9**]). In this paper we investigate the transfer of the Krull and Gabriel dimensions of T-modules when considered as R-modules and of R-modules when their tensor product with T is formed. For right modules this has been done in [**6**] for the case when M is semimaximal (that is, an intersection of a finite number of maximal right ideals). We obtain essentially the same results making use only of the hypotheses that $M = M^2$ and that $(T/M)_R$ has finite length (for the Krull dimension) or $(T/M)_T$ and $(R/M)_R$ are semiartinian (for the Gabriel dimension). For left modules there are some results due to Teply [**9**] who studied the transfer of the Krull dimension for cyclic left modules. In view of the difficulties (see [**2**, p. 416]) concerning the transfer of left-handed chain conditions, it is surprising that the comparatively weak assumptions of R/M being left artinian (for the Krull dimension) and left semi-artinian (for the Gabriel dimension) assure an almost perfect transfer of these dimensions with one exception: passing down from T to R need not preserve the existence of the Krull dimension even if M is maximal, R/M is a field, and T is a left and right noetherian ring.

In each of the sections, the results obtained for modules are applied to show that, with the exception mentioned above, the respective dimension of T equals that of R if either dimension exists. In this context we mention a recent theorem of Armendariz and Fisher ([**1**, Theorem 1.1]) which shows that in case M is generative the respective dimension of T is less than or equal to that of R, provided the latter exists.

We briefly recall some of the definitions used in this article. The *Krull dimension* of a right S-module M is defined recursively by setting K-dim (M) =

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-1 for $M = 0$ and $K\text{-dim}(M) = \alpha$ for an ordinal $\alpha \geqq 0$ if $K\text{-dim}(M) \nless \alpha$ and there is no infinite properly descending chain of submodules $M_1 \supseteq \ldots \supseteq M_i \supseteq M_{i+1} \supseteq \ldots$ with $K\text{-dim}(M_i/M_{i+1}) \nless \alpha$ for all i. We define $r.K\text{-dim}(S) = K\text{-dim}(S)_S$. The *Gabriel dimension* of M is the smallest ordinal for which M is in one of the following subcategories of the category mod-S of all right S-modules: $(\text{mod-}S)_0 = \{0\}$; if α is an ordinal $\geqq 0$, then $(\text{mod-}S)_{\alpha+1}$ is the smallest localizing subcategory of mod-S containing all S-modules which have finite length when considered as objects of the quotient category mod-$S/(\text{mod-}S)_\alpha$; and for a limit ordinal α, $(\text{mod-}S)_\alpha$ is the smallest localizing subcategory containing all $(\text{mod-}S)_\beta$ for $\beta < \alpha$. If $(\text{mod-}S)_\alpha = \text{mod-}S$ for some α, the least such α is called the *right Gabriel dimension of* S and is denoted by $r.G\text{-dim}(S)$. For detailed information on these dimensions we refer to [**3**] and [**4**]. We list only a few that play a major role in our investigations:

(i) If N is a submodule of a module M, then $K\text{-dim}(M) = \max(K\text{-dim}(M/N), K\text{-dim}(N))$ if either side exists [**3**, Lemma 1.1].
(ii) The analogue of (i) for the Gabriel dimension [**4**, Lemma 1.3].
(iii) If M has Krull dimension, then it has finite uniform dimension [**5**, Proposition 4].
(iv) A module M has Krull dimension if and only if it has Gabriel dimension and every homomorphic image of M has finite uniform dimension [**4**, Theorem 2.5].
(v) If the ring S has right Krull dimension and M is a right S-module with Krull dimension, then $K\text{-dim}(M) \leqq r.K\text{-dim}(S)$ [**3**, Corollary 4.4].
(vi) If the ring S has right Gabriel dimension, then $G\text{-dim}(M) \leqq r.G\text{-dim}(S)$ for any right S-module M [**4**, Lemma 1.3].

In [**9**], Teply has shown that if M is a generative right ideal of T and R is a subidealizer of M in T, then the class $\mathscr{T}_M = \{{}_RA \mid MA = 0\}$ is a TTF-class, and there exists a class $\mathscr{F}_M$ of left R-modules such that $(\mathscr{T}_M, \mathscr{F}_M)$ is a perfect faithful torsion theory with T being the quotient ring of R. Throughout this paper we refer to this torsion theory as to the M-torsion theory. The maximal M-torsion submodule of a left R-module A is denoted by $t_M(A) = \{a \in A \mid Ma = 0\}$. Details concerning torsion-theoretical results can be found in [**8**].

1. Gabriel dimension of right modules. Whenever R is the full idealizer of a semimaximal right ideal M of a ring T, it may be assumed (see [**7**, Proposition 1.7]) that $TM = T$. The ring $R/M \simeq \text{End}_T(T/M)$ is then also semisimple artinian. Thus the hypotheses of $(T/M)_T$ and $(R/M)_R$ being semiartinian and of M being idempotent used in this section are considerably weaker than the assumption of the semimaximality of M. We obtain, however, virtually the same results as in [**6**]. We start by studying the transfer of the Gabriel dimension when right T-modules are viewed as right R-modules.

LEMMA 1.1. *Let R be a subidealizer of a right ideal M of T such that $(R/M)_R$ is semiartinian. Then any simple right T-module is semiartinian as a right R-module.*

Proof. For the simple module C_T define the right R/M-module $D = \{x \in C | xM = 0\}$. As $xM = C$ for all $x \in C - D$, $(C/D)_R$ is simple. The result follows since D_R is semiartinian.

