BOUNDARY COMPONENTS OF RIEMANN SURFACES*

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Introduction. The boundary components of an abstract Riemann surface were defined by B. v. Kérékjártó [7] and utilized in the book [14] written by S. Stoïlow.[1] It is the purpose of the present paper to investigate their images under conformal mapping and to solve the Dirichlet problem with boundary values distributed on them.

Suppose that the universal covering surface of a Riemann surface $\mathfrak{R}$ is of hyperbolic type, that is, conformally equivalent to the disc $U: |z| < 1$. The work [15] by M. Tsuji shows that the linear measure of the image on $\Gamma: |z| = 1$ of the set $\mathfrak{C}_{\mathfrak{R}}$ of all boundary components of $\mathfrak{R}$ is 0 or 2π according as $\mathfrak{R}$ has a null or positive boundary. The writer in [8] studied topologically the image on Γ of each boundary component. In Chapter I of the present paper we shall continue this study.

The set $\mathfrak{C}_{\mathfrak{R}}$ may be regarded as a topological space, as was done by Stoïlow [14]. We are naturally led to consider the Dirichlet problem on $\mathfrak{R}$ with boundary values on $\mathfrak{C}_{\mathfrak{R}}$, with respect to this topology. We shall treat this problem in Chapter II by the Perron-Brelot's method; it was proposed in [8] but left open there.[2]

Chapter I. Boundary Correspondence

1. Definition of boundary components. Throughout this paper let $\mathfrak{R}$ be an

Received December 20, 1953.

* This is the work indicated at the footnote 5) of [10]. The essential part of the present paper was first reported to the Annual Meeting of Japanese Mathematical Society held in Tokyo, Japan, in June, 1952, and then to the conference at Michigan, U.S.A., in 1953 (see [11]).

1) Kérékjártó and Stoïlow called them Randstücke and éléments-frontières respectively. The writer used the term "ideal boundary component" in [8] but now drops the word "ideal."

2) It was pointed out in the lecture given by M. Brelot at the conference at Michigan in 1953 (see [4]) that the solution of this problem follows also from the results in [5].

https://doi.org/10.1017/S0027763000018067 Published online by Cambridge University Press

open Riemann surface of connectivity at least three.[3] We take an exhaustion $\{\mathfrak{R}_n\}$, $\mathfrak{R}_n^a \subset \mathfrak{R}_{n+1}$, of $\mathfrak{R}$, where $\mathfrak{R}_n^a$ means the closure of $\mathfrak{R}_n$ and is compact in $\mathfrak{R}$. We select $\{\mathfrak{R}_n\}$ so that each $\mathfrak{R}_n$ is bounded by a finite number of simple closed analytic curves, each of which divides $\mathfrak{R}$ into two non-simply-connected domains, and that any two of the boundary curves of $\mathfrak{R}_1$ are not homotopic to each other. Let the boundary curves of $\mathfrak{R}_1$ be $\gamma_1, \gamma_2, \ldots, \gamma_\mu$, and the domain, outside $\mathfrak{R}_1$ and bounded by γ_i, be D_i. Let the boundary curves of $\mathfrak{R}_2$, lying in D_i, be $\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,\mu(i)}$, and the domain, outside $\mathfrak{R}_2$ and bounded by $\gamma_{i,j}$, be $D_{i,j}$, and so on. Thus we get domains $\{D_{i,j,\ldots,k}\}$ such that $D_{i,j,\ldots,k} \supset D_{i,j,\ldots,k,\ldots,l}$, and their relative boundaries $\{\gamma_{i,j,\ldots,k}\}$ in $\mathfrak{R}$. We shall call these domains *elementary domains.* Here we may, and do, add an assumption that $\gamma_{i,j,\ldots,k}$ and $\gamma_{i,j,\ldots,k,l}$ are not homotopic to each other on $\mathfrak{R}$ unless $D_{i,j,\ldots,k}$ is doubly-connected.

With each nested infinite sequence $D_i \supset D_{i,j} \supset \ldots$ we associate a *boundary component* and call the sequence the *determining sequence* of the component. Let $D_{i,j,\ldots,k}$ be an elementary domain, and add to it all boundary components determined by the sequences which begin with $D_i \supset D_{i,j} \supset \ldots \supset D_{i,j,\ldots,k}$. We denote this set by $\overline{D}_{i,j,\ldots,k}$. We take both all $\{\overline{D}_{i,j,\ldots,k}\}$ and a countable open base of $\mathfrak{R}$ as a countable open base in the space $\mathfrak{R} + \mathfrak{C}_{\mathfrak{R}}$, where $\mathfrak{C}_{\mathfrak{R}}$ denotes the set of all boundary components of $\mathfrak{R}$. Then it is easily shown that $\mathfrak{R} + \mathfrak{C}_{\mathfrak{R}}$ is a compact space with respect to this topology. Since it is a Hausdorff space, it is normal and hence metrizable. Further $\mathfrak{C}_{\mathfrak{R}}$ is a null-dimensional space as is seen from the definition.

Let $P_{\mathfrak{C}}$ be a boundary component and $\{D_{i,j,\ldots,k}\}$ its determining sequence. $P_{\mathfrak{C}}$ is said to belong to the *first class* if and only if one of $\{D_{i,j,\ldots,k}\}$ is of planar character, and, otherwise, to the *second class.* We now suppose that one of $\{D_{i,j,\ldots,k}\}$ is of planar character, and map it conformally and in a one-to-one manner onto a plane domain D by the Koebe's uniformization theorem. If the images in D of the boundaries $\{\gamma_{i,j,\ldots,k}\}$ converge to an isolated boundary point of D, $P_{\mathfrak{C}}$ is called *parabolic.* Any non-parabolic boundary component, regardless

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[3] This means that, if the genus of $\mathfrak{R}$ is zero, $\mathfrak{R}$ shall be conformally equivalent to a plane domain with at least three boundary components.

If we admit the case when $\mathfrak{R}$ is simply- or doubly-connected, we must make special mentions of these cases often in the sequel, while the treatment of them is easy. Therefore we omit these cases in this paper.

of its class, is called *hyperbolic.*

2. Regular and singular points with respect to a Fuchsian or Fuchsoid group. Under our assumption that $\mathfrak{R}$ is of connectivity at least three, the universal covering surface $\mathfrak{R}^\infty$ of $\mathfrak{R}$ is mapped conformally onto $U: |z| < 1$. We shall denote the corresponding function mapping U onto $\mathfrak{R}$ by $f(z)$, and the Fuchsian or Fuchsoid group, with respect to which $f(z)$ is automorphic, by $\mathfrak{G}$. This group does not contain any elliptic substitution. The fixed points of parabolic and hyperbolic substitutions of $\mathfrak{G}$ lie on $\Gamma: |z| = 1$ and are called *parabolic* and *hyperbolic fixed points* respectively. If there is a sequence of points $\{z_n\}$ in U such that z_n tends to a point z_0 on Γ and $f(z_n)$ tends to a point P_0 of $\mathfrak{R}$ as $n \to \infty$, z_0 is called a *singular point.* If at z_0 there exists no such sequence, z_0 is called a *regular point.* It can be shown as in the case where $\mathfrak{R}$ is a plane domain (cf. [1]) that the set of all singular points coincides with the closure of the set of all fixed points on Γ. Hence the set of all regular points is decomposed into disjoint open arcs, and each of them is called a *regular arc,* and its closure a *closed regular arc.* If the end-points of a regular arc are fixed points, the arc is called a *completely regular arc.* Under our assumption on $\mathfrak{R}$ every completely regular arc has two fixed points of a single hyperbolic substitution as its end-points.

