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THE POLYNOMIAL NUMERICAL INDEX OF A BANACH SPACE

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Abstract In this paper, we introduce the *polynomial numerical index of order* k of a Banach space, generalizing to k-homogeneous polynomials the 'classical' numerical index defined by Lumer in the 1970s for linear operators. We also prove some results. Let k be a positive integer. We then have the following:

(i) $n^{(k)}(C(K)) = 1$ for every scattered compact space K.

(ii) The inequality $n^{(k)}(E) \geqslant k^{k/(1-k)}$ for every *complex* Banach space E and the constant $k^{k/(1-k)}$ is sharp.

(iii) The inequalities
$$n^{(k)}(E) \leqslant n^{(k-1)}(E) \leqslant \frac{k^{(k+(1/(k-1)))}}{(k-1)^{k-1}} n^{(k)}(E)$$
for every Banach space E.

(iv) The relation between the polynomial numerical index of c_0, l_1, l_∞ sums of Banach spaces and the infimum of the polynomial numerical indices of them.

(v) The relation between the polynomial numerical index of the space $C(K, E)$ and the polynomial numerical index of E.

(vi) The inequality $n^{(k)}(E^{**}) \leqslant n^{(k)}(E)$ for every Banach space E.

Finally, some results about the numerical radius of multilinear maps and homogeneous polynomials on $C(K)$ and the disc algebra are given.

Keywords: polynomial numerical index; numerical radius; Aron–Berner extension; homogeneous polynomials; Banach spaces

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1. Introduction

Let E and F be real or complex Banach spaces. We write $\mathring{B}_E$, B_E and S_E for the open unit ball, the closed unit ball and the unit sphere of E, respectively. The dual space of

E is denoted by E^*. Let $k \in \mathbb{N}$. We let $\mathcal{L}(^kE : F)$ denote the Banach space of continuous k-linear mappings of $E^k := E \times \cdots \times E$ into F, endowed with the norm

$$\|A\| = \sup\{\|A(x_1, \dots, x_k)\| : x_j \in B_E,\ j = 1, \dots, k\}.$$

A mapping $P : E \to F$ is called a continuous k-homogeneous polynomial if there is an $A \in \mathcal{L}(^kE : F)$ such that $P(x) = A(x, \dots, x)$ for all $x \in E$. Each such P has a unique associated continuous symmetric k-linear map $\check{P}$ of E^k into F. We let $\mathcal{P}(^kE : F)$ denote the Banach space of continuous k-homogeneous polynomials of E into F, endowed with the polynomial norm $\|P\| = \sup_{x \in B_E} \|P(x)\|$. When F is the scalar field $\mathbb{R}$ or $\mathbb{C}$, we denote this space by $\mathcal{P}(^kE)$. Note that $\mathcal{P}(^1E : E) = \mathcal{L}(^1E : E)$ is the space of bounded linear operators on E. (See [**10**] for a general background on the theory of polynomials on an infinite-dimensional Banach space.) Let

$$\Pi(E) = \{(x, x^*) : x \in S_E,\ x^* \in S_{E^*},\ x^*(x) = 1\}.$$

For each $P \in \mathcal{P}(^kE : E)$, the *numerical range of* P is the subset $V(P)$ of the scalar field defined by

$$V(P) = \{x^*(Px) : (x, x^*) \in \Pi(E)\}.$$

In [**5**] the *numerical radius* of P is given by

$$v(P) = \sup\{|\lambda| : \lambda \in V(P)\}$$

and the numerical radius of a homogeneous polynomial on some classes of Banach spaces was computed. It is clear that v is a seminorm on $\mathcal{P}(^kE : E)$ and $v(P) \leqslant \|P\|$ for every $P \in \mathcal{P}(^kE : E)$. We introduce the polynomial numerical index of order k of a Banach space, generalizing to k-homogeneous polynomials the 'classical' numerical index defined by Lumer in the 1970s for linear operators. It is natural to consider the *polynomial numerical index of order* k of the space E, namely the constant $n^{(k)}(E)$ defined by

$$n^{(k)}(E) = \inf\{v(P) : P \in S_{\mathcal{P}(^kE:E)}\}.$$

Equivalently, $n^{(k)}(E)$ is the greatest constant $c \geqslant 0$ such that $c\|P\| \leqslant v(P)$ for every $P \in \mathcal{P}(^kE : E)$. Note that $0 \leqslant n^{(k)}(E) \leqslant 1$, and $n^{(k)}(E) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms on $\mathcal{P}(^kE : E)$. It is obvious that if E_1, E_2 are isometrically isomorphic Banach spaces, then $n^{(k)}(E_1) = n^{(k)}(E_2)$.

The concept of the numerical index (in our terminology, the polynomial numerical index of order 1) was first suggested by Lumer [**17**]. He gave a theory of the numerical range or bounded linear operators on a Banach space. This is a very successful generalization of the classical theory, in which only Hilbert spaces are considered. At that time, it was known that a Hilbert space of dimension greater than 1 has numerical index $\frac{1}{2}$ in the complex case and 0 in the real case. Several years later, Duncan *et al.* [**11**] proved that L-spaces and M-spaces have numerical index 1. McGregor [**18**] obtained necessary and sufficient conditions such that a finite-dimensional normed space has numerical index 1. The disc algebra is another example of a Banach space with numerical index 1 [**8**, Theorem 3.3]. Crabb *et al.* [**7**] investigated some extremal problems in the theory of numerical ranges. Recently, Lopez *et al.* [**16**] investigated necessary conditions for a real Banach space to have numerical index 1. Martin and Paya [**19**] studied the numerical index of vector-valued function spaces. For general information and background on numerical ranges we refer to the books by Bonsall and Duncan [**3**, **4**]. Further developments in the Hilbert space case can be found in [**13**].