PROPOSITION 1.2. *Let R be a subidealizer of a right ideal M of T such that $(T/M)_T$ and $(R/M)_R$ are semiartinian, and let B be a right T-module. Then $G\text{-dim}(B)_T = G\text{-dim}(B)_R$ if either side exists.*

Proof. Assume first that $G\text{-dim}(B)_T = \alpha$. We want to show that $G\text{-dim}(B)_R \leqq \alpha$. The case $\alpha = 0$ is trivial; so let $\alpha \geqq 1$ and assume $G\text{-dim}(X)_R \leqq \beta$ for every right T-module X with $G\text{-dim}(X)_T = \beta < \alpha$. In view of Lemmas 3.1 and 1.4 of [**4**] we only have to establish $G\text{-dim}(X)_R \leqq \gamma$ for every γ-simple right T-module X, where γ is a nonlimit ordinal $\leqq \alpha$. Let $0 \neq N_R \subseteq X_R$. If $NM = 0$, then NT is a homomorphic image of a direct sum of copies of the semiartinian right T-module T/M; hence NT contains a simple T-submodule C. But X_T is Gabriel-simple; so $\gamma = 1$ and $C = X$. Hence $G\text{-dim}(X)_R = 1$ by Lemma 1.1. If $NM \neq 0$, then $G\text{-dim}(X/NM)_T = \beta < \gamma \leqq \alpha$, and hence $G\text{-dim}(X/NM)_R \leqq \beta$ by induction hypothesis. But then

$$G\text{-dim}(X/N)_R \leqq G\text{-dim}(X/NM)_R \leqq \beta < \gamma \leqq \alpha;$$

so $G\text{-dim}(X)_R \leqq \gamma$.

Conversely, let $G\text{-dim}(B)_R = \alpha$. For $\alpha = 0$ there is nothing to show; so let $\alpha \geqq 1$ and assume that $G\text{-dim}(X)_T \leqq \beta$ for every right T-module X with $G\text{-dim}(X)_R = \beta < \alpha$. In order to obtain $G\text{-dim}(B)_T \leqq \alpha$ we employ Lemma 1.4 of [**4**] and establish the existence of a T-submodule $N'/N \neq 0$ with $G\text{-dim}(N'/N)_T \leqq \alpha$ in every nonzero homomorphic image B/N of B_T. Let xR be a γ-simple R-submodule of B/N, where γ is a nonlimit ordinal $\leqq \alpha$. If $xM = 0$, then xT is semiartinian, and we can set $N'/N = xT$. Otherwise we have $G\text{-dim}(xM/Y)_R < \gamma$ for every nonzero T-module Y of xM; hence $G\text{-dim}(xM/Y)_T < \gamma$ by induction hypothesis. But then $G\text{-dim}(xM)_T \leqq \gamma \leqq \alpha$, and hence we take $N'/N = xM$.

We now turn to the transfer of the Gabriel dimension of a right R-module A when it is "turned into a T-module" by forming its tensor product with ${}_RT$.

LEMMA 1.3. *Let R be a subidealizer of an idempotent right ideal M of T, and let A be a right R-module. Then*

(i) *The canonical map $f : AM \rightarrow AM \otimes_R T$ is an isomorphism.*

(ii) *If A_R is flat, then $h : AM \otimes_R T \rightarrow A \otimes_R T$ is a monomorphism.*

Proof. (i) The sequence

$$AM \simeq AM \otimes_R R \xrightarrow{f} AM \otimes_R T \rightarrow AM \otimes_R T/R \rightarrow 0$$

is exact. Since $M = M^2$, we get

$$AM \otimes_R (T/R) = AM^2 \otimes_R (T/R) = AM \otimes_R M(T/R) = AM \otimes_R 0 = 0;$$

so f is an epimorphism. If $m : AM \otimes_R T \to AMT = AM$ denotes the multiplication map, then mf is the identity on AM; so f is also a monomorphism.

(ii) Consider the commutative diagram

$$\begin{array}{ccc} 0 \to AM & \longrightarrow & A \\ f \downarrow & & \downarrow g \\ AM \otimes_R T & \xrightarrow{h} & A \otimes_R T, \end{array}$$

where f is an isomorphism by (i). If A_R is flat, then g is a monomorphism; so h must be a monomorphism.

PROPOSITION 1.4. *Let R be a subidealizer of an idempotent right ideal M of T such that $(T/M)_T$ and $(R/M)_R$ are semiartinian, and let A be a right R-module with Gabriel dimension. Then*

$$G\text{-dim}(A \otimes_R T)_T \leqq G\text{-dim}(A)_R.$$

Proof. We may assume $A \otimes_R T \neq 0$. By Lemma 1.3 we have $AM \simeq AM \otimes_R T$; hence

$$G\text{-dim}(A)_R \geqq G\text{-dim}(AM)_R = G\text{-dim}(AM \otimes_R T)_R = G\text{-dim}(AM \otimes_R T)_T$$

by Proposition 1.2. If we can show that $G\text{-dim}((A/AM) \otimes_R T)_T \leqq 1$, then our claim will follow from Lemma 1.3 of [**4**] and the exactness of the sequence

$$AM \otimes_R T \to A \otimes_R T \to (A/AM) \otimes_R T \to 0.$$

As a right T-module $(A/AM) \otimes_R T$ is generated by elements of the form $x \otimes 1$ with $x \in A/AM$, and these elements are annihilated by M. Each $(x \otimes 1)T$ is thus a homomorphic image of $(T/M)_T$ and hence semiartinian. Hence $((A/AM) \otimes_R T)_T$ is semiartinian; that is, $G\text{-dim}(A/AM \otimes_R T)_T \leqq 1$.

The inequality in 1.4 can be strict. In fact, whenever M is generative, then $T \otimes_R T \simeq T$ under the canonical map induced by the multiplication in T (see [**7**, Lemma 2.1]). This implies $(T/R) \otimes_R T = 0$; hence we get

$$0 = G\text{-dim}((T/R) \otimes_R T)_T < G\text{-dim}(T/R)_R = 1$$

by assuming the validity of all our other hypotheses, and, of course, $T \neq R$. We do not know if this can also happen for $A \otimes_R T \neq 0$. The answer is certainly negative for a semimaximal M with R being the full idealizer (see Proposition 4 of [**6**]), but then it comes as a consequence of the flatness of ${}_RT$. For the case of a semisimple artinian $\mathbf{I}(M)/M$, however, the flatness of ${}_RT$ and the semimaximality of M are equivalent properties by Proposition 1.3 of [**2**].