Since the boundary curve of any elementary domain is not homotopic to zero on $\mathfrak{R}$, its image in U consists of curves terminating at certain parabolic or hyperbolic fixed points. For later use we prove

LEMMA 1. *Under the mapping $\mathfrak{R}^\infty \to U$ the image of any simple closed curve γ on $\mathfrak{R}$, non-homotopic to zero on $\mathfrak{R}$, consists of curves which have no end-points in common on Γ. Furthermore, let γ' be another simple closed curve on $\mathfrak{R}$, which is homotopic neither to zero nor to γ on $\mathfrak{R}$, and disjoint from γ. Then any two respective image-curves of γ and γ' have no end-points in common on Γ.*

Proof. We select a finite or infinite number of simple closed curves $\{C_n\}$ on $\mathfrak{R}$, which are disjoint both from γ and γ' and from each other, such that, by cutting $\mathfrak{R}$ along them $\mathfrak{R}$ becomes a domain $\mathfrak{R}'$ of planar character. We take infinitely many replicas of $\mathfrak{R}'$ and connect them along the opposite edges of $\{C_n\}$ such that two replicas have at most one curve in common and no free edges are left. The resulting surface is a Schottky covering surface of $\mathfrak{R}$ and is

https://doi.org/10.1017/S0027763000018067 Published online by Cambridge University Press

mapped conformally and in a one-to-one manner onto a plane domain B. Under this mapping γ and γ' are transformed to at most countably many disjoint simple closed curves $\{L\}$ in B, which are neither homotopic to zero nor to each other in B. The universal covering surface B^∞ of B is conformally equivalent to $\mathfrak{R}^\infty$ and hence we can interpose B^∞ between the mapping $\mathfrak{R}^\infty \to U$. The images in U of γ and γ' may be regarded as the images of $\{L\}$ under the mapping $B^\infty \to U$. Since the assertion in our lemma is known to be true in the case where $\mathfrak{R}$ is a plane domain, it follows that Lemma 1 holds good.

3. Images of boundary components. Let $P_{\mathfrak{C}}$ be a point of $\mathfrak{C}_{\mathfrak{R}}$, and $\{D^{(n)}\}$ its determining sequence of domains with boundaries $\{\gamma^{(n)}\}$. Under the mapping of $\mathfrak{R}^\infty$ onto U we can choose a nested sequence $\{G_n\}$ of simply-connected domains which are images of $\{D^{(n)}\}$: $D^{(n)} = f(G_n)$. As $n \to \infty$ the closure G_n^a of G_n tends to a point or to a closed arc on Γ. We shall call this an *α-image*, or distinctively a *point-image* or an *arc-image* of $P_{\mathfrak{C}}$, and the nested sequence $\{G_n\}$ the *fundamental sequence* of the α-image. Each G_n is bounded by one or countably many cross-cuts[4] which are images of $\gamma^{(n)}$, and one of them separates $z=0$ from G_n, where we suppose that $z=0$ lies in a certain image of $\mathfrak{R}_1$ (= the first domain of the exhaustion). We denote this separating cross-cut by l_n, and call $\{l_n\}$ the *fundamental sequence* of cross-cuts of the α-image. On account of Lemma 1 these cross-cuts have no end-points in common unless $D^{(n)}$ is doubly-connected.

We shall say that two fundamental sequences $\{G_n\}$ and $\{G'_n\}$ are *different* if there is a number n_0 such that $G_{n_0} \neq G'_{n_0}$. Then it is clear that G_n and G'_n $(n \geqq n_0)$ are disjoint from each other.

THEOREM 1. *Let $\mathfrak{R}$ be an open Riemann surface of connectivity at least three. On mapping $\mathfrak{R}^\infty$ onto U, the different fundamental sequences determine disjoint α-images on Γ.*

Proof. Let $\{G_n\}$ and $\{G'_n\}$ be two different fundamental sequences, and let $G_{n_0} \neq G'_{n_0}$. Each boundary cross-cut of G_{n_0} borders a domain which is some component of the open set $U - G^a_{n_0}$, and G'_{n_0} is located in one of them, say in M with boundary cross-cut l. We connect any point on l with any point of G'_{n_0} by a curve running inside M, and let l' be the boundary cross-cut of G'_{n_0} which

[4] The end-points of a cross-cut may, or may not, coincide.

https://doi.org/10.1017/S0027763000018067 Published online by Cambridge University Press

the curve meets at the first time. Since l and l' have no end-points in common by Lemma 1, and since G_{n_0} and G'_{n_0} lie along the opposite edges of them, G_{n_0} and G'_{n_0} have no points in common. Therefore the two α-images $\bigcap_n G_n^a$ and $\bigcap_n G_n'^a$ are disjoint from each other.

4. Correspondence between boundary components and their α-images. We assumed in § 1 that $\gamma^{(n+1)}$ is not homotopic to $\gamma^{(n)}$ unless $D^{(n)}$ is doubly-connected. It is immediately shown that $P_{\mathfrak{C}}$ is an isolated boundary component of the first class if and only if some domains of its determining sequence are doubly-connected. For such boundary components we have

THEOREM 2. *Let $\mathfrak{R}$ be an open Riemann surface of connectivity at least three. Under the mapping of $\mathfrak{R}^\infty$ onto U, the α-images on Γ of a parabolic boundary component consist of a class of equivalent*[5] *parabolic fixed points, and conversely, any such class forms the α-images of a certain parabolic boundary component. The isolated hyperbolic boundary components of the first class and the classes of equivalent closed completely regular arcs correspond to each other in the similar manner.*

Remark. Each domain of the fundamental sequence of such an α-image, with the exception of a finite number of domains, is bounded by a single cross-cut terminating at the point-image (i.e., at the parabolic fixed point), or at the two end-points of the arc-image (i.e., at the two hyperbolic fixed points).

The proof of this theorem was given in [8], Chap. III.[6] Other proofs for the correspondence of parabolic boundary components may be found in [9] and [12].

Next we shall study the α-images of a non-isolated boundary component of the first class or of any boundary component of the second class.

Let $P_{\mathfrak{C}}$ be such a boundary component and $\{D^{(n)}\}$ its determining sequence with relative boundaries $\{\gamma^{(n)}\}$. Since the connectivity of $D^{(n)}$ is infinite, each image of $D^{(n)}$ is bounded by countably many cross-cuts, which are images of

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5) When a point or a set on $U+\Gamma$ is transformed to another point or set by a substitution of $\mathfrak{G}$, the two points or sets are called equivalent (under $\mathfrak{G}$) to each other.

6) At this juncture the writer wishes to correct some errors in Chap. III of [8].
p. 112, line 6 is to be read as: small, the inside of W_c defined below is divided into...
p. 112, line 18 is to be read as: sufficiently small, the inside of W_0 is divided into...
p. 114, line 4 is to be read as: ...components of the first class of R correspond to...

$\gamma^{(n)}$. Furthermore each image of $D^{(n)}$ contains a countable set of the images of $D^{(n+1)}$ in it. Therefore, the ways to select a sequence of nested images of $\{D^{(n)}\}$ have the power of the continuum. Thus the power of the α-images on Γ of $P_{\mathfrak{C}}$ is of the continuum, because different fundamental sequences determine disjoint α-images according to Theorem 1. Since the possible set of arc-images is countable, there are always an uncountable number of point-images of $P_{\mathfrak{C}}$.

Let now z_0 be a point-image on Γ, which is not a parabolic fixed point, and $\{l_n\}$ the fundamental sequence of cross-cuts determining it. If z_0 were a hyperbolic fixed point, there would exist a closed curve C in $\mathfrak{R}$ whose image c terminates at z_0. Then there would be a number n_0 such that, for any $n > n_0$, l_n intersects c in U. However, this contradicts the fact that the images $\{\gamma^{(n)}\}$ of $\{l_n\}$ do not intersect C on $\mathfrak{R}$ for n sufficiently large. Hence z_0 can not be a fixed point, but, since clearly it is not a regular point, it is a non-fixed singular point.

Let us turn to an arc-image $\widehat{z_1 z_2}$ on Γ, which is not a completely regular arc. For the same reason as above, both z_1 and z_2 are non-fixed singular points. Further no inner point of $\widehat{z_1 z_2}$ is a singular point, because near any singular point there are always hyperbolic fixed points. Thus $\widehat{z_1 z_2}$ is a closed regular arc with two non-fixed singular end-points. Obviously the disjoint arc-images are countable in number. So then, equivalent (under $\mathfrak{G}$) arc-images being brought together into a class, how many such classes correspond to one boundary component? We can show without difficulty by examples that the number may be finite or countable.