In § 2 of this paper we prove the following results. Let k be a positive integer. Then we have the following:

(i) $n^{(k)}(C(K)) = 1$ for every positive integer k and every scattered compact space K.

(ii) The inequality $n^{(k)}(E) \geqslant k^{k/(1-k)}$ for every *complex* Banach space E and the constant $k^{k/(1-k)}$ is sharp.

(iii) The inequalities

$$n^{(k)}(E) \leqslant n^{(k-1)}(E) \leqslant \frac{k^{(k+(1/(k-1)))}}{(k-1)^{k-1}} n^{(k)}(E)$$

for every Banach space E.

(iv) The relation between the polynomial numerical index of c_0, l_1, l_∞ sums of Banach spaces and the infimum of the polynomial numerical indices of them.

(v) The relation between the polynomial numerical index of the space $C(K, E)$ and the polynomial numerical index of E.

(vi) The inequality $n^{(k)}(E^{**}) \leqslant n^{(k)}(E)$ for every Banach space E.

In § 3 some results about the numerical radius of multilinear maps and homogeneous polynomials on $C(K)$ and the disc algebra are given.

2. Properties of the polynomial numerical index of order k

It was proved in [**5**, Theorem 3.1 (ii)] that $n^{(k)}(c_0) = n^{(k)}(c) = n^{(k)}(l_\infty) = 1$ for every positive integer k, where c is the Banach space of convergent sequences in $\mathbb{C}$.

Given a Banach space E, we denote by $\mathcal{A}(B_E)$ the Banach space of all functions $f : B_E \to \mathbb{C}$ which are holomorphic on $\mathring{B}_E$ and uniformly continuous on B_E, endowed with the supremum norm. Recall that a mapping P is said to be a continuous polynomial on E if it can be represented as a sum

$$P = P_0 + P_1 + \cdots + P_m,$$

where $P_j \in \mathcal{P}(^j E)$ for $j = 0, \dots, m$. The vector space of all continuous polynomials on P is always a dense subspace of $\mathcal{A}(B_E)$.

Lemma 2.1 (see Theorem 3.3 in [6]). *Let K be a scattered compact Hausdorff space. If T is an element of $\mathcal{A}(B_{C(K)})$, then*

$$\|T\| = \sup\{|T(f)| : f \in \operatorname{ext} B_{C(K)}\},$$

where $\operatorname{ext} B_{C(K)}$ is the set of all extreme points of $B_{C(K)}$.

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Theorem 2.2. *Let K be a scattered compact space. For every positive integer k, we have $n^{(k)}(C(K)) = 1$.*

Proof. It suffices to show that $\|P\| = v(P)$ for every $P \in \mathcal{P}({}^kC(K) : C(K))$. Let $P \in \mathcal{P}({}^kC(K) : C(K))$. Let $\varepsilon > 0$ be given. We can choose $f_0 \in B_{C(K)}$ and $t_0 \in K$ such that $|P(f_0)(t_0)| > \|P\| - \varepsilon$. Define a continuous k-homogeneous polynomial $Q : C(K) \to \mathbb{C}$ by $Q(f) = P(f)(t_0)$ $(f \in C(K))$. By Lemma 2.1 there exists $g_0 \in \operatorname{ext} B_{C(K)}$ such that $|Q(g_0)| > \sup_{f\in B_{C(K)}} |Q(f)| - \varepsilon$. Then $|g_0(t)| = 1$ for every $t \in K$. It follows that

$$\begin{aligned}\|P\| - 2\varepsilon < |P(f_0)(t_0)| - \varepsilon &\leqslant \sup_{f\in B_{C(K)}} |P(f)(t_0)| - \varepsilon \\ &= \sup_{f\in B_{C(K)}} |Q(f)| - \varepsilon < |Q(g_0)| = |P(g_0)(t_0)| \\ &= |\operatorname{sgn}(\delta_{t_0}(g_0))\delta_{t_0} P(g_0)| \leqslant v(P),\end{aligned}$$

which shows that $\|P\| = v(P)$ because $(g_0, \operatorname{sgn}(\delta_{t_0}(g_0))\delta_{t_0}) \in \Pi(C(K))$. □

Theorem 2.3. *Let E be a complex Banach space. For every positive integer k, we have*

$$n^{(k)}(E) \geqslant k^{k/(1-k)}$$

and the constant $k^{k/(1-k)}$ is sharp.

Proof. By [**14**, Theorem 1], it is true that $\|P\| \leqslant k^{k/(k-1)} v(P)$ for each $P \in \mathcal{P}({}^kE : E)$. This follows from the fact that

$$v\left(\frac{P}{\|P\|}\right) = \frac{1}{\|P\|} v(P)$$

and the definition of $n^{(k)}(E)$. In [**14**, § 7] it is proved that for every $k \in \mathbb{N}$ there is a two-dimensional space E with $n^{(k)}(E) = k^{k/(1-k)}$. □

Lemma 2.4. *Let E be a Banach space. Let $P \in \mathcal{P}({}^kE : E)$, $x \in B_E$. For $1 \leqslant m < k$, we have*

$$v(\hat{D}^m P(x)) \leqslant \frac{k^{(k+(k/(k-1)))} m!}{(k-m)^{k-m} m^m} v(P),$$

where $\hat{D}^m P(x) \in \mathcal{P}({}^{k-m}E : E)$ is defined by $\hat{D}^m P(x)(y) = \check{P}(x^m, y^{k-m})$ for $x, y \in E$.