PROPOSITION 1.5. *Let R be a subidealizer of an idempotent right ideal M of T such that $(T/M)_T$ and $(R/M)_R$ are semiartinian, and let A be a right R-module with* $\mathrm{Tor}_1^R(A, T/R) = 0$. *Then $G\text{-dim}(A)_R = G\text{-dim}(A \otimes_R T)_T$ if either side exists.*

Proof. In view of Proposition 1.4 we only have to show that if $A \otimes_R T$ has Gabriel dimension, then $G\text{-dim}(A \otimes_R T)_T \geqq G\text{-dim}(A)_R$. But as $G\text{-dim}(A \otimes_R T)_T = G\text{-dim}(A \otimes_R T)_R$ by Proposition 1.2 and as the canonical map $A \to A \otimes_R T$ is a monomorphism by $\operatorname{Tor}_1^R(A, T/R) = 0$, the result follows from Lemma 1.3 of [**4**].

THEOREM 1.6. *Let R be a subidealizer of an idempotent right ideal M of T such that $(T/M)_T$ and $(R/M)_R$ are semiartinian. Then $r.G\text{-dim}(R) = r.G\text{-dim}(T)$ if either side exists.*

Proof. Since R_R is flat and $(R \otimes_R T)_T \simeq T_T$, it follows from Proposition 1.5 that $G\text{-dim}(T)_T = G\text{-dim}(R \otimes_R T)_T = G\text{-dim}(R)_R$ if any of these dimensions is defined. The result now follows from Lemma 1.3 of [**4**].

The preceding theorem does not remain true without assuming $(R/M)_R$ to be semiartinian, not even when M is semimaximal.

Example 1.7. Let $\mathbf{Q}$ denote the field of rationals, and let $\mathbf{Z}$ stand for the ring of integers. Let

$$T = \begin{bmatrix} \mathbf{Q} & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ \mathbf{Q} & \mathbf{Q} \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} \mathbf{Z} & 0 \\ \mathbf{Q} & \mathbf{Q} \end{bmatrix}.$$

Then R is a proper subidealizer of M, and M is semimaximal. We have $r.G\text{-dim}(T) = 1$, but $r.G\text{-dim}(R) \geqq r.G\text{-dim}(R/M) = r.G\text{-dim}(\mathbf{Z}) = 2$. Since the same reasoning applies on the left-hand side, this example will also show that the "left semiartinian" hypothesis cannot be dropped in Section 3.

Example 1.8. Let $\alpha \geqq 2$ be a nonlimit ordinal, and let D be a commutative domain of characteristic zero with $G\text{-dim}(D) = \alpha$. For the existence of D we refer to Theorem 9.8 of [**3**] and Corollary 3.5 of [**4**]. Let F denote the field of quotients of D, and let $T = F(y)[x]$ with $xy - yx = 1$. It is well-known that T is a simple hereditary noetherian domain; whence $r.K\text{-dim}(T) = 1$ and $r.G\text{-dim}(T) = 2$. The right ideal $M = xT$ is maximal and generative; its idealizer is $\mathbf{I}_T(M) = F + M$. The ring $R = D + M$ is a subidealizer of M, and $R/M \simeq D$ is not semiartinian. We claim that T_R is $(\alpha + 1)$-simple, and this will prove that $r.G\text{-dim}(R) = \alpha + 1$. Since T is a domain, any nonzero R-submodule of T_R contains a copy of R_R; so it cannot be β-simple for a nonlimit ordinal $\beta \leqq \alpha$. This shows that T_R does not have a nonzero R-submodule X with $G\text{-dim}(X)_R \leqq \alpha$. Now let N be a nonzero R-submodule of T_R. Since T is hereditary noetherian prime, the right T-module T/NM is artinian. To see that $G\text{-dim}(T/N)_R \leqq \alpha$, it is thus sufficient to show that every simple right T-module X satisfies $G\text{-dim}(X)_R \leqq \alpha$. Let $Y = \{x \in X | xM = 0\}$. Since Y is an R/M-module, we have $G\text{-dim}(Y)_R \leqq G\text{-dim}(R/M)_R = \alpha$. If $x \in X - Y$, then $xM = X$ as X is a simple T-module. Thus X/Y is a simple R-module; so $G\text{-dim}(X/Y)_R = 1$. Hence $G\text{-dim}(X)_R \leqq \max\{\alpha, 1\} = \alpha$ by [**4**, Lemma 1.3].

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It is clear that if D is chosen to be a commutative domain without Gabriel dimension (see [**4**, p. 470]), then the subidealizer R in Example 1.8 does not have Gabriel dimension. We do not know if similar pathologies are possible when R is the full idealizer of M.

2. Krull dimension of right modules. It is a consequence of Corollary 1.5 in Robson's paper [**7**] that $(T/R)_R$ is semisimple if R is the idealizer of the semimaximal right ideal M. Together with the semisimplicity of R/M, this makes $(T/M)_R$ a module of finite length. We show in this section that just this condition together with $M = M^2$ guarantees a well behaved transfer of the Krull dimension on the right-hand side.

LEMMA 2.1. *Let R be a subidealizer of a right ideal M of T such that $(T/M)_R$ has finite length, and let B be a right T-module with Krull dimension. If A is an R-submodule of B, then $(A/AM)_R$ has finite length.*

Proof. Obviously $(T/M)_T$ has finite length. Now $(AT/AM)_T$ is a homomorphic image of a direct sum of copies of $(T/M)_T$ and thus has finite Loewy length. It is also a homomorphic image of a submodule of B_T; so it has Krull dimension. But then $(AT/AM)_T$ has finite length, and thus it is a homomorphic image of finitely many copies of $(T/M)_T$. Thus the finite length of $(T/M)_R$ implies the finite length of $(AT/AM)_R$ and also the finite length of its submodule $(A/AM)_R$.