We summarize the results in

THEOREM 3. *Let $\mathfrak{R}$ be an open Riemann surface, and let $P_{\mathfrak{C}}$ be a non-isolated boundary component of the first class or a boundary component of the second class. On mapping $\mathfrak{R}^{\infty}$ onto U, the set of the α-images of $P_{\mathfrak{C}}$ consists of an uncountable set of non-fixed singular points and, possibly, of an at most countable set of classes of equivalent closed non-completely regular arcs.*

5. Paths converging to the boundary of $\mathfrak{R}$. We shall call a curve $C = \{P(t)\ ;\ 0 \leqq t < 1\}$, lying in $\mathfrak{R}$ and tending to the boundary of $\mathfrak{R}$ as $t \to 1$, a *path converging to the boundary of* $\mathfrak{R}$ or simply a *path*. It is seen easily that a path converges to a certain point of $\mathfrak{C}_{\mathfrak{R}}$ with respect to the topology of

$\mathfrak{R}+\mathfrak{C}_{\mathfrak{R}}$. Let $\{G_n\}$ be a fundamental sequence in U and $c=\{z(t)\ ;\ 0\leqq t<1\}$ be a curve in U. If, for every G_n, there is a number t_0, $0<t_0<1$, such that $\{z(t)\ ;\ t_0\leqq t<1\}$ lies in G_n, c will be said to converge to the (point- or arc-) image determined by $\{G_n\}$. We can show easily that any image of a path converging to $P_{\mathfrak{C}}$ converges to a certain α-image of $P_{\mathfrak{C}}$. Conversely, for any image α of a boundary component $P_{\mathfrak{C}}$ there is a curve which converges to α. It is obvious that its image on $\mathfrak{R}$ converges to $P_{\mathfrak{C}}$. Therefore, to a set of curves on $\mathfrak{R}$ which converge to $P_{\mathfrak{C}}$, there corresponds a set of curves in U which converge to the α-images of $P_{\mathfrak{C}}$, and vice versa.

Let $C_1=\{P_1(t)\ ;\ 0\leqq t<1\}$ and $C_2=\{P_2(t)\ ;\ 0\leqq t<1\}$ be two paths on $\mathfrak{R}$ which converge to the same point $P_{\mathfrak{C}}$ of $\mathfrak{C}_{\mathfrak{R}}$, and $\{K(t)\ ;\ 0\leqq t<1\}$ be a set of curves on $\mathfrak{R}$ such that, for every t, $0<t<1$, $K(t)$ connects $P_1(t)$ with $P_2(t)$ and tends to $P_{\mathfrak{C}}$ as $t\to 1$. If $\{K(t)\}$ can be chosen such that the closed curve, consisting of four parts: $\{P_1(\tau)\ ;\ 0\leqq\tau\leqq t\}$, $K(t)$, $\{P_2(\tau)\ ;\ 0\leqq\tau\leqq t\}$ and $K(0)$, is homotopic to zero on $\mathfrak{R}$ for every t $(0<t<1)$, C_1 and C_2 will be called *homotopic* to each other on $\mathfrak{R}$.[7]

THEOREM 4. *If and only if two paths are homotopic to each other on $\mathfrak{R}$, their image-curves in U converge to the images, equivalent under $\mathfrak{G}$, of a boundary component to which the paths converge.*

Proof. Let $C_1=\{P_1(t)\}$ and $C_2=\{P_2(t)\}$ be homotopic to each other and $\{K(t)\}$ a set of curves with the property stated above. Let $c_1=\{z_1(t)\}$ be an image of C_1 in U and α the α-image to which c_1 converges. We take the function-element at $P_1(0)$, corresponding to the point $z_1(0)$, continue it analytically along the curve $K(0)+C_2$ and denote the image of C_2 by c_2. Since the closed curve, consisting of four parts: $\{P_1(\tau)\ ;\ 0\leqq\tau\leqq t\}$, $K(t)$, $\{P_2(\tau)\ ;\ 0\leqq\tau\leqq t\}$ and $K(0)$, is homotopic to zero, we obtain images $\{k(t)\}$ of $\{K(t)\}$ which connect c_1 and c_2. It is obvious that, given a domain G_n of the fundamental sequence determining α, $k(t)$ lies in G_n for t sufficiently large. This shows that an end-part of c_2 lies in G_n. Hence c_2 converges to α.

Conversely, let C_1 and C_2 be two curves whose images $c_1=\{z_1(t)\}$ and $c_2=\{z_2(t)\}$ converge to the same α-image with fundamental sequence $\{G_n\}$.

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[7] Compare it with the definition of accessible boundary points by R. Nevanlinna. See [6].

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We can select $\{t_n\}$, $t_n \to 1$, such that $z_1(t_n)$ and $z_2(t_n)$ lie in G_n. We connect $z_1(t)$ with $z_2(t)$ for t, $t_n < t < t_{n+1}$, by curves running in G_n. Their images play the role of $\{K(t)\}$ required for the homotopicity of C_1 and C_2.

No difference has been seen, up to this place, between the non-isolated boundary components of the first class and the boundary components of the second class, so far as their images on Γ are concerned. However, it is desirable that a certain effective characterization of the distinction between these images is obtained.

6. Points of Γ which are not contained in α-images. We know that all parabolic fixed points and all closed regular arcs are α-images, and that no hyperbolic fixed point is contained in an α-image unless it is an end-point of a certain completely regular arc. In this section we shall study on non-fixed points on Γ which are not contained in α-images.

First we consider an arbitrary connected component of the image in U of $\mathfrak{R}_n$, and denote it by D. This domain is bounded by a countable set of cross-cuts and by their accumulating points. The boundary δ of D is regarded as a Jordan curve which has no double points except for possible parabolic fixed points on it. Since the set of singular points on δ is a perfect set, it has the power of the continuum. Fixed points being countable, all non-fixed singular points on δ are uncountable. Suppose that such a non-fixed singular point z_0 is contained in an α-image, and let $\{l_k\}$ be the fundamental sequence of cross-cuts which determines this α-image. Then $\{l_k\}$ intersect D except for a finite number of them. However, on $\mathfrak{R}$ the images of $\{l_k\}$ do not intersect $\mathfrak{R}_n$ for sufficient large k. Thus we get a contradiction and it is shown that any non-fixed singular point on δ is not contained in any α-image.

Next let z_0 be a non-fixed point on Γ which is neither contained in any α-image nor on the boundary of any component of the image of any $\mathfrak{R}_n$. We take all images of the curves $\{\gamma_{i,j,\dots,k}\}$ which separate z_0 from $z=0$ and enumerate them in $\{l_k\}$ so that l_k separates $z=0$ from l_{k+1}. By Lemma 1 these $\{l_k\}$ have different end-points. If the images of $\{l_k\}$ on $\mathfrak{R}$ were contained in a certain $\mathfrak{R}_n$, z_0 would be a boundary point of a component of the image of the $\mathfrak{R}_n$. Therefore the images of $\{l_k\}$ are not compact in $\mathfrak{R}$. Since z_0 is not contained in any α-image, the images on $\mathfrak{R}$ of $\{l_k\}$ do not converge to a boundary component. The image on $\mathfrak{R}$ of any curve terminating at z_0 is neither compact in $\mathfrak{R}$ nor a path in the sense of §5. Conversely, if every curve terminating at z_0 has this property, or if there is a sequence $\{l_k\}$ of cross-cuts whose images on $\mathfrak{R}$ are neither compact in $\mathfrak{R}$ nor converge to a boundary component, z_0 is a non-fixed point which is not contained in any α-image. The ways of selection of such $\{l_k\}$ have the power of the continuum if $\mathfrak{R}$ is of infinite connectivity. Thus we have

THEOREM 5. *Let $\mathfrak{R}$ be an open Riemann surface with connectivity at least three. On mapping $\mathfrak{R}^\infty$ onto U, the set of non-fixed points, which lie on the boundary of a component of the image of $\mathfrak{R}_n$ but not contained in any α-image, have the power of the continuum, for every n. If and only if $\mathfrak{R}$ has infinite connectivity, there are non-fixed points of the power of the continuum, which are neither contained in α-images nor lie on the boundary of any component of the image of any $\mathfrak{R}_n$.*

Chapter II. Dirichlet Problem

7. Perron-Brelot's method. We shall treat in this chapter the Dirichlet problem on $\mathfrak{R}$ with boundary functions on $\mathfrak{C}_{\mathfrak{R}}$ by means of the Perron-Brelot's method (cf. [2], [3]. See also footnote 2)).