Proof. By a result of Harris [**15**, Corollary 3] and Theorem 2.3, it follows that

$$\begin{aligned}v(\hat{D}^m P(x)) \leqslant \|\hat{D}^m P(x)\| &\leqslant \frac{k^k m!}{(k-m)^{k-m} m^m} \|P\| \\ &\leqslant \frac{k^k m!}{(k-m)^{k-m} m^m} k^{k/(k-1)} v(P).\end{aligned}$$

□

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Proposition 2.5. *Let E be a Banach space. For every positive integer $k \geqslant 2$, we have*

$$n^{(k)}(E) \leqslant n^{(k-1)}(E) \leqslant \frac{k^{(k+(1/(k-1)))}}{(k-1)^{k-1}} n^{(k)}(E).$$

Proof. First we will prove the left inequality, $n^{(k)}(E) \leqslant n^{(k-1)}(E)$, for every Banach space E and every $k \geqslant 2$.

Indeed, let $\alpha = n^{(k)}(E)$. Let $Q \in S_{\mathcal{P}(^{k-1}E:E)}$. Let $\{x_i\} \subset S_E$ such that $\|Q(x_i)\| \to 1$ as $i \to \infty$. Define $P_i(x) = x_i^*(x)Q(x)$ for $x \in E$, where $x_i^* \in E^*$,with $\|x_i^*\| = x_i^*(x_i) = 1$ for every positive integer i. Then $P_i \in \mathcal{P}(^kE : E)$. Note that $\|P_i\| \to 1$ as $i \to \infty$. Since $v(P_i/\|P_i\|) \geqslant \alpha$, we have

$$\begin{aligned}\alpha\|P_i\| &\leqslant v(P_i)\\ &= \sup_{(x,x^*)\in \Pi(E)} |x^*(P_i(x))|\\ &= \sup_{(x,x^*)\in \Pi(E)} |x_i^*(x)|\,|x^*(Q(x))|\\ &\leqslant \sup_{(x,x^*)\in \Pi(E)} |x^*(Q(x))|\\ &= v(Q).\end{aligned}$$

Taking the limit as $i \to \infty$, we get $\alpha \leqslant v(Q)$. Since $Q \in S_{\mathcal{P}(^{k-1}E:E)}$ was arbitrary, we obtain the left inequality.

In order to prove the right inequality, let $P \in S_{\mathcal{P}(^kE:E)}$, $x \in S_E$. By Lemma 2.4, it follows that

$$\begin{aligned}n^{(k)}(E) \leqslant n^{(k-1)}(E) &\leqslant v\biggl(\frac{\hat{D}P(x)}{\|\hat{D}P(x)\|}\biggr) = \frac{1}{\|\hat{D}P(x)\|} v(\hat{D}P(x))\\ &\leqslant \frac{k^{(k+(k/(k-1)))}}{(k-1)^{k-1}} \frac{v(P)}{\|\hat{D}P(x)\|}.\end{aligned}$$

We claim that

$$\inf_{P\in S_{\mathcal{P}(^kE:E)},\, x\in S_E} \frac{v(P)}{\|\hat{D}P(x)\|} \leqslant \frac{1}{k} n^{(k)}(E).$$

Let

$$I = \inf_{P\in S_{\mathcal{P}(^kE:E)},\, x\in S_E} \frac{v(P)}{\|\hat{D}P(x)\|}.$$

Then

$$\begin{aligned}I &= \inf_{P\in S_{\mathcal{P}(^kE:E)}} \biggl\{ v(P) \inf_{x\in S_E} \frac{1}{\|\hat{D}P(x)\|}\biggr\}\\ &= \inf_{P\in S_{\mathcal{P}(^kE:E)}} \biggl\{ v(P) \frac{1}{\sup_{x\in S_E} \|\hat{D}P(x)\|}\biggr\}.\end{aligned}$$

We show that

$$\sup_{x\in S_E} \|\hat{D}P(x)\| \geqslant k.$$

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Indeed,

$$\sup_{x \in S_E} \|\hat{D}P(x)\| = \sup_{x \in S_E} \Big(\sup_{y \in S_E} \|\hat{D}P(x)(y)\| \Big)$$
$$= k \sup_{x,y \in S_E} \|\check{P}(x^{k-1}y)\| \geqslant k \sup_{x \in S_E} \|P(x)\| = k.$$

So,

$$I \leqslant \inf_{P \in S_{\mathcal{P}(^kE:E)}} \left\{ v(P)\frac{1}{k} \right\} = \frac{1}{k} n^{(k)}(E).$$

Therefore,

$$n^{(k)}(E) \leqslant n^{(k-1)}(E) \leqslant \frac{k^{(k+(k/(k-1)))}}{(k-1)^{k-1}} I \leqslant \frac{k^{(k+(1/(k-1)))}}{(k-1)^{k-1}} n^{(k)}(E).$$

□

Remark 2.6. If l_2 is a real Hilbert space, then $n^{(k)}(l_2) = 0$ for every $k \geqslant 2$.

Remark 2.7. If l_2 is a complex Hilbert space, then $n^{(k)}(l_2) \leqslant \frac{1}{2}$ for every $k \geqslant 2$.

The proof of the following proposition is almost the same as the one given in [**19**, Proposition 1].