PROPOSITION 2.2. *Let R be a subidealizer of a right ideal M of T such that $(T/M)_R$ has finite length, and let B be a right T-module. Then $K\text{-dim}(B)_T = K\text{-dim}(B)_R$ if either side exists.*

Proof. If B_R has Krull dimension, then clearly B_T has Krull dimension and $K\text{-dim}(B)_R \geqq K\text{-dim}(B)_T$. Assuming that $K\text{-dim}(B)_T = \alpha$, we wish to show that $K\text{-dim}(B)_R \leqq \alpha$, and we proceed by induction on α. The case $\alpha = -1$ is trivial; so let $\alpha \geqq 0$. For a descending chain

(*) $B = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_i \supseteq A_{i+1} \supseteq \ldots$

of R-submodules of B, we form the chain

(**) $BM = A_0M \supseteq A_1M \supseteq \ldots \supseteq A_iM \supseteq A_{i+1}M \supseteq \ldots$

of T-submodules of B. If $\alpha = 0$, then (**) becomes stationary, and hence there is an index n such that

$$A_{n+k}M = A_nM \subseteq A_{n+k} \subseteq A_n \quad \text{for all } k \geqq 0.$$

Since each $(A_i/A_iM)_R$ has finite length by Lemma 2.1, this implies that the chain (*) becomes stationary; so $K\text{-dim}\ (A_i/A_{i+1})_R = -1$ for all sufficiently large i. If $\alpha > 0$, then there is an index j such that $K\text{-dim}(A_iM/A_{i+1}M)_T = \beta_i < \alpha$ for all $i \geqq j$, and hence $K\text{-dim}(A_iM/A_{i+1}M)_R \leqq \beta_i < \alpha$ by induction.

It now follows from Lemma 1.1 of [3] and our Lemma 2.1 that

$$K\text{-dim}(A_i/A_{i+1})_R \leqq K\text{-dim}(A_i/A_{i+1}M)_R$$
$$= \max\{K\text{-dim }(A_i/A_iM)_R, K\text{-dim}(A_iM/A_{i+1}M)_R\}$$
$$\leqq \max\{0, \beta_i\} < \alpha \quad \text{for all } i \geqq j.$$

Hence $K\text{-dim}(B)_R \leqq \alpha$.

PROPOSITION 2.3. *Let R be a subidealizer of an idempotent right ideal M of T such that $(T/M)_R$ has finite length. If A is a right R-module with Krull dimension, then*

$$K\text{-dim}(A \otimes_R T)_T \leqq K\text{-dim}(A)_R.$$

Proof. We may assume that $A \neq 0$, and we get

$$K\text{-dim}(A)_R \geqq K\text{-dim}(AM)_R = K\text{-dim}(AM \otimes_R T)_R$$
$$= K\text{-dim}(AM \otimes_R T)_T$$

from Lemma 1.3 and Proposition 2.2. By Lemma 1.1 of [3] it will be sufficient to establish that $((A/AM) \otimes_R T)_T$ has finite length. Since $(A/AM)_R$ is a module with Krull dimension over the right artinian ring R/M, it has finite length by Corollary 4.4 of [3]. Our hypothesis also implies the finite length of $(T/R)_R$; so $((A/AM) \otimes_R (T/R))_R$ has finite length. It now follows from the exactness of the sequence

$$A/AM \rightarrow (A/AM) \otimes_R T \rightarrow (A/AM) \otimes_R (T/R) \rightarrow 0$$

that $(A/AM) \otimes_R T$ has finite length as an R-module and hence also as a T-module.

The remark following the proof of Proposition 1.4 can also be used here to show that the inequality in the preceding result can actually be strict. The following two results are proved in complete analogy to Proposition 1.5 and Theorem 1.6.

PROPOSITION 2.4. *Let R be a subidealizer of an idempotent right ideal of M of T such that $(T/M)_R$ has finite length, and let A be a right R-module with* $\text{Tor}_1{}^R(A, T/R) = 0$. *Then* $K\text{-dim}(A)_R = K\text{-dim}(A \otimes_R T)_T$ *if either side exists.*

THEOREM 2.5. *Let R be a subidealizer of an idempotent right ideal M of T such that $(T/M)_R$ has finite length. Then* $r.K\text{-dim}(R) = r.K\text{-dim}(T)$ *if either side exists.*

The following example shows that Theorem 2.5 does not remain valid if we only assume that $(T/M)_T$ and $(R/M)_R$ are of finite length.

Example 2.6. Let $\mathbf{R}$ and $\mathbf{Q}$ denote the fields of real and rational numbers respectively. Set

$$T = \begin{bmatrix} \mathbf{R} & \mathbf{R} \\ \mathbf{R} & \mathbf{R} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ \mathbf{R} & \mathbf{R} \end{bmatrix}, \quad R = \begin{bmatrix} \mathbf{Q} & 0 \\ \mathbf{R} & \mathbf{R} \end{bmatrix}.$$

Then M is a generative, semimaximal right ideal of T, and $R/M \simeq \mathbf{Q}$ is a field. T is artinian, but R does not have right Krull dimension because R_R does not have finite uniform dimension. If, for example, the real numbers r_i form a basis of the $\mathbf{Q}$-vector space $\mathbf{R}$, then the right ideals

$$N_i = \begin{bmatrix} 0 & 0 \\ r_i\mathbf{Q} & 0 \end{bmatrix}$$

form an infinite direct sum.

3. Gabriel dimension of left modules. For this section and the following one we assume M to be a generative right ideal T; so by Lemma 2.1 of [**7**] the ring T will be a finite left localization of any subidealizer R of M. Although this fact has generally not resulted in as many strong connections between R and T as one might expect, it is nevertheless very useful for our investigation of the transfer of the Gabriel dimension and the Krull dimension of left modules.

The following result appears in [**9**] (Lemma 3.2); we present a more straightforward proof.