For a real-valued function φ (admitting $\pm\infty$) defined on $\mathfrak{C}_{\mathfrak{R}}$, the lower class $\mathfrak{U}_\varphi^{\mathfrak{C}}$ is defined by all continuous subharmonic functions $\{u(P)\}$ bounded from above on $\mathfrak{R}$ such that $\overline{\lim}_{P \to P_{\mathfrak{C}}} u(P) \leqq \varphi(P_{\mathfrak{C}})$, where $\overline{\lim}$ is taken with respect to the topology introduced in §1 and $-\infty$ is added to $\mathfrak{U}_\varphi^{\mathfrak{C}}$. The upper cover $\underline{H}_\varphi^{\mathfrak{C}}(P)$ (= the supremum at each point) of $U_\varphi^{\mathfrak{C}}$ is harmonic or equal to the constant $+\infty$ or $-\infty$ on $\mathfrak{R}$ on account of the Perron-Brelot's principle. Similarly the upper class $\mathfrak{V}_\varphi^{\mathfrak{C}}$ and its lower cover $\overline{H}_\varphi^{\mathfrak{C}}(P)$ are defined for superharmonic functions, and $\overline{H}_\varphi^{\mathfrak{C}}(P)$ has the similar character as $\underline{H}_\varphi^{\mathfrak{C}}(P)$. On account of the maximum principle there holds $\underline{H}_\varphi^{\mathfrak{C}}(P) \leqq \overline{H}_\varphi^{\mathfrak{C}}(P)$ and the equality at a point induces their identity. When $\underline{H}_\varphi^{\mathfrak{C}}(P) \equiv \overline{H}_\varphi^{\mathfrak{C}}(P)$ we shall denote it by $H_\varphi^{\mathfrak{C}}(P)$ and call it the general solution, and if, in addition, it is finite, φ will be called a resolutive boundary function. For any fixed point P, $H_\varphi^{\mathfrak{C}}(P)$ is a positive ($\geqq 0$ for $\varphi \geqq 0$) linear functional defined on the class of resolutive boundary functions.

Our main concern in this chapter is to decide the class of resolutive boundary functions.

https://doi.org/10.1017/S0027763000018067 Published online by Cambridge University Press

When $\mathfrak{R}$ has a null boundary there is no non-constant continuous subharmonic function bounded from above on $\mathfrak{R}$ (Lemma 1, 2 of [8]). Therefore $\underline{H}_{\varphi}^{\mathfrak{C}}(P) = \inf_{\mathfrak{C}_{\mathfrak{R}}} \varphi$ and $\overline{H}_{\varphi}^{\mathfrak{C}}(P) = \sup_{\mathfrak{C}_{\mathfrak{R}}} \varphi$, and hence $\underline{H}_{\varphi}^{\mathfrak{C}}(P) \neq \overline{H}_{\varphi}^{\mathfrak{C}}(P)$ unless φ is a constant. Accordingly we assume hereafter that $\mathfrak{R}$ *has a positive boundary.*

Preparing for later use, we shall prove

LEMMA 2. *Let* $\{\varphi_n\}$ $(n = 1, 2, \ldots)$ *be non-negative resolutive boundary functions on* $\mathfrak{C}_{\mathfrak{R}}$ *and put* $\sum_{n=1}^{\infty} \varphi_n = \varphi$. *Then*

$$\underline{H}_{\varphi}^{\mathfrak{C}}(P) \equiv \overline{H}_{\varphi}^{\mathfrak{C}}(P) \equiv \sum_{n=1}^{\infty} H_{\varphi_n}^{\mathfrak{C}}(P). \tag{1}$$

Proof. Given $\varepsilon > 0$ and $P_0 \in \mathfrak{R}$, we take a non-negative function $u_n(P)$ of class $\mathfrak{U}_{\varphi_n}^{\mathfrak{C}}$ and $v_n(P)$ of class $\mathfrak{B}_{\varphi_n}^{\mathfrak{C}}$ such that $v_n(P_0) - u_n(P_0) < \varepsilon/2^n$. If $\sum_{n=1}^{\infty} u_n(P_0) = \infty$, let N be a number for which $\sum_{n=1}^{N} u_n(P_0)$ is greater than any assigned positive number M. It is easily shown that $\sum_{n=1}^{N} u_n(P) \in \mathfrak{U}_{\varphi}^{\mathfrak{C}}$, whence $M < \sum_{n=1}^{N} u_n(P_0) \leqq \underline{H}_{\varphi}^{\mathfrak{C}}(P_0)$. From this relation, equalities (1) follow at once.

We next suppose $\sum_{n=1}^{\infty} u_n(P_0) < \infty$. Since $\sum_{n=1}^{\infty} v_n(P_0) < \sum_{n=1}^{\infty} u_n(P_0) + \sum_{n=1}^{\infty} \varepsilon/2^n = \sum_{n=1}^{\infty} u_n(P_0) + \varepsilon$, the series $\sum_{n=1}^{\infty} v_n(P_0)$ converges. Hence $\sum_{n=1}^{\infty} v_n(P)$ is a superharmonic function on R and $\varliminf_{P \to P_{\mathfrak{C}}} \sum_{n=1}^{\infty} v_u(P) \geqq \varphi(P_{\mathfrak{C}})$, but it might be discontinuous. We replace it, therefore, by the harmonic function, with the same boundary value, in every $\mathfrak{R}_{2n+2} - \mathfrak{R}_{2n}^a$ $(n \geqq 0,\ \mathfrak{R}_0 = \text{empty set})$, and then harmonize the resulting function again in every $\mathfrak{R}_{2n+1} - \mathfrak{R}_{2n-1}^a$ $(n \geqq 1)$. Thus obtained function $v_0(P)$ belongs to $\mathfrak{B}_{\varphi}^{\mathfrak{C}}$, and $v_0(P) \leqq \sum_{n=1}^{\infty} v_n(P)$. We take a sufficiently large number p so that $\sum_{n=p+1}^{\infty} v_n(P_0) < \varepsilon$. Then

$$\sum_{n=1}^{\infty} v_n(P_0) - \sum_{n=1}^{p} u_n(P_0) = \sum_{n=p+1}^{\infty} v_n(P_0) + \sum_{n=1}^{p} v_n(P_0) - \sum_{n=1}^{p} u_n(P_0) < 2\varepsilon. \tag{2}$$

Since $\sum_{n=1}^{p} u_n(P)$ belongs to the class $\mathfrak{U}_{\varphi}^{\mathfrak{C}}$, there hold

$$\sum_{n=1}^{p} u_n(P) \leqq \underline{H}_{\varphi}^{\mathfrak{C}}(P) \leqq \overline{H}_{\varphi}^{\mathfrak{C}}(P) \leqq v_0(P) \leqq \sum_{n=1}^{\infty} v_n(P). \tag{3}$$

On the other hand we have

$$\sum_{n=1}^{p} u_n(P) \leqq \sum_{n=1}^{\infty} H_{\varphi_n}^{\mathfrak{C}}(P) \leqq \sum_{n=1}^{\infty} v_n(P). \tag{4}$$

ε being arbitrarily small, (1) follows from (2), (3) and (4).

8. Lower and upper integrals. We shall define in this section lower and upper integrals for arbitrary real-valued functions, which may take $\pm\infty$, in order to make use of them to represent $\underline{H}_{\varphi}^{\mathfrak{C}}(P)$ and $\overline{H}_{\varphi}^{\mathfrak{C}}(P)$.