Proposition 2.8. *For every Banach space E_λ and every positive integer k, we have*

(1) $n^{(k)}\Big(\Big[\bigoplus_{\lambda \in \Lambda} E_\lambda\Big]_{c_0}\Big) \leqslant \inf_{\lambda \in \Lambda} n^{(k)}(E_\lambda)$;

(2) $n^{(k)}\Big(\Big[\bigoplus_{\lambda \in \Lambda} E_\lambda\Big]_{l_1}\Big) \leqslant \inf_{\lambda \in \Lambda} n^{(k)}(E_\lambda)$;

(3) $n^{(k)}\Big(\Big[\bigoplus_{\lambda \in \Lambda} E_\lambda\Big]_{l_\infty}\Big) \leqslant \inf_{\lambda \in \Lambda} n^{(k)}(E_\lambda)$.

Proof. We prove only (2) because the proofs of (1) and (3) are similar. Let $P \in \mathcal{P}(^kX : X)$ with $\|P\| = 1$. Let $Q \in \mathcal{P}(^kX \oplus_1 Y : X \oplus_1 Y)$ be such that $Q(x,y) = (P(x), 0)$. Then $\|Q\| = 1$. Given $\varepsilon > 0$, there exist $(x,y) \in S_{X \oplus_1 Y}$ and $(x^*, y^*) \in S_{(X \oplus_1 Y)^*}$ such that $x^*(x) + y^*(y) = \|x^*\|\,\|x\| + \|y^*\|\,\|y\| = 1$ and

$$n^{(k)}(X \oplus_1 Y) - \varepsilon \leqslant |(x^*, y^*)Q(x,y)|$$
$$= |x^*(P(x))| \leqslant \frac{1}{\|x^*\|\,\|x\|^k} |x^*(P(x))| = \left| \frac{x^*}{\|x^*\|} P\left(\frac{x}{\|x\|}\right) \right| \leqslant v(P),$$

because $(1/\|x^*\|\,\|x\|^k) \geqslant 1$. Thus $n^{(k)}(X \oplus_1 Y) \leqslant n^{(k)}(X)$. □

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In $l_1(\mu, \mathbb{R}) = l_1$, the inequality in Proposition 2.8 (2) is strict because $n^{(2)}(\mathbb{R}) = 1$ and $n^{(2)}(l_1(\mu, \mathbb{R})) \leqslant \frac{1}{2}$. Indeed, as in [**5**], let $P \in \mathcal{P}(^2 l_1 : l_1)$ be defined by

$$P(x) = (\tfrac{1}{2}x_1^2 + 2x_1x_2, -\tfrac{1}{2}x_2^2 - x_1x_2, 0, 0, \dots) \quad (\text{for } x = (x_i) \in l_1).$$

Then it is not difficult to show that $\|P\| = 1$ and $v(P) = \frac{1}{2}$. Thus $n^{(2)}(l_1) \leqslant \frac{1}{2}$.

The following lemma can be deduced from Corollary 2 of [**14**].

Lemma 2.9. *Let K be a compact Hausdorff space and let k be a positive integer. Let $Q \in \mathcal{P}(^k C(K, E) : C(K, E))$. Then*

$$v(Q) = \sup\{|x^*(Q(f)(t))| : f \in S_{C(K,E)},\ t \in K,\ x^* \in S_{E^*},\ x^*(f(t)) = 1\}.$$

The proof of the following theorem is almost the same as the one given in [**19**, Theorem 5].

Proposition 2.10. *Let K be a compact Hausdorff space. For every positive integer k, we have $n^{(k)}(C(K, E)) \leqslant n^{(k)}(E)$.*

Proof. Let $P \in \mathcal{P}(^k E : E)$ with $\|P\| = 1$. Define $Q \in \mathcal{P}(^k C(K, E) : C(K, E))$ by

$$Q(f)(t) = P(f(t)) \quad (t \in K,\ f \in C(K, E)).$$

Then $\|Q\| = 1$. So, $v(Q) \geqslant n^{(k)}(C(K, E))$. By Lemma 2.9, given $\varepsilon > 0$, we can find $f \in S_{C(K,E)}$, $t \in K$, $x^* \in S_{E^*}$ such that $x^*(f(t)) = 1$ and

$$|x^*(P(f(t)))| = |x^*(Q(f)(t))| > n^{(k)}(C(K, E)) - \varepsilon.$$

Thus $n^{(k)}(C(K, E)) \leqslant n^{(k)}(E)$. □

Let E and F be Banach spaces. A bounded k-homogeneous polynomial P has an extension $\bar{P} \in \mathcal{P}(^k E^{**} : F^{**})$ to the bidual E^{**} of E, which is called the Aron–Berner extension of P (see [**1**]). In fact, $\bar{P}$ is defined in the following way. We first start with the complex-valued bounded k-homogeneous polynomial $P \in \mathcal{P}(^k E)$. Let A be the bounded symmetric k-linear form on E corresponding to P. We can extend A to an k-linear form $\bar{A}$ on the bidual E^{**} in such a way that, for each fixed j, $1 \leqslant j \leqslant k$, and, for each fixed $x_1, \dots, x_{j-1} \in E$ and $z_{j+1}, \dots, z_m \in E^{**}$, the linear form

$$z \to \bar{A}(x_1, \dots, x_{j-1}, z, z_{j+1}, \dots, z_k), \quad z \in E^{**},$$

is weak* continuous. By this weak* continuity A can be extended to a k-linear form $\bar{A}$ on E^{**}, beginning with the last variable and working backwards to the first. Then the restriction

$$\bar{P}(z) = \bar{A}(z, \dots, z)$$

is called the Aron–Berner extension of P. In particular, Davie and Gamelin [**9**] proved that $\|P\| = \|\bar{P}\|$. It is also worth remarking that $\bar{A}$ is not symmetric in general.