LEMMA 3.1. *Let R be the subidealizer of a generative right ideal M of T, and let A be a left T-module. Then the following statements hold.*

(i) *${}_TA$ is simple if and only if ${}_R(MA)$ is simple.*

(ii) *${}_R(MA)$ is an essential submodule of ${}_RA$.*

Proof. (i) If ${}_TA$ is simple and $0 \neq a \in A$, then $W = \{x \in A | Mx \subseteq Ra\}$ is a nonzero T-submodule of A; so $W = A$. Thus $MA \subseteq Ra$ for any nonzero element a in A; so ${}_R(MA)$ is certainly simple. Conversely, assume that ${}_R(MA)$ is simple, and let $0 \neq {}_TX \subseteq {}_TA$. As M is generative, $MX \neq 0$, and hence $MX = MA$. Thus $X = TX = TMX = TMA = TA = A$, which proves that ${}_TA$ is simple.

(ii) As M is generative, we get $0 \neq Mx \subseteq Rx \cap MA$ for any nonzero element x of A.

PROPOSITION 3.2. *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is semiartinian, and let B be a left T-module. Then $G\text{-dim}_R(B) = G\text{-dim}_T(B)$ if either side exists.*

Proof. First we show that $G\text{-dim}_T(B) = \alpha$ implies that $G\text{-dim}_R(B) \leqq \alpha$. By Lemmas 1.4 and 3.1 of [**4**] it will be sufficient to establish that $G\text{-dim}_R(Y) \leqq \alpha$ for any γ-simple left T-module Y, where γ denotes a nonlimit ordinal $\leqq \alpha$. If $\alpha = 1$, then ${}_R(MY)$ is simple by Lemma 3.1, and as the R/M-module

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${}_R(Y/MY)$ is semiartinian, we obtain $G\text{-dim}_R(Y) \leqq 1$. Now let $\alpha > 1$, and assume that $G\text{-dim}_R(X) \leqq \beta$ for every left T-module X with $G\text{-dim}_T(X) = \beta < \alpha$. Let ${}_TY$ be γ-simple for some nonlimit ordinal $\gamma \leqq \alpha$. If $0 \neq {}_RN \subseteq Y$, then $G\text{-dim}_T(Y/TN) = \delta < \gamma \leqq \alpha$; so $G\text{-dim}_R(Y/TN) \leqq \delta < \gamma \leqq \alpha$ by induction hypothesis. Since TN/N is an R/M-module, $G\text{-dim}_R(TN/N) \leqq 1$. Hence we get

$$G\text{-dim}_R(Y/N) = \max\{G\text{-dim}_R(Y/TN), G\text{-dim}_R(TN/N)\} \leqq \max\{\delta, 1\} < \alpha$$

by Lemma 1.3 of [**4**]. Thus $G\text{-dim}_R(Y) \leqq \alpha$.

Conversely, assume $G\text{-dim}_R(B) = \alpha$. For the case $\alpha = 1$ let B/A be an arbitrary T-homomorphic image of ${}_TB$, and let C be a simple R-submodule of B/A. Since M is generative, $MC \neq 0$, and hence $MC = C$. Thus TC is a simple T-submodule of B/A by Lemma 3.1. Thus ${}_TB$ is semiartinian; that is, $G\text{-dim}_T(B) = 1$. Now let $\alpha > 1$. We assume that $G\text{-dim}_T(X) \leqq \beta$ for every left T-module X with $G\text{-dim}_R(X) = \beta < \alpha$, and we wish to show that $G\text{-dim}_T(B) \leqq \alpha$. By Lemma 1.4 of [**4**] this can be achieved by finding a nonzero submodule N'/N with $G\text{-dim}_T(N'/N) \leqq \alpha$ in any T-homomorphic image $B/N \neq 0$ of ${}_TB$. Let $Rx \neq 0$ be a γ-simple R-submodule of B/N, where γ is a nonlimit ordinal $\leqq \alpha$, and let ${}_TY$ be a nonzero submodule of Tx. We get $G\text{-dim}_R(Tx/(Y + Mx)) \leqq 1$ because R/M is semiartinian. Since $Mx = M(Tx)$ is an essential submodule of ${}_R(Tx)$ by Lemma 3.1, we get $Mx \cap Y \neq 0$, and since Mx is also a γ-simple R-module, we obtain $G\text{-dim}_R((Y + Mx)/Y) = G\text{-dim}_R(Mx/(Y \cap Mx)) < \gamma$. If now $\gamma = 1$, then $G\text{-dim}_R(Tx/Y) \leqq 1$, and hence $G\text{-dim}_T(Tx/Y) \leqq 1$ by the case $\alpha = 1$. If $\gamma > 1$, then $G\text{-dim}_R(Tx/Y) < \gamma$; hence $G\text{-dim}_T(Tx/Y) < \gamma$ by induction. Thus we get $G\text{-dim}_T(Tx/Y) < \alpha$ in either case, and as this is valid for any nonzero T-submodule Y of Tx, it follows that $G\text{-dim}_T(Tx) \leqq \alpha$; so we can set $N'/N = Tx$.

PROPOSITION 3.3. *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is semiartinian, and let A be a left R-module. Then the following statements hold.*

(i) *If A is a nonzero M-torsion module, then* $G\text{-dim}_R(A) = 1$ *and* $G\text{-dim}_T(T \otimes_R A) = 0$.

(ii) *If A is not M-torsion or if $A = 0$, then* $G\text{-dim}_R(A) = G\text{-dim}_T(T \otimes_R A)$ *if either side exists.*

Proof. (i) If ${}_RA$ is M-torsion and nonzero, then as an R/M-module it is semiartinian; that is, $G\text{-dim}_R(A) = 1$. As M is generative, we get $T \otimes_R A = TM \otimes_R A = T \otimes_R MA = T \otimes_R 0 = 0$.