Let $\mathfrak{E}$ be a σ-algebra[8] of sets in $\mathfrak{C}_{\mathfrak{R}}$, and suppose that it contains all open sets and hence the Borel class $\mathfrak{B}$ in $\mathfrak{C}_{\mathfrak{R}}$. Let $\mu(E)$ be a finite-valued measure defined on $\mathfrak{E}$. It will be called regular if, for every set $E \in \mathfrak{E}$, $\mu(E) = \inf\{\mu(G)\,;\ G \supset E,\ G = \text{an open set}\}$. On the other hand $\mu(E)$ will be called complete if the conditions $E \in \mathfrak{E}$, $E' \subset E$ and $\mu(E) = 0$ imply that $E' \in \mathfrak{E}$.

Let now $\mu(E)$ be a finite-valued regular complete measure defined on $\mathfrak{E}$. For an arbitrary real-valued function φ on $\mathfrak{C}_{\mathfrak{R}}$ we define the *lower integral* $\underline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi(Q)\,d\mu(Q)$ by

$$\sup\left\{\int_{\mathfrak{C}_{\mathfrak{R}}} \psi(Q)\,d\mu(Q)\ ;\ \psi \text{ is bounded from above and } \mathfrak{E}\text{-measurable, and } \psi \leqq \varphi\right\},$$

where the value $-\infty$ may be taken by ψ. Similarly we define the *upper integral* $\overline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \psi(Q)\,d\mu(Q)$ by

$$\inf\left\{\int_{\mathfrak{C}_{\mathfrak{R}}} \psi(Q)\,d\mu(Q)\ ;\ \psi \text{ is bounded from below and } \mathfrak{E}\text{-measurable, and } \psi \leqq \varphi\right\},$$

where the value $+\infty$ may be taken by ψ. Then we have

LEMMA 3.

$$\underline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi\,d\mu = \sup\left\{\int_{\mathfrak{C}_{\mathfrak{R}}} \psi\,d\mu\ ;\ \psi \textit{ is bounded from above and upper semicontinuous, and } \psi \leqq \varphi\right\},$$

where $-\infty$ *may be taken by* ψ, *and*

$$\overline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi\,d\mu = \inf\left\{\int_{\mathfrak{C}_{\mathfrak{R}}} \psi\,d\mu\ ;\ \psi \textit{ is bounded from below and lower semicontinuous, and } \psi \geqq \varphi\right\},$$

where $+\infty$ *may be taken by* ψ.

From the definition of lower (resp. upper) integral we see that it is neces-

[8] That is, $\mathfrak{E}$ is a non empty class of sets closed under the formation of complements and countable unions.

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sary to prove this lemma only for every $\mathfrak{C}$-measurable function φ bounded from above (resp. below). For such φ, however, the lemma is valid in virtue of the regularity of μ.[9]

Since μ is a complete measure we can show without difficulty

LEMMA 4. *If φ is $(\mathfrak{C}, \mu)$-integrable,[10] there holds*

$$\underline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi d\mu = \overline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi d\mu = \int_{\mathfrak{C}_{\mathfrak{R}}} \varphi d\mu.$$

Conversely, if $-\infty < \underline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi d\mu = \overline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \varphi d\mu < +\infty$, *then φ is $(\mathfrak{C}, \mu)$-integrable in the narrow sense.*

COROLLARY. *Let χ_A be the characteristic function of a set $A \subset \mathfrak{C}_{\mathfrak{R}}$. Then A belongs to $\mathfrak{C}$ if and only if* $\underline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \chi_A d\mu = \overline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \chi_A d\mu$.

9. Resolutivity of the characteristic function of any open set. Let $D = D_{i,j,\ldots,k}$ be an elementary domain on $\mathfrak{R}$ and γ be its relative boundary in $\mathfrak{R}$. We take the characteristic function of the set $\overline{D} - D \subset \mathfrak{C}_{\mathfrak{R}}$ as a boundary function on $\mathfrak{C}_{\mathfrak{R}}$, and denote it by χ. *We shall show its resolutivity* in the first place.

Let $\omega(P, \gamma, D)$ and $\omega(P, \gamma, D^{ac})$ be the harmonic measures of γ with respect to the domains D and $D^{ac} = \mathfrak{R} - (D \cup \gamma)$. Since $\mathfrak{R}$ has a positive boundary, one of them is not a constant and its infimum is zero. Suppose $\inf_{D^{ac}} \omega(P, \gamma, D^{ac}) = 0$. Then $0 \leqq \inf_{\mathfrak{R}} (\overline{H}_\chi^{\mathfrak{C}}(P) - \underline{H}_\chi^{\mathfrak{C}}(P)) \leqq \inf_{D^{ac}} \overline{H}_\chi^{\mathfrak{C}}(P) \leqq \inf_{D^{ac}} \omega(P, \gamma, D^{ac}) = 0$. If $\inf_D \omega(P, \gamma, D) = 0$, then $0 \leqq \inf_{\mathfrak{R}} ((1 - \underline{H}_\chi^{\mathfrak{C}}(P)) - (1 - \overline{H}_\chi^{\mathfrak{C}}(P))) \leqq \inf_D (1 - \underline{H}_\chi^{\mathfrak{C}}(P)) \leqq \inf_D \omega(P, \gamma, D) = 0$. Thus there holds always

$$\inf_{\mathfrak{R}} (\overline{H}_\chi^{\mathfrak{C}}(P) - \overline{H}_\chi^{\mathfrak{C}}(P)) = 0. \tag{5}$$

We define a boundary function ψ for domain D by $\overline{H}_\chi^{\mathfrak{C}}(P)$ on γ and by 1 on $\overline{D} \cap \mathfrak{C}_{\mathfrak{R}}$. The function $\underline{H}_\psi^D(P)$ is defined as the upper cover of the lower class $\mathfrak{U}_\psi^D$ consisting of all continuous subharmonic functions $\{u_\psi(P)\}$, $0 \leqq u_\psi(P) \leqq 1$, with the property that $\overline{\lim}_{P \to Q \in \gamma} u_\psi(P) \leqq \underline{H}_\chi^{\mathfrak{C}}(Q)$. We define $\overline{H}_\psi^D(P)$ in a similar way. Then the inequality $\overline{H}_\psi^D(P) \leqq \overline{H}_\chi^{\mathfrak{C}}(P)$ holds clearly. Since $\overline{H}_\chi^{\mathfrak{C}}(P) \leqq 1$, it belongs

9) See the proof of Vitali-Carathéodory's theorem in [13].

10) The value of the integral may be infinite. When it is finite, we shall say that the function is integrable in the narrow sense.

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to $\mathfrak{U}_\psi^D$. Hence $\overline{H}_\chi^{\mathfrak{C}}(P) \leqq \underline{H}_\psi^D(P) \leqq \overline{H}_\psi^D(P) \leqq \overline{H}_\chi^{\mathfrak{C}}(P)$. Thus $H_\psi^D(P) = \overline{H}_\chi^{\mathfrak{C}}(P)$ in D. Accordingly $\overline{H}_\chi^{\mathfrak{C}}(P)$ is equal to the upper cover of $\mathfrak{U}_\psi^D$ in D.