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Next, for a vector-valued k-homogeneous polynomial $P \in \mathcal{P}(^kE : F)$, the Aron–Berner extension $\bar{P} \in \mathcal{P}(^kE^{**} : F^{**})$ is defined as follows: given $z \in E^{**}$ and $w \in F^*$,

$$\bar{P}(z)(w) = \overline{w \circ P}(z).$$

For $x \in E$, we define $\delta_x : E^* \to \mathbb{C}$ by $\delta_x(x^*) = x^*(x)$ for each $x^* \in E^*$. Then $\delta_x \in E^{**}$.

Let $\langle x_\alpha \rangle$ be a net in E and let $x_0^{**} \in E^{**}$. We say that $\langle x_\alpha \rangle$ converges polynomial* to x_0^{**} if, for every $P \in \mathcal{P}(^kE)(k \in \mathbb{N})$, we have that $P(x_\alpha)$ converges to $\bar{P}(x_0^{**})$, where $\bar{P}$ is the Aron–Berner extension of P.

A function $f : E^{**} \to F^*$ is called (pol*,w*)-continuous if $x_0^{**} \in E^{**}$ and $\langle x_\alpha \rangle$ is a net in E such that $\langle x_\alpha \rangle$ converges polynomially* to x_0^{**}, then $\langle f(\delta_{x_\alpha}) \rangle$ converges weakly* to $f(x_0^{**})$.

The proof of the following theorem is very close to the one given in [**4**, Theorem 17.2].

Theorem 2.11. *Let E be a Banach space. Let $P \in \mathcal{P}(^kE^{**} : E^{**})$ $(n \geqslant 1)$ be (pol*,w*)-continuous. Let*

$$LV(P) := \{P(x'')(x') : (x', x'') \in \Pi(E^*)\}$$

and

$$lV(P) := \{\delta_{x'}(P(\delta_x)) : (x, x') \in \Pi(E)\}.$$

Then $lV(P) \subset V(P) \subset \overline{lV(P)}$, so $\overline{lV(P)} = \overline{V(P)}$.

Proof. We may assume that $\|P\| = 1$. Clearly, $lV(P) \subset V(P)$.

Claim 2.12. $V(P) \subset \overline{LV(P)}$.

Let $\lambda \in V(P)$. Then $\lambda = x_0'''(P(x_0''))$ for some $(x_0'', x_0''') \in \Pi(E^{**})$. Let $0 < \varepsilon < 1$. By the uniform continuity of P on $B_{E^{**}}$ there is a $0 < \delta < \frac{1}{3}\varepsilon$ such that, for $x'', y'' \in B_{E^{**}}$ with $\|x'' - y''\| < \delta$, we have $\|P(x'') - P(y'')\| < \frac{1}{3}\varepsilon$. Since B_{E^*} is w*-dense in $B_{E^{***}}$, there exists $x_0' \in B_{E^*}$ such that

$$|\delta_{x_0'}(P(x_0'')) - x_0'''(P(x_0''))| = |\lambda - \delta_{x_0'}(P(x_0''))| < \delta$$

and

$$|\delta_{x_0'}(x_0'') - x_0'''(x_0'')| = |1 - x_0''(x_0')| < \tfrac{1}{4}\delta^2.$$

By the Bishop–Phelps–Bollobas theorem [**2**] there exist $(y_0', y_0'') \in \Pi(E^*)$ such that $\|x_0' - y_0'\| < \delta$ and $\|x_0'' - y_0''\| < \delta$. Thus $\delta_{y_0'}(P(y_0'')) \in LV(P)$. It follows that

$$\begin{aligned}
&|\lambda - \delta_{y_0'}(P(y_0''))| \\
&\qquad \leqslant |\lambda - \delta_{x_0'}(P(x_0''))| + |P(x_0'')(x_0') - P(x_0'')(y_0')| + |P(x_0'')(y_0') - P(y_0'')(y_0')| \\
&\qquad < \delta + \|P(x_0'')\|\,\|x_0' - y_0'\| + \|P(x_0'') - P(y_0'')\| \\
&\qquad < 3\delta < \varepsilon,
\end{aligned}$$

which shows that $\lambda \in \overline{LV(P)}$. Thus $V(P) \subset \overline{LV(P)}$.

Claim 2.13. $LV(P) \subset \overline{lV(P)}$.

Let $\beta \in LV(P)$. Then $\beta = P(x_0'')(x_0') = \delta_{x_0'}(P(x_0''))$ for some $(x_0', x_0'') \in \Pi(E^*)$. Let $0 < \varepsilon < 1$. By the Davie–Gamelin theorem [**9**] (B_E is pol*-dense in $B_{E^{**}}$) there is a net $\langle x_\alpha \rangle$ in B_E such that δ_{x_α} converges pol* to x_0''. Then $\delta_{x_0'}(\delta_{x_\alpha}) = x_0'(x_\alpha)$ converges to $\delta_{x_0'}(x_0'') = x_0''(x_0') = 1$. Let $Q = \delta_{x_0'} \circ P \in \mathcal{P}(^kE^{**})$. Since $P \in \mathcal{P}(^kE^{**} : E^{**})$ is (pol*,w*)-continuous, $Q(\delta_{x_\alpha}) = \delta_{x_0'}(P(\delta_{x_\alpha}))$ converges to $Q(x_0'') = \delta_{x_0'}(P(x_0'')) = \beta$. Choose x_{α_0} such that

$$|\beta - \delta_{x_0'}(P(\delta_{x_{\alpha_0}}))| < \delta \quad \text{and} \quad |x_0'(x_{\alpha_0}) - 1| < \tfrac{1}{4}\delta^2.$$