(ii) Since the case $A = 0$ is trivial, we assume that $A \neq 0$ and $A \supset t_M(A)$. By (i) we have $G\text{-dim}_R(t_M(A)) \leqq 1$, and hence

(*) $G\text{-dim}_R(A) = G\text{-dim}_R(A/t_M(A))$

if either side exists. By Lemma 0.1 of [**9**] the M-torsion theory is perfect; so

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we get the monomorphism $A/t_M(A) \xrightarrow{f} T \otimes_R (A/t_M(A))$, and as

$$M(T \otimes_R (A/t_M(A))) \subseteq \operatorname{Im}(f),$$

we have $G\text{-dim}_R((T \otimes_R (A/t_M(A)))/\operatorname{Im}(f)) \leqq 1$. Thus

$$(^{**})\ G\text{-dim}_R(A/t_M(A)) = G\text{-dim}_R(\operatorname{Im}(f)) = G\text{-dim}_R(T \otimes_R (A/t_M(A)))$$

if any of these exists. Since T_R is projective by Lemma 2.1 of [**7**], then $T \otimes_R (A/t_M(A)) \simeq (T \otimes_R A)/(T \otimes_R t_M(A)) = T \otimes_R A$ by (i). By Proposition 3.2 we get

$$(^{***})\ G\text{-dim}_R(T \otimes_R (A/t_M(A))) = G\text{-dim}_R(T \otimes_R A) = G\text{-dim}_T(T \otimes_R A),$$

provided any of these dimensions exist. Putting (*), (**), and (***) together gives the desired conclusion.

THEOREM 3.4. *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is semiartinian. Then $l.G\text{-dim}(R) = l.G\text{-dim}(T)$ if either side exists.*

Proof. Since R is certainly not M-torsion, we apply part (ii) of Proposition 3.3 and obtain

$$G\text{-dim}_R(R) = G\text{-dim}_T(T \otimes_R R) = G\text{-dim}_T(T)$$

whenever one of the ordinals exists. The result now follows from Lemma 1.3 of [**4**].

The hypothesis that ${}_R(R/M)$ is semiartinian cannot be dropped from Theorem 3.4. This is shown by the following example which is a continuation of Example 1.8.

Example 3.5. Let T, R, and M be as in Example 1.8, and let S denote the full idealizer of M in T. We claim that ${}_RT$ is $(\alpha + 1)$-simple. Since $G\text{-dim}_R(R/M) = \alpha$, and since R/M is a proper homomorphic image of ${}_RR$, the left R-module ${}_RR$ is certainly not β-simple for any nonlimit ordinal $\beta \leqq \alpha$. If X is a nonzero R-submodule of ${}_RT$, then any Gabriel-simple R-submodule of X contains a copy of ${}_RR$; so $G\text{-dim}_R(X) \nless \alpha$. Now let L be a nonzero R-submodule of ${}_RT$. We wish to show that $G\text{-dim}_R(T/L) \leqq \alpha$. Since TL/L is an R/M-module, and since T/TL is artinian as T is hereditary noetherian prime, this will follow if we can show that every simple left T-module X satisfies $G\text{-dim}_R(X) \leqq \alpha$. Now $G\text{-dim}_S(X) = G\text{-dim}_T(X) = 1$ by Proposition 3.2. Hence we need only show that each simple left S-module Y satisfies $G\text{-dim}_R(Y) \leqq \alpha$. Let $Z = \{y \in Y | My = 0\}$. Then Z is an R/M-module; whence $G\text{-dim}_R(Z) \leqq \alpha$. If $y \in Y - Z$, then $My = Y$ as ${}_SY$ is simple. Thus Y/Z is a simple left R-module. The assertion now follows from [**4**, Lemma 1.3]. Since ${}_RT$ is $(\alpha + 1)$-simple, then $l.G\text{-dim}(R) = \alpha + 1$, but $l.G\text{-dim}(T) = 2$.

We end this section by listing an obvious corollary.

COROLLARY 3.6. *If F is any unital subfield of the center of a ring T with left Gabriel dimension, and if M is any generative right ideal of T, then* $l.G$-dim $(F + M) = l.G$-dim(T).

4. Krull dimension of left modules. In the preceding three sections it has always been sufficient to put certain restrictions on M, T, and R in order to assure a satisfactory transfer of the dimensions between R-modules and T-modules; no reference had to be made to the internal structure of the modules. This changes drastically as we consider the Krull dimension of left modules. While we still get $K\text{-dim}_R(B) = K\text{-dim}_T(B)$ for a left T-module B if $K\text{-dim}_R(B)$ exists, the existence of $K\text{-dim}_T(B)$ does not grant the existence of $K\text{-dim}_R(B)$ without the presence of certain other properties of B. This can happen even when very stringent conditions are assumed for M, T, and R.

PROPOSITION 4.1. *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is artinian, and let $\alpha \geqq 0$ be an ordinal. The following statements are equivalent for the left T-module B.*

(1) (i) $K\text{-dim}_T(B) = \alpha$.

(ii) *Every M-torsion submodule of a homomorphic image of ${}_RB$ has finite uniform dimension.*

(2) $K\text{-dim}_R(B) = \alpha$.

Proof. (1) → (2): Since ${}_TB$ is a module with Krull dimension, it has Gabriel dimension by Corollary 2.2 of [**4**], and then ${}_RB$ has Gabriel dimension by Proposition 3.2. Thus ${}_R(TA/MA)$ has Gabriel dimension for any R-submodule A of B. Each R-homomorphic image of TA/MA has finite uniform dimension by (1) (ii); so ${}_R(TA/MA)$ is a module with Krull dimension by Theorem 2.5 of [**4**]. But as R/M is assumed to be left artinian, this implies that $K\text{-dim}_R (TA/MA) \leqq 0$. Let now

$$(*)\ B = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_i \supseteq A_{i+1} \supseteq \ldots$$

be a descending chain of R-submodules of B. Then

$$(**)\ B = TA_0 \supseteq TA_1 \supseteq \ldots \supseteq TA_i \supseteq TA_{i+1} \supseteq \ldots$$