Next let ψ' be the boundary function of D defined by $\underline{H}_\chi^{\mathfrak{C}}(P)$ on γ and by 1 on $\overline{D} \cap \mathfrak{C}_{\mathfrak{R}}$. The identity $\underline{H}_{\psi'}^D(P) \equiv \overline{H}_{\psi'}^D(P)$ is shown easily, but some consideration is necessary in order to prove $H_{\psi'}^D(P) \equiv \underline{H}_\chi^{\mathfrak{C}}(P)$. It is clear that $\underline{H}_\chi^{\mathfrak{C}}(P) \leqq H_{\psi'}^D(P)$. Suppose now that $\underline{H}_\chi^{\mathfrak{C}}(P) < H_{\psi'}^D(P)$ and put $H_{\psi'}^D(P_0) - \underline{H}_\chi^{\mathfrak{C}}(P_0) = a > 0$ at an arbitrarily fixed point $P_0 \in D$. Let $u(P)$ be a function of the lower class $\mathfrak{U}_\chi^{\mathfrak{C}}$ such that $0 \leqq \underline{H}_\chi^{\mathfrak{C}}(P) - u(P) < a$ on γ. If a boundary function ψ'' of D is defined by $u(P)$ on γ and by 1 on $\overline{D} \cap \mathfrak{C}_{\mathfrak{R}}$, obviously holds $u(P) \leqq \underline{H}_{\psi''}^D(P) \equiv \overline{H}_{\psi''}^D(P)$. We replace $u(P)$ by $H_{\psi''}^D(P)$ in D. Then the resulting function on $\mathfrak{R}$ still belongs to $\mathfrak{U}_\chi^{\mathfrak{C}}$. Hence $H_{\psi''}^D(P) \leqq \underline{H}_\chi^{\mathfrak{C}}(P)$ in D. Especially at P_0, $H_{\psi'}^D(P_0) - H_{\psi''}^D(P_0) = a + \underline{H}_\chi^{\mathfrak{C}}(P_0) - H_{\psi''}^D(P_0) \geqq a$. However, this contradicts the inequality $H_{\psi'}^D(P) - H_{\psi''}^D(P) \leqq \max_\gamma (\psi' - \psi'') < a$, which holds everywhere in D. Thus it is shown that $H_{\psi'}^D(P) = \underline{H}_\chi^{\mathfrak{C}}(P)$ in D. Therefore in D the function $\underline{H}_\chi^{\mathfrak{C}}(P)$ equals the lower cover of upper class $\mathfrak{B}_{\psi'}^D$.

Let $u_\psi(P) \in \mathfrak{U}_\psi^D$ and $v_{\psi'}(P) \in \mathfrak{B}_{\psi'}^D$. The bounded continuous subharmonic function $u_\psi(P) - v_{\psi'}(P)$ has the upper limit less than $\overline{H}_\chi^{\mathfrak{C}}(Q) - \underline{H}_\chi^{\mathfrak{C}}(Q) \geqq 0$ as $P \to Q \in \gamma$ and its upper limit as $P \to P_{\mathfrak{C}} \in \overline{D} \cap \mathfrak{C}_{\mathfrak{R}}$ is non-positive. Hence $u_\psi(P) - v_{\psi'}(P) \leqq \max_{Q \in \gamma}(\overline{H}_\chi^{\mathfrak{C}}(Q) - \underline{H}_\chi^{\mathfrak{C}}(Q))$. Since the upper cover of the left hand side is $\overline{H}_\chi^{\mathfrak{C}}(P) - \underline{H}_\chi^{\mathfrak{C}}(P)$, we have $\overline{H}_\chi^{\mathfrak{C}}(P) - \underline{H}_\chi^{\mathfrak{C}}(P) \leqq \max_{Q \in \gamma} (\overline{H}_\chi^{\mathfrak{C}}(Q) - \underline{H}_\chi^{\mathfrak{C}}(Q))$ in D. Similarly we can show that this inequality holds in D^{ac}, too. In virtue of the maximum principle this function reduces to a constant on $\mathfrak{R}$, and this constant must be zero by (5). Therefore $\underline{H}_\chi^{\mathfrak{C}}(P) \equiv \overline{H}_\chi^{\mathfrak{C}}(P)$. That is, χ is a resolutive boundary function.

Let now G be any open subset of $\mathfrak{C}_{\mathfrak{R}}$. We took in §1 $\{\overline{D}_{i,j,\dots,k} \cap \mathfrak{C}_{\mathfrak{R}}\}$ and the empty set ϕ as a countable open base of $\mathfrak{C}_{\mathfrak{R}}$. We shall denote this class by $\mathfrak{D}$. Each of them is open and closed, and any two of them are disjoint from each other or one is contained in the other. Therefore G is represented as a countable disjoint union of them: $G = \sum_n D_n$. On account of Lemma 2 we have

$$(6) \qquad \underline{H}_{\chi G}^{\mathfrak{C}}(P) = \overline{H}_{\chi G}^{\mathfrak{C}}(P) = \sum_n H_{\chi G_n}^{\mathfrak{C}}(P).$$

We may state

THEOREM 6. *Let $\mathfrak{R}$ be an open Riemann surface with positive boundary.*

Then the characteristic function of any open or closed subset of $\mathfrak{C}_{\mathfrak{R}}$ *is resolutive.*

The resolutivity of the characteristic function χ_F of any closed set $F \subset \mathfrak{C}_{\mathfrak{R}}$ follows from the equality $\chi_F = 1 - \chi_G$, where $G = \mathfrak{C}_{\mathfrak{R}} - F$ and is an open set.

10. Integral representation. Given a point $P \in R$, we define a set function for an arbitrary set $X \subset \mathfrak{C}_{\mathfrak{R}}$ by

$$\overline{\mu}^P(X) = \inf\left\{\sum_{i=1}^{\infty} H^{\mathfrak{C}}_{\chi G_i}(P) \ ; \ X \subset \bigcup_{i=1}^{\infty} G_i \text{ and every } G_i \text{ is open}\right\}. \tag{7}$$

This is an outer measure on $\mathfrak{C}_{\mathfrak{R}}$ and $0 \leqq \overline{\mu}^P(X) \leqq 1$. Since $H^{\mathfrak{C}}_{\chi_{\bigcup_i G_i}}(P) \leqq H^{\mathfrak{C}}_{\sum_i \chi_{G_i}}(P) = \sum_i H^{\mathfrak{C}}_{\chi G_i}(P)$ by Lemma 2, we may write (7) in the following manner:

$$\overline{\mu}^P(X) = \inf\{H^{\mathfrak{C}}_{\chi G}(P) \ ; \ X \subset G \text{ and } G \text{ is open}\}.$$

Therefore $\overline{\mu}^P(G) = H^{\mathfrak{C}}_{\chi G}(P)$ for any open set G.

In order to prove that any open set is measurable with respect to this outer measure, it is sufficient to prove the measurability of any set of $\mathfrak{D}$, because any open set is the countable union of certain sets of $\mathfrak{D}$ and the class $\mathfrak{E}^P$ of all $\overline{\mu}^P$-measurable sets is a σ-algebra. We take any $D_0 \in \mathfrak{D}$. We have only to prove $\overline{\mu}^P(X) \geqq \overline{\mu}^P(X \cap D_0) + \overline{\mu}^P(X - D_0)$ for any set $X \subset \mathfrak{C}_{\mathfrak{R}}$. For given $\varepsilon > 0$, we choose an open set G such that $G \supset X$ and $\overline{\mu}^P(X) + \varepsilon \geqq H^{\mathfrak{C}}_{\chi G}(P)$. Since both $G \cap D_0$ and $G - D_0$ are open, there holds $H^{\mathfrak{C}}_{\chi G}(P) = H^{\mathfrak{C}}_{\chi G \cap D_0}(P) + H^{\mathfrak{C}}_{\chi G - D_0}(P) \geqq \overline{\mu}^P(X \cap D_0) + \overline{\mu}^P(X - D_0)$. Hence $\overline{\mu}^P(X) + \varepsilon \geqq \overline{\mu}^P(X \cap D_0) + \overline{\mu}^P(X - D_0)$. ε being arbitrarily small, it is concluded that any set of $\mathfrak{D}$ is $\overline{\mu}^P$-measurable. Thus any open set and hence any Borel set is $\overline{\mu}^P$-measurable.

Let us regard $\overline{\mu}^P$ as a measure defined on $\mathfrak{E}^P$ and denote it by μ^P. This is clearly regular and complete. Conversely, it can be shown that if a regular complete measure is defined on a σ-algebra of sets in $\mathfrak{C}_{\mathfrak{R}}$ containing all Borel sets, and if its value is equal to $H^{\mathfrak{C}}_{\chi G}(P)$ for every open set G, then the σ-algebra coincides with $\mathfrak{E}^P$ and the measure does with μ^P.