By the Bishop–Phelps–Bollobas theorem [**2**] there is $(y_0, y_0') \in \Pi(E)$ such that $\|x_{\alpha_0} - y_0\| < \delta$ and $\|x_0' - y_0'\| < \delta$. Then $\delta_{y_0'}(P(\delta_{y_0})) \in lV(P)$. We have

$$\begin{aligned}
&|\beta - \delta_{y_0'}(P(\delta_{y_0}))| \\
&\quad \leqslant |\beta - \delta_{x_0'}(P(\delta_{x_{\alpha_0}}))| + |\delta_{x_0'}(P(\delta_{x_{\alpha_0}})) - \delta_{x_0'}(P(\delta_{y_0}))| + |\delta_{x_0'}(P(\delta_{y_0})) - \delta_{y_0'}(P(\delta_{y_0}))| \\
&\quad < \delta + \|P(\delta_{x_{\alpha_0}}) - P(\delta_{y_0})\| + \|P(\delta_{y_0})\| \, \|x_0' - y_0'\| \\
&\quad < 3\delta < \varepsilon,
\end{aligned}$$

which shows that $\beta \in \overline{lV(P)}$. Thus $LV(P) \subset \overline{lV(P)}$. Thus, by Claims 2.12 and 2.13, $V(P) \subset \overline{lV(P)}$. □

Corollary 2.14. *Let E be a Banach space and let k be a positive integer. Let $Q \in \mathcal{P}(^kE : E)$. Then $\overline{V(Q)} = \overline{V(\bar{Q})}$, where $\bar{Q}$ is the Aron–Berner extension of Q.*

Proof. Since $\bar{Q}$ is (pol*,w*)-continuous and $lV(\bar{Q}) = V(Q)$, the corollary is proven. □

Corollary 2.15. *Let E be a Banach space. For every positive integer k, we have $n^{(k)}(E^{**}) \leqslant n^{(k)}(E)$.*

Proof. For every $Q \in \mathcal{P}(^kE : E)$ with $\|Q\| = 1$ there is the Aron–Berner extension $\bar{Q} \in \mathcal{P}(^kE^{**} : E^{**})$ of Q. Davie and Gamelin [**9**] proved that $\|Q\| = 1 = \|\bar{Q}\|$ and Corollary 2.14 shows that $v(Q) = v(\bar{Q})$, which proves the corollary. □

3. Numerical radius of a multilinear map and a polynomial on $C(K)$ and the disc algebra

In [**5**] the *numerical radius of a k-linear mapping* $A \in \mathcal{L}(^kE : E)$ is defined by

$$v(A) = \sup\{|x^*(A(x_1, \dots, x_k))| : (x_1, \dots, x_k, x^*) \in \Pi(E^k)\},$$

where

$$\Pi(E^k) = \{(x_1, \dots, x_k, x^*) : \|x_j\| = \|x^*\| = 1 = x^*(x_j),\ j = 1, \dots, k\}.$$

Theorem 3.1. *Let K be a compact Hausdorff space and let $P \in \mathcal{P}(^kC(K) : C(K))$ ($k \in \mathbb{N}$). Then $v(P) = \|P\|$ or $v(\check{P}) \geqslant \|P\|$, where $\check{P}$ is the symmetric k-linear map associated with P.*

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Proof. Let $\varepsilon > 0$. Assume that $\|P\| = 1$. Then there exist $f_0 \in C(K)$ with $\|f_0\| = 1$ and $t_0 \in K$ such that $f_0(t_0) \neq 0$ and $|P(f_0)(t_0)| > 1-\varepsilon$. Let U be an open neighbourhood of t_0 with $0 \notin f_0(U)$. By Urysohn's lemma there is a continuous function $\pi : K \to [0,1]$ such that $\pi(t_0) = 1$, $\pi(K - U) = 0$. Now define ψ on K by $\psi(t) = 0$ when $f_0(t) = 0$ and

$$\psi(t) = \frac{f_0(t)}{|f_0(t)|}\sqrt{1 - |f_0(t)|^2}\pi(t),$$

where $f_0(t) \neq 0$. Then $\psi \in C(K)$. Let $g_0 = f_0 + \mathrm{i}\psi$, $h_0 = f_0 - \mathrm{i}\psi$, so that $g_0, h_0 \in C(K)$, $f_0 = \frac{1}{2}(g_0 + h_0)$ and $|g_0(t_0)| = |h_0(t_0)| = \|g_0\| = \|h_0\| = 1$. Note that

$$\begin{aligned} 1 - \varepsilon < |P(f_0)(t_0)| &\leqslant \frac{1}{2^k} \sum_{0 \leqslant j \leqslant k} {}_kC_j |\check{P}(g_0^j h_0^{k-j})(t_0)| \\ &= \frac{1}{2^k}\bigg(|P(g_0)(t_0)| + |P(h_0)(t_0)| + \sum_{1 \leqslant j \leqslant k-1} {}_kC_j |\check{P}(g_0^j h_0^{k-j})(t_0)|\bigg), \end{aligned}$$