is a chain of T-submodules. If $\alpha = 0$, then $TA_i = TA_{i+1}$ for some j and all $i \geqq j$. But then

$$\ldots = TA_{j+1} = TA_j \supseteq A_j \supseteq A_{j+1} \supseteq \ldots \supseteq MA_j = MA_{j+1} = \ldots,$$

and as $K\text{-dim}_R(TA_j/MA_j) \leqq 0$ by the above, we see that the chain (*) becomes stationary. For $\alpha > 0$ there is an index j such that

$$K\text{-dim}_T(TA_i/TA_{i+1}) = \beta_i < \alpha \quad \text{for all } i \geqq j;$$

whence

$$K\text{-dim}_R(TA_i/TA_{i+1}) \leqq \beta_i < \alpha \quad \text{for all } i \geqq j$$

by the inductive hypothesis. Together with $K\text{-dim}_R(TA_{i+1}/MA_{i+1}) \leqq 0$ and $TA_i \supseteq A_i \supseteq A_{i+1} \supseteq MA_{i+1}$ we get from Lemma 1.1 of [**3**] that

$$K\text{-dim}_R(A_i/A_{i+1}) \leqq \max(\beta_i, 0) < \alpha \quad \text{for all } i \geqq j.$$

Thus $K\text{-dim}_R(B) \leqq \alpha = K\text{-dim}_T(B)$, and the reverse inequality is obvious.

(2) $\rightarrow$ (1): If $K\text{-dim}_R(B) = \alpha$, then $K\text{-dim}_T(B) \leqq \alpha$, and by Proposition 4 of [**5**] every homomorphic image of ${}_RB$ has finite uniform dimension. But then $K\text{-dim}_T(B) = \alpha$ by the discussion of (1) $\rightarrow$ (2).

PROPOSITION 4.2. *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is artinian, and let A be a nonzero left R-module with Krull dimension. Then ${}_T(T \otimes_R A)$ has Krull dimension and the following statements hold.*

(i) $K\text{-dim}_T(T \otimes_R A) = -1$ *if and only if A is M-torsion. In this case* $K\text{-dim}_R(A) = 0$.

(ii) $K\text{-dim}_R(A) = K\text{-dim}_T(T \otimes_R A)$ *if A is not M-torsion.*

Proof. (i) If A is M-torsion, then A is an R/M-module; whence $K\text{-dim}_R(A) = 0$ by Corollary 4.4 of [**3**]. Since the M-torsion theory is perfect (see [**9**, Lemma 0.1]), then $T \otimes_R A = 0$ if and only if A is M-torsion by Ex. 7 on page 242 of [**8**]; whence the result follows.

(ii) It follows from (i) that $K\text{-dim}_R(A) = K\text{-dim}_R(A/t_M(A))$, and that $T \otimes_R (A/t_M(A)) \simeq T \otimes_R A$. Furthermore, since M is generative, the mapping $X \rightarrow MX$ from the lattice of T-submodules of $T \otimes_R F$ into the lattice of R-submodules of F preserves strict inequalities for any M-torsion-free left R-module F. Putting these facts together, we obtain $K\text{-dim}_T(T \otimes_R A) \leqq K\text{-dim}_R(A)$. To obtain equality, we proceed by induction on $\alpha = K\text{-dim}_T (T \otimes_R A)$. Assume first that $\alpha = 0$, and let

$$(*)\ A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_i \supseteq A_{i+1} \supseteq \ldots$$

be a descending chain of R-submodules of A. As T_R is projective and thus flat by Lemma 2.1 of [**7**], we obtain the chain

$$(**)\ T \otimes_R A \supseteq T \otimes_R A_1 \supseteq \ldots \supseteq T \otimes_R A_i \supseteq T \otimes_R A_{i+1} \supseteq \ldots$$

of T-submodules of $T \otimes_R A$. Since $T \otimes_R A$ is artinian, there is an index n such that $T \otimes_R A_i = T \otimes_R A_{i+1}$ and hence $T \otimes_R (A_i/A_{i+1}) = 0$ for all $i \geqq n$. By (i) this implies that A_i/A_{i+1} is M-torsion; that is, $MA_i \subseteq A_{i+1}$ for all $i \geqq n$. Since $M = M^2$, it follows that

$$A_n \supseteq A_{n+1} \supseteq \ldots \supseteq A_{n+k} \supseteq \ldots \supseteq MA_n.$$

As A_n/MA_n has Krull dimension and is M-torsion, the chain (*) becomes stationary; so $K\text{-dim}_R(A) = 0$. Now let $\alpha \geqq 1$, and consider again the descending chains (*) and (**) of R-submodules of A and T-submodules of

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$T \otimes_R A$ respectively. It follows that there is an index n such that

$$K\text{-dim}_T(T \otimes_R (A_i/A_{i+1})) = K\text{-dim}_T((T \otimes_R A_i)/(T \otimes_R A_{i+1})) < \alpha$$

for all $i \geqq n$; so $K\text{-dim}_R(A_i/A_{i+1}) < \alpha$ by induction. Thus $K\text{-dim}_R(A) \leqq \alpha$.

The following example, which is discussed in [**7**], shows that even though $T \otimes_R A$ has Krull dimension as a left T-module whenever ${}_RA$ has Krull dimension, $T \otimes_R A$ need not have Krull dimension as an R-module. It also shows that condition (1) (ii) of Proposition 4.1 is not superfluous.

Example 4.3. Let F be a field of characteristic zero, let $T = F(y)[x]$ with $xy - yx = 1$, let $M = xT$, and let $R = F + xT$. Obviously, M is a maximal right ideal of T, and as T is a simple hereditary noetherian domain, M is generative and $l.K\text{-dim}(T) = 1$. By Theorem 7.4 of [**7**] we also have $l.K\text{-dim}$ $(R) = 1$. Finally $R/M \simeq F$ is a field. As now $\sum_{i=0}^{\infty} R(y^i + xT)$ is a direct sum of nonzero R-submodules of the left F-module T/xT, the module ${}_RT \simeq {}_R(T \otimes_R R)$ has no Krull dimension.