Let φ *be any upper semicontinuous function bounded from above.* First we assume $\varphi < 0$. We take a sequence $0 = a_1 > a_2 > \ldots \to -\infty$, and put $\{P ; \varphi(P) < a_n\} = G_n$ $(n = 1, 2, \ldots)$ and $\{P ; \varphi(P) = -\infty\} = F_0$. Then each G_n is open and F_0 is closed, and $\mathfrak{C}_{\mathfrak{R}}$ is equal to the disjoint union $\sum_{n=1}^{\infty}(G_n - G_{n+1}) + F_0$. We define boundary functions as follows:

$$\varphi_n = a_n(\chi_{G_n} - \chi_{G_{n+1}}) - \chi_{F_0} \text{ and } \varphi'_n = a_{n+1}(\chi_{G_n} - \chi_{G_{n+1}}) - \chi_{F_0} \quad (n = 1, 2, \ldots).$$

Each of them is non-positive and resolutive, and there holds $\sum_{n=1}^{\infty}\varphi_n \geqq \varphi \geqq \sum_{n=1}^{\infty}\varphi'_n$ on $\mathfrak{C}_{\mathfrak{R}}$. Hence by Lemma 2 we have

$$\sum_{n=1}^{\infty}(a_n\mu^P(G_n - G_{n+1}) - \mu^P(F_0)) \geqq \overline{H}_{\varphi}^{\mathfrak{C}}(P) \geqq \underline{H}_{\varphi}^{\mathfrak{C}}(P)$$
$$\geqq \sum_{n=1}^{\infty}(a_{n+1}\mu^P(G_n - G_{n+1}) - \mu^P(F_0)).$$

Since both the first and the last members tend to the same limit $\int_{\mathfrak{C}_{\mathfrak{R}}}\varphi d\mu^P$ as $\sup_n (a_n - a_{n+1}) \to 0$, there holds

$$\underline{H}_{\varphi}^{\mathfrak{C}}(P) = \overline{H}_{\varphi}^{\mathfrak{C}}(P) = \int_{\mathfrak{C}_{\mathfrak{R}}}\varphi d\mu^P. \tag{8}$$

If $\varphi < M < \infty$, then $\underline{H}_{\varphi-M}^{\mathfrak{C}}(P) = \overline{H}_{\varphi-M}^{\mathfrak{C}}(P) = \int_{\mathfrak{C}_{\mathfrak{R}}}(\varphi - M)d\mu^P$, whence (8) holds for any upper semicontinuous function φ bounded from above. Since $\underline{H}_{-\varphi}^{\mathfrak{C}}(P) = -\overline{H}_{\varphi}^{\mathfrak{C}}(P)$, the same relation is true for any lower semicontinuous function φ bounded from below.

Finally let φ be an arbitrary function. We define a boundary function $\psi(P_{\mathfrak{C}})$ by $\lim_{P \to P_{\mathfrak{C}}} u(P)$ $(\leqq \varphi(P_{\mathfrak{C}}))$ for any $u(P) \in \mathfrak{U}_{\varphi}^{\mathfrak{C}}$. Then ψ is an upper semicontinuous function bounded from above for which (8) holds, and the inequality $u(P) \leqq H_{\psi}^{\mathfrak{C}}(P) \leqq \underline{H}_{\varphi}^{\mathfrak{C}}(P)$ is valid on $\mathfrak{R}$. Therefore the upper cover $\underline{H}_{\varphi}^{\mathfrak{C}}(P)$ of $\mathfrak{U}_{\varphi}^{\mathfrak{C}}$ is equal to the upper cover of $\{H_{\psi}^{\mathfrak{C}}(P)\}$, where $\{\psi\}$ are upper semicontinuous functions, bounded from above and not greater than φ. Making use of Lemma 3 and (8) there follows $\underline{H}_{\varphi}^{\mathfrak{C}}(P) = \sup_{\psi} H_{\psi}^{\mathfrak{C}}(P) = \sup_{\psi}\int_{\mathfrak{C}_{\mathfrak{R}}}\psi d\mu^P = \underline{\int}_{\mathfrak{C}_{\mathfrak{R}}}\varphi d\mu^P$. Similarly we obtain $\overline{H}_{\varphi}^{\mathfrak{C}}(P) = \overline{\int}_{\mathfrak{C}_{\mathfrak{R}}}\varphi d\mu^P$.

THEOREM 7. *Let $\mathfrak{R}$ be an open Riemann surface with positive boundary, and φ be an arbitrary real-valued function defined on $\mathfrak{C}_{\mathfrak{R}}$. For any given $P \in \mathfrak{R}$, a σ-algebra $\mathfrak{E} \supset \mathfrak{B}$ in $\mathfrak{C}_{\mathfrak{R}}$ and a regular complete measure μ^P defined on $\mathfrak{E}$ are determined uniquely by the requirement that $\mu^P(G) = H_{\chi_G}^{\mathfrak{C}}(P)$ for every open set G, where $\mathfrak{E}$ does not depend upon P. Furthermore there hold*

$$\underline{H}_{\varphi}^{\mathfrak{C}}(P) = \underline{\int}_{\mathfrak{C}_{\mathfrak{R}}}\varphi d\mu^P \quad \textit{and} \quad \overline{H}_{\varphi}^{\mathfrak{C}}(P) = \overline{\int}_{\mathfrak{C}_{\mathfrak{R}}}\varphi d\mu^P.$$

Taking Lemma 4 into account, we have

COROLLARY. *A boundary function φ is resolutive if and only if it is $(\mathfrak{E}, \mu^P)$-*

https://doi.org/10.1017/S0027763000018067 Published online by Cambridge University Press

integrable in the narrow sense.

In Theorem 7, it is left unproved that $\mathfrak{G}$ does not depend upon $P \in \mathfrak{R}$. If a set E belongs to $\mathfrak{G}^{P_0}$ for $P_0 \in \mathfrak{R}$, there follows $\underline{H}^{\mathfrak{C}}_{\chi_E}(P_0) = \overline{H}^{\mathfrak{C}}_{\chi_E}(P_0) = \mu^{P_0}(E)$. Then $\underline{H}^{\mathfrak{C}}_{\chi_E}(P) = \overline{H}^{\mathfrak{C}}_{\chi_E}(P) = \underline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \chi_E d\mu^P = \overline{\int}_{\mathfrak{C}_{\mathfrak{R}}} \chi_E d\mu^P$ everywhere on $\mathfrak{R}$, which shows by Corollary of Lemma 4 that E belongs to $\mathfrak{G}^P$ for any $P \in \mathfrak{R}$. Theorem 7 is thus proved completely.

11. Relation between the solutions on $\mathfrak{R}$ and those in U. As we studied in Chapter I, some set of points on Γ corresponds to each $P_{\mathfrak{C}} \in \mathfrak{C}_{\mathfrak{R}}$ under the mapping $\mathfrak{R}^\infty \to U$, and is called the α-images of $P_{\mathfrak{C}}$. According to Theorem 1 the α-images of different boundary components have no points in common with each other. Denote the set of points of all α-images by E. Then its linear measure is 2π.[11] For, if we regard a Green's function on $\mathfrak{R}$ as function in U, it has the limit zero along almost all radii and the set E_0 of the end-points of such radii is contained in E.

The Dirichlet problem with boundary values defined almost everywhere on Γ can be treated always in U by the Perron-Brelot's method; for instance, it is known that the general solution for a Lebesgue integrable boundary function is the Poisson integral with the boundary value, and that a boundary function is resolutive if and only if it is Lebesgue integrable in the narrow sense.

Given a boundary function $\varphi(P_{\mathfrak{C}})$ on $\mathfrak{C}_{\mathfrak{R}}$, we give the value $\varphi(P_{\mathfrak{C}})$ to the α-images of $P_{\mathfrak{C}}$ on Γ. In such a manner φ is transformed to a function on E. We shall call it the function on Γ corresponding to φ. In this section we prove

THEOREM 8. *Let $\mathfrak{R}$ be an open Riemann surface with positive boundary and of connectivity at least three. Let φ be any real function on $\mathfrak{C}_{\mathfrak{R}}$ and, on mapping $\mathfrak{R}^\infty$ onto U, use the same notation φ to denote the corresponding function on Γ. Then there hold*

$$\underline{H}^{\mathfrak{C}}_{\varphi}(f(z)) = \underline{H}^{U}_{\varphi}(z) \quad \textit{and} \quad \overline{H}^{\mathfrak{C}}_{\varphi}(f(z)) = \overline{H}^{U}_{\varphi}(z).$$

Proof. We supply the function φ, defined on E, with the value zero on $\Gamma - E$, and use the same notation φ to denote the resulting function on Γ. The ambiguity of the range of definition of φ will not infer nor arouse any confusion.