where ${}_kC_j = k!/(j!(k-j)!)$. So we have $|P(g_0)(t_0)| > 1 - \varepsilon$ or $|P(h_0)(t_0)| > 1 - \varepsilon$ or $|\check{P}(g_0^j h_0^{k-j})(t_0)| > 1 - \varepsilon$ for some $1 \leqslant j \leqslant k - 1$. Note that

$$1 - \varepsilon < |P(g_0)(t_0)| = |\operatorname{sgn}(g_0(t_0))\delta_{t_0}(P(g_0))| \leqslant v(P)$$

or

$$1 - \varepsilon < |P(h_0)(t_0)| = |\operatorname{sgn}(h_0(t_0))\delta_{t_0}(P(h_0))| \leqslant v(P)$$

or

$$1 - \varepsilon < |\check{P}(g_0^j h_0^{k-j})(t_0)| = |\delta_{t_0}(\check{P}((\operatorname{sgn}(g_0(t_0))g_0)^j(\operatorname{sgn}(h_0(t_0))h_0)^{k-j}))| \leqslant v(\check{P}),$$

because

$$(g_0, \operatorname{sgn}(g_0(t_0))\delta_{t_0}), (h_0, \operatorname{sgn}(h_0(t_0))\delta_{t_0}), (\operatorname{sgn}(g_0(t_0))g_0, \delta_{t_0}), (\operatorname{sgn}(h_0(t_0))h_0, \delta_{t_0}) \in \Pi(C(K)).$$

Thus $v(P) > 1 - \varepsilon$ or $v(\check{P}) > 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $v(P) = \|P\|$ or $v(\check{P}) \geqslant \|P\|$. □

Theorem 3.2. *Let A_D be the disc algebra. Let $L \in \mathcal{L}({}^kA_D : A_D)$ $(k \in \mathbb{N})$. Then $v(L) = \|L\|$.*

Proof. Let $\varepsilon > 0$. Assume that $\|L\| = 1$. It suffices to prove theorem in the case $n = 2$. There exist $f_1, f_2 \in A_D$ with $\|f_1\| = \|f_2\| = 1$ and such that $\|L(f_1, f_2)\| > 1 - \varepsilon$. Since L is uniformly continuous on the closed unit ball $B_{A_D} \times B_{A_D}$, there is a $\delta > 0$ such that, for all $f_i, g_i \in B_{A_D}$ $(i = 1, 2)$ with $\|f_i - g_i\| < \delta$, we have $\|L(f_1, f_2) - L(g_1, g_2)\| < \varepsilon$. By a theorem of Fischer [**12**] there exist $\alpha_1, \dots, \alpha_l$, $\beta_1, \dots, \beta_m$ with $\alpha_j \geqslant 0$, $\beta_n \geqslant 0$,

$$\sum_{1 \leqslant j \leqslant l} \alpha_j = \sum_{1 \leqslant n \leqslant m} \beta_n = 1,$$

and finite Blaschke products $g_1, \dots, g_l$, $h_1, \dots, h_m$ such that

$$\Big\| f_1 - \sum_{1\leqslant j\leqslant l} \alpha_j g_j \Big\| < \delta \quad \text{and} \quad \Big\| f_2 - \sum_{1\leqslant n\leqslant m} \beta_n h_n \Big\| < \delta.$$

Clearly,

$$\Big\| L(f_1, f_2) - L\Big(\sum_{1\leqslant j\leqslant l} \alpha_j g_j, \sum_{1\leqslant n\leqslant m} \beta_n h_n \Big) \Big\| < \varepsilon,$$

so

$$\Big\| L\Big(\sum_{1\leqslant j\leqslant l} \alpha_j g_j, \sum_{1\leqslant n\leqslant m} \beta_n h_n \Big) \Big\| > 1 - 2\varepsilon.$$

Choose $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that

$$\Big| L\Big(\sum_{1\leqslant j\leqslant l} \alpha_j g_j, \sum_{1\leqslant n\leqslant m} \beta_n h_n \Big)(z_0) \Big| = \Big\| L\Big(\sum_{1\leqslant j\leqslant l} \alpha_j g_j, \sum_{1\leqslant n\leqslant m} \beta_n h_n \Big) \Big\|.$$

Note that $|g_j(z_0)| = |h_n(z_0)| = 1$ for all j, n. We have

$$1 - 2\varepsilon < \Big| L\Big(\sum_{1\leqslant j\leqslant l} \alpha_j g_j, \sum_{1\leqslant n\leqslant m} \beta_n h_k n \Big)(z_0) \Big| \leqslant \sum_{1\leqslant j\leqslant l,\ 1\leqslant n\leqslant m} \alpha_j \beta_n |L(g_j, h_n)(z_0)|.$$

Since

$$\sum_{1\leqslant j\leqslant l,\ 1\leqslant n\leqslant m} \alpha_j \beta_n = \Big(\sum_{1\leqslant j\leqslant l} \alpha_j \Big) \Big(\sum_{1\leqslant n\leqslant m} \beta_n \Big) = 1,$$

we have $|L(g_{j_0}, h_{n_0})(z_0)| > 1 - 2\varepsilon$ for some j_0, n_0. It follows that

$$1 - 2\varepsilon < |L(g_{j_0}, h_{n_0})(z_0)| = |\delta_{z_0} L(\overline{g_{j_0}(z_0)} g_{j_0}, \overline{h_{n_0}(z_0)} h_{n_0})| \leqslant v(L)$$

because

$$(\overline{g_{j_0}(z_0)} g_{j_0}, \delta_{z_0}), (\overline{h_{n_0}(z_0)} h_{n_0}, \delta_{z_0}) \in \Pi(A_D).$$