As an application of Proposition 4.2 and our results in Section 3 we now give a different proof of Theorem 2.3 of [**9**].

THEOREM 4.4. (Teply [**9**]) *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is artinian. The following properties of the proper left ideal L of T are equivalent.*

(1) $K\text{-dim}_R(R/ML) = \alpha$.

(2) (i) $K\text{-dim}_T(T/L) = \alpha$.

(ii) *For every left ideal H of T containing L the left R-module $(H \cap R)/MH$ has finite length.*

Proof. Since T_R is projective ([**7**, Lemma 2.1]) and hence flat, the sequence

$$0 \to T \otimes_R ML \to T \otimes_R R \to T \otimes_R (R/ML) \to 0$$

is exact. By flatness of T_R we also have $T \otimes_R ML \simeq TML = TL = L$ as left T-modules under the multiplication map. Since $T \otimes_R R \simeq T$ canonically, we get $T \otimes_R (R/ML) \simeq T/L$.

(1) $\to$ (2): Since M is generative and $T \neq L$, R/ML cannot be M-torsion. The remark above and Proposition 4.2 yield

$$K\text{-dim}_R(R/ML) = K\text{-dim}_T(T \otimes_R(R/ML)) = K\text{-dim}_T(T/L).$$

If ${}_TH \supseteq {}_TL$, then $(H \cap R)/MH$ is a subfactor of R/ML and as such it is a left R-module with Krull dimension. As it is an R/M-module and as R/M is assumed to be left artinian, $(H \cap R)/MH$ has finite length.

(2) $\to$ (1): In view of the preceding argument we only have to establish the existence of $K\text{-dim}_R$ (R/ML). For this, we note that as a module with Krull dimension ${}_T(T/L)$ has Gabriel dimension by Corollary 2.2 of [**4**]; so ${}_R(T/L)$ has Gabriel dimension by our Proposition 3.2. The module ${}_R(L/ML)$

has Gabriel dimension because it is an R/M-module, and therefore ${}_R(R/ML)$ has Gabriel dimension by Lemma 1.3 of [**4**]. By Theorem 2.5 of [**4**] it will also have Krull dimension if we can show that each of its factor modules has finite uniform dimension. Assume the existence of an infinite direct sum $\bigoplus_{i\in I} Y_i/Y$ of nonzero R-submodules of R/Y with ${}_RY \supseteq ML$. Without loss of generality, we may assume the Y_i/Y to be either M-torsion or M-torsion-free. Let X/Y denote the sum of all M-torsion modules among the Y_i/Y. Then $MTX = MX \subseteq Y$ and $X \subseteq TX \cap R$; thus X/Y is a subfactor of $(TX \cap R)/MTX$ and is therefore of finite length because of (2)(ii). Thus we may assume all the Y_i/Y to be M-torsion-free. But then as T is the quotient ring of R with respect to the perfect M-torsion theory, $T \otimes_R (Y_i/Y) \neq 0$ for all $i \in I$; hence $T \otimes_R (\bigoplus_{i\in I} Y_i/Y) \simeq \bigoplus_{i\in I} (T \otimes_R (Y_i/Y))$ does not have finite uniform dimension as a left T-module. But because of the flatness of T_R, $T \otimes_R (\bigoplus_{i\in I} Y_i/Y)$ is a subfactor of $T \otimes_R R/ML \simeq T/L$, which is a left T-module with Krull dimension. By Proposition 4 of [**5**], this is impossible.

COROLLARY 4.5. (Teply [**9**]) *Let R be a subidealizer of a generative right ideal M of T such that ${}_R(R/M)$ is artinian. If R has left Krull dimension, then so does T and* $l.K\text{-dim}(T) = l.K\text{-dim}(R)$.

Proof. Specialize the preceding result to $L = 0$.

We end this paper with an example to show that contrary to what one might expect in view of Theorems 1.6, 2.5, and 3.4, the Krull dimension is generally not passed down from T to R.

Example 4.6. Let $D = \bar{Z}_2[t; \rho]$ be the twisted polynomial ring with coefficients in the algebraic closure $\bar{Z}_2$ of the two-element field Z_2, where the automorphism ρ of $\bar{Z}_2$ is given by ρ: $a \to a^2$. It is well known that D is a principal right and left ideal domain and that the only non-zero two-sided ideals of D are those of the form t^nD, where n is an integer $\geqq 0$. It is easy to check that $(t+1)D$ is a generative maximal right ideal of D with idealizer $\mathbf{I}((t+1)D) = Z_2 + (t+1)D$. We now set

$$T = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}, \quad M = \begin{bmatrix} (t+1)D & D \\ 0 & D \end{bmatrix}, \quad \text{and}$$

$$R = \begin{bmatrix} Z_2 + (t+1)D & D \\ 0 & D \end{bmatrix} = \mathbf{I}(M).$$

Then T is left and right noetherian, M is a generative maximal right ideal, and $R/M \simeq Z_2$ is a field. But R does not have left Krull dimension. For consider the left ideals

$$L = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & (t+1)D \\ 0 & 0 \end{bmatrix}$$

of R. As $ML \subseteq K$, the lattice of left R-submodules of L/K is isomorphic to the lattice of left Z_2-submodules of L/K. But since $L/K \simeq D/(t+1)D \simeq \bar{Z}_2$ as Z_2-modules and since $\bar{Z}_2$ is not a finite extension of Z_2, the left R-module L/K does not have finite uniform dimension.

We note that our Theorems 1.6, 2.5, and 3.4 can all be applied to the rings in this example, and as $l.K\text{-dim}(T) = 1 = r.K\text{-dim}(T)$, we get $r.K\text{-dim}(R) = 1$ and $l.G\text{-dim}(R) = l.G\text{-dim}(T) = 2 = r.G\text{-dim}(R) = r.G\text{-dim}(T)$.

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