We take any function $u(P)$ of class $\mathfrak{U}^{\mathfrak{C}}_{\varphi}$ and denote $u(f(z))$ by $\tilde{u}(z)$. This

[11] This fact with the following proof was given in [15].

is bounded from above, continuous and subharmonic in U, and $\overline{\lim}\ \tilde{u}(z) \leqq \varphi$ along every radius with an end-point on E_0. Let us put $\overline{\lim_{r\to 1}}\ \tilde{u}(re^{i\theta}) = \psi(e^{i\theta})$. This is measurable on Γ and bounded from above. From the inequality

$$\tilde{u}(re^{i\theta}) \leqq \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(\rho e^{i\xi}) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \xi)} d\xi \quad (z = re^{i\theta},\ \rho > r),$$

we have by Fatou's lemma

$$\tilde{u}(re^{i\theta}) \leqq \overline{\lim_{\rho\to 1}} \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(\rho e^{i\xi}) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \xi)} d\xi$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \overline{\lim_{\rho\to 1}}\ \tilde{u}(\rho e^{i\xi}) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \xi)} d\xi = H_{\psi}^{U}(re^{i\theta}).$$

Since $\psi(e^{i\theta}) \leqq \varphi(e^{i\theta})$ on E_0, there holds $H_{\psi}^{U}(z) \leqq \underline{H}_{\varphi}^{U}(z)$. Hence $\tilde{u}(z) \leqq \underline{H}_{\varphi}^{U}(z)$. Consequently $\underline{H}_{\varphi}^{\mathfrak{C}}(f(z)) \leqq \underline{H}_{\varphi}^{U}(z)$. Similarly we have $\overline{H}_{\varphi}^{\mathfrak{C}}(f(z)) \geqq \overline{H}_{\varphi}^{U}(z)$. If, therefore, φ is a $(\mathfrak{C}, \mu^P)$-integrable function, there holds the equality

$$H_{\varphi}^{\mathfrak{C}}(f(z)) \equiv H_{\varphi}^{U}(z).$$

When a point or an arc $\alpha \subset \Gamma$ is an image of a point $P_{\mathfrak{C}}$ of $\mathfrak{C}_{\mathfrak{R}}$, this correspondence will be denoted by $P_{\mathfrak{C}} = f(\alpha)$, and, for any set X on Γ, $f(X)$ will defined by $\{f(\alpha)\,;\ \alpha \cap X \neq \phi\} \subset \mathfrak{C}_{\mathfrak{R}}$. Given a closed set $F \subset \Gamma$, the $\overline{\mu}^P$-measurability of $f(F)$ is shown as follows:

We enumerate the images in U of the elementary domains $\{D_i\}(1 \leqq i \leqq \mu)$ in an arbitrary way : $G_1, G_2, \ldots.$ In G_k, which is an image of D_i, let the enumerated images of $\{D_{i,j}\}$ $(1 \leqq j \leqq \mu(i))$ be $G_{k,1}, G_{k,2}, \ldots.$ In such a manner any image α on Γ of a $P_{\mathfrak{C}}$ is determined by a nested sequence of domains $G_k \supset G_{k,l} \supset \ldots$, or by a sequence of numbers $k, l, \ldots.$ We put $A_{k,l,\ldots,m} = \overline{D}_{i,j,\ldots,h} \cup \mathfrak{C}_{\mathfrak{R}}$, if $G_{k,l,\ldots,m}$ is an image of $D_{i,j,\ldots,h}$ and if the intersection of $F \cap E$ with the boundary $G^b_{k,\ldots,m}$ of $G_{k,\ldots,m}$ is not empty. Otherwise we put $A_{k,\ldots,m} = \phi$ for any finite sequence of positive integers. Then if $\alpha \cap F \neq \phi$, and if $\{G_{k,\ldots,m}\}$ is the determining sequence of α, $f(\alpha) = A_k \cap A_{k,l} \cap \ldots \cap A_{k,l,\ldots,m} \cap \ldots \in \mathfrak{C}_{\mathfrak{R}}$. But if $\alpha \cap F = \phi$, and if $\{G_{k',l',\ldots,m'}\}$ is the determining sequence, then $A_{k'} \cap A_{k',l'} \cap \ldots \cap A_{k',l',\ldots,m'} \cap \ldots = \phi$, because F is a closed set and hence there is a $G_{k',l',\ldots,m'}$, such that $G^b_{k',l',\ldots,m'} \cap F \cap E = \phi$. Hence $f(F) = \bigcup_{\{k,l,\ldots,m,\ldots\}} (A_k \cap A_{k,l} \cap \ldots \cap A_{k,l,\ldots,m} \cap \ldots)$. The set of the right hand side is the nucleus of the Souslin's graph $\{A_{k,l,\ldots,m}\}$. Since every $A_{k,l,\ldots,m}$ is $\overline{\mu}^P$-measurable, the nucleus $f(F)$ is so too (cf. [13], pp. 47-50).

Let now φ be any real-valued function on $\mathfrak{C}_{\mathfrak{R}}$, and ψ be an upper semicontinuous function, bounded from above on Γ and not greater than φ. We define a function $\psi'(P_{\mathfrak{C}})$ on $\mathfrak{C}_{\mathfrak{R}}$ by sup $\psi(z)$, where $z \in \alpha$ with $f(\alpha) = P_{\mathfrak{C}}$. For any number k there holds $f(\{z\,;\,\psi(z) > k\}) = \{P_{\mathfrak{C}}\,;\,\psi'(P_{\mathfrak{C}}) > k\}$. Since $\{z\,;\,\psi(z) > k\}$ is a countable union of closed sets, the left hand side is a $\bar{\mu}^P$-measurable set. Therefore $\psi'(P_{\mathfrak{C}})$ is a $\bar{\mu}^P$-measurable function bounded from above on $\mathfrak{C}_{\mathfrak{R}}$. We use the same letter ψ' to denote the corresponding function on Γ. Then $H^{\mathfrak{C}}_{\psi'}(f(z)) = H^{U}_{\psi'}(z)$ and $\psi \leqq \psi' \leqq \varphi$ on Γ. Hence $H^{U}_{\psi}(z) \leqq H^{U}_{\psi'}(z) = H^{\mathfrak{C}}_{\psi'}(f(z)) \leqq \underline{H}^{\mathfrak{C}}_{\varphi}(f(z))$. By the arbitrariness of $\psi \leqq \varphi$ there is concluded that $\underline{H}^{U}_{\varphi}(z) \leqq \underline{H}^{\mathfrak{C}}_{\varphi}(f(z))$. Since the reverse inequality has already been obtained, there follows the equality. Similarly we get $\overline{H}^{\mathfrak{C}}_{\varphi}(f(z)) = \overline{H}^{U}_{\varphi}(z)$. Thus our theorem is proved.

Taking Theorem 7 and Corollary of Lemma 4 into account, we have

COROLLARY. *A set X in $\mathfrak{C}_{\mathfrak{R}}$ belongs to $\mathfrak{C}$ if and only if its image on Γ is linearly measurable.*

Let us give a remark to our present paper. When $\mathfrak{R}$ is a plane domain surrounded by curves, each curve is a point with respect to the topology of $\mathfrak{C}_{\mathfrak{R}}$. This shows that in some cases the points of $\mathfrak{C}_{\mathfrak{R}}$ are too wide to be defined as boundary points of $\mathfrak{R}$. It is desirable, therefore, to study conformal mappings and Dirichlet problems for boundary points of $\mathfrak{R}$ defined more finely than the points of $\mathfrak{C}_{\mathfrak{R}}$.[12]

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[12] For instance, for Martin's boundary points. cf. [12].

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