Thus $v(L) > 1 - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $v(L) = \|L\|$. □

Theorem 3.3. *Let A_D be the disc algebra. Let $P \in \mathcal{P}(^k A_D : A_D)$ $(k \in \mathbb{N})$. Then $v(P) = \|P\|$ or $v(\check{P}) \geqslant \|P\|$, where $\check{P}$ is the symmetric k-linear map associated with P.*

Proof. Let $\varepsilon > 0$. Assume that $\|P\| = 1$. Then there exist $f_0 \in A_D$ with $\|f_0\| = 1$ such that $\|P(f_0)\| > 1 - \varepsilon$. Since P is uniformly continuous on the closed unit ball B_{A_D}, there is a $\delta > 0$ such that, for all $f, g \in B_{A_D}$ with $\|f - g\| < \delta$, we have $\|P(f) - P(g)\| < \varepsilon$. By a theorem of Fischer [**12**] there exist $\alpha_1, \dots, \alpha_n$ with $\alpha_j \geqslant 0$,

$$\sum_{1\leqslant j\leqslant n} \alpha_j = 1,$$

and finite Blaschke products $g_1, \dots, g_n$ such that

$$\bigg\| f_0 - \sum_{1 \leqslant j \leqslant n} \alpha_j g_j \bigg\| < \delta.$$

Clearly,

$$\bigg\| P(f_0) - P\bigg(\sum_{1 \leqslant j \leqslant n} \alpha_j g_j \bigg) \bigg\| < \varepsilon,$$

so

$$\bigg\| P\bigg(\sum_{1 \leqslant j \leqslant n} \alpha_j g_j \bigg) \bigg\| > 1 - 2\varepsilon.$$

Choose $z_0 \in C$ with $|z_0| = 1$ such that

$$\bigg| P\bigg(\sum_{1 \leqslant j \leqslant n} \alpha_j g_j \bigg)(z_0) \bigg| = \bigg\| P\bigg(\sum_{1 \leqslant j \leqslant n} \alpha_j g_j \bigg) \bigg\|.$$

Note that $|g_j(z_0)| = 1$ for all $j = 1, \dots, n$. We have

$$\begin{aligned} 1 - 2\varepsilon & \\ &< \bigg| P\bigg(\sum_{1 \leqslant j \leqslant n} \alpha_j g_j \bigg)(z_0) \bigg| \\ &\leqslant \sum_{i_1 + \dots + i_l = k} \frac{k!}{i_1! \cdots i_l!} |\check{P}((\alpha_{i_1} g_1)^{i_1} \cdots (\alpha_{i_l} g_l)^{i_l})(z_0)| \\ &= \bigg(\sum_{1 \leqslant j \leqslant n} \alpha_j^k |P(g_j)(z_0)| + \sum_{i_1 + \dots + i_l = k,\ i_j < k} \frac{k!}{i_1! \cdots i_l!} \alpha_{i_1}^{i_1} \cdots \alpha_{i_l}^{i_l} |\check{P}((g_1)^{i_1} \cdots (g_k)^{i_l})(z_0)| \bigg). \end{aligned}$$

Since

$$\sum_{1 \leqslant j \leqslant n} \alpha_j^k + \sum_{i_1 + \dots + i_k = l,\ i_j < k} \frac{k!}{i_1! \cdots i_l!} \alpha_{i_1}^{i_1} \cdots \alpha_{i_l}^{i_l} = (i_1 + \dots + i_l)^k = 1,$$

we have $|P(g_j)(z_0)| > 1 - 2\varepsilon$ for some j or $|\check{P}((g_1)^{i_1} \dots (g_k)^{i_l})(z_0)|$ for some i_j with $i_1 + \dots + i_l = k$, $i_j < k$. It follows that

$$\begin{aligned} 1 - 2\varepsilon &< |\check{P}((\alpha_{i_1} g_1)^{i_1} \cdots (\alpha_{i_l} g_l)^{i_l})(z_0)| \\ &= |\delta_{z_0}(\check{P}((\overline{g_1(z_0)} g_1)^{i_1} \cdots (\overline{g_l(z_0)} g_l)^{i_l}))(z_0)| \leqslant v(\check{P}) \end{aligned}$$

or

$$1 - 2\varepsilon < |P(g_j)(z_0)| = |\overline{g_j(z_0)} \delta_{z_0}(P(g_j))| \leqslant v(P)$$

because

$$(g_j, \overline{g_j(z_0)} \delta_{z_0}), (\overline{g_{i_1}(z_0)} g_{i_1}, \delta_{z_0}), \dots, (\overline{g_{i_l}(z_0)} g_{i_l}, \delta_{z_0}) \in \mathit{\Pi}(A_D).$$

Thus $v(P) > 1 - 2\varepsilon$ or $v(\check{P}) > 1 - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $v(P) = \|P\|$ or $v(\check{P}) \geqslant \|P\|$. □

Recall that (a complex) M-space with order unit is isometrically isomorphic to $C(K)$ for some compact Hausdorff space K.

Corollary 3.4. *Let E be an M-space with order unit. Let $P \in \mathcal{P}(^kE : E)$ $(k \in \mathbb{N})$. Then $v(P) = \|P\|$ or $v(\check{P}) \geqslant \|P\|$, where $\check{P}$ is the symmetric k-linear map associated with P.*

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