ON k-CONJUGACY IN A GROUP

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All elements mentioned herein are in a group G. A well-known definition states that x and y are conjugate if there exists an element a such that $y = a^{-1}xa$. Conjugacy is an equivalence relation in G. In the present paper this will be called 1-conjugacy.

When $k > 1$, the following definition is in effect: y is k-conjugate to $x (y \underset{k}{\sim} x)$ if there exist r and s such that $y = r^{-1}xs$ and $s \underset{k-1}{\sim} r$. While k-conjugacy is not generally an equivalence relation, it will be seen that there are groups (for example, all finite groups) in which it is so for some $k > 1$. Moreover, this concept is related to that of the number of commutators required to express each element of the commutator subgroup.

These properties are easily verified:

P1. If $y \underset{k}{\sim} x$, then $y \underset{n}{\sim} x$ for every $n > k$.

P2. k-conjugacy is reflexive.

P3. k-conjugacy is symmetric.

P4. If $y \underset{k}{\sim} x$, then $y^{-1} \underset{k}{\sim} x^{-1}$.

To prove P1, let $y \underset{k}{\sim} x$. Then $y = x^{-1}xy$, so $y \underset{k+1}{\sim} x$, and the result follows by induction. Also, since $x \underset{1}{\sim} x$ is always true, P2 is a direct consequence of P1 for every k.

P3 and P4 are proved simultaneously by induction. If $y \underset{1}{\sim} x$, then $x \underset{1}{\sim} y$ and $y^{-1} \underset{1}{\sim} x^{-1}$, so both are true when $k = 1$. Now assume P3 and P4 when $k = m$, letting $v \underset{m}{\sim} u$ and $y = u^{-1}xv$. Then $x = uyv^{-1}$, but $v^{-1} \underset{m}{\sim} u^{-1}$, so $x \underset{m+1}{\sim} y$. Also $y^{-1} = v^{-1}x^{-1}u$, but $u \underset{m}{\sim} v$, so $y^{-1} \underset{m+1}{\sim} x^{-1}$. Thus the results are true when $k = m+1$ and hence for every k.

If 1 is the identity element of G, and $x = a^{-1}b^{-1}ab$, then $1 = axb^{-1}a^{-1}b$, so $1 \underset{2}{\sim} x$. Conversely, if $1 = c^{-1}yd^{-1}cd$, then $y = cd^{-1}c^{-1}d$. Therefore 1 is 2-conjugate to an element if and only if the element is a commutator.

Theorem 1. *2-conjugacy is not always transitive.*

Proof. There exist groups in which the product of two commutators is not necessarily a commutator. Let $x = (a^{-1}b^{-1}ab)(c^{-1}d^{-1}cd)$ be such an element. Then $dxd^{-1} = d(a^{-1}b^{-1}ab)c^{-1}d^{-1}c$, so $dxd^{-1} \underset{2}{\sim} a^{-1}b^{-1}ab$. Now $1 \underset{2}{\sim} a^{-1}b^{-1}ab$, but 1 is not 2-conjugate to dxd^{-1} since the latter, like x, is not a commutator.

Theorem 2. *$y \sim_{k} x$ if and only if there exist $a_1, \ldots, a_{k+1}$ such that $x = a_1 \ldots a_{k+1}$ and $y = a_{k+1} \ldots a_1$.*

Proof. Let $x = a_1 \ldots a_{k+1}$ and $y = a_{k+1} \ldots a_1$. Since $a_2 a_1 = a_1^{-1}(a_1 a_2) a_1$, $y \sim_{1} x$ when $k = 1$. Assume that $y \sim_{m} x$ when $k = m$. Then

$$a_{m+2} \ldots a_1 = (a_1 \ldots a_{m+1})^{-1}(a_1 \ldots a_{m+2})(a_{m+1} \ldots a_1),$$

and $y \sim_{m+1} x$ when $k = m+1$. Therefore $y \sim_{k} x$ for every k.

Conversely, let $y \sim_{k} x$. When $k = 1$, $y = a^{-1}(xa)$ for some a, and $x = (xa)a^{-1}$. Now assume the result when $k = m$. Let $y \sim_{m+1} x$, or $y = u^{-1}xv$, $v \sim_{m} u$. By the inductive hypothesis, $u = b_1 \ldots b_{m+1}$, $v = b_{m+1} \ldots b_1$. Select b_{m+2} such that $x = (b_1 \ldots b_{m+1}) b_{m+2}$. Then $y = b_{m+2} \ldots b_1$, and the theorem is proved.

Let C be the commutator subgroup of G. It is known from (2) that C is the set of all elements expressible in the form $a^{-1} \ldots a_k^{-1} a_1 \ldots a_k$, for some k. This result may be strengthened by the following:

Theorem 3. (a) *Any product of k commutators is expressible in the form $a_1^{-1} \ldots a_{2k}^{-1} a_1 \ldots a_{2k}$.*

(b) *Conversely, the element $a_1^{-1} \ldots a_m^{-1} a_1 \ldots a_m$, in which $m = 2k$ or $2k+1$, may be written as a product of k commutators.*

Proof. (a) This is true when $k = 1$; assume it when $k = n$. Let $c_1, \ldots, c_n$, and $u^{-1}v^{-1}uv$ be commutators. Then

$$\begin{aligned} &u^{-1}v^{-1}uv(c_1 \ldots c_n) \\ &\quad = u^{-1}v^{-1}uv(a_1^{-1} \ldots a_{2n}^{-1} a_1 \ldots a_{2n}) \\ &\quad = (a_1 v^{-1})^{-1}(uva_1^{-1})^{-1}(a_1 v^{-1} u^{-1} v)^{-1} a_2^{-1} \ldots a_{2n}^{-1}(a_1 v^{-1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (uva_1^{-1})(a_1 v^{-1} u^{-1} v) a_2 \ldots a_{2n} \\ &\quad = b_1^{-1} \ldots b_{2n+2}^{-1} b_1 \ldots b_{2n+2}. \end{aligned}$$

(b) Since $a_1^{-1} a_2^{-1} a_1 a_2$ and

$$a_1^{-1} a_2^{-1} a_3^{-1} a_1 a_2 a_3 = (a_2 a_1)^{-1}(a_2 a_3)^{-1}(a_2 a_1)(a_2 a_3)$$

are both commutators, the statement is true when $m = 2$ and $m = 3$. Suppose it is true when $m = r$. Then

$$\begin{aligned} &a_1^{-1} \ldots a_{r+2}^{-1} a_1 \ldots a_{r+2} \\ &\quad = a_1^{-1} \ldots a_r^{-1} a_1 \ldots a_r (a_{r+1} a_1 \ldots a_r)^{-1}(a_{r+1} a_{r+2})^{-1}(a_{r+1} a_1 \ldots a_r)(a_{r+1} a_{r+2}). \end{aligned}$$

By the inductive hypothesis, the expression $a_1^{-1} \ldots a_r^{-1} a_1 \ldots a_r$ may be written as $\frac{1}{2}r$ or $\frac{1}{2}(r-1)$ commutators, according as r is even or odd. The remaining expressions constitute a single commutator, and the proof is complete.

Remark: Clearly, when m is odd, the product $a_1^{-1} \ldots a_m^{-1} a_1 \ldots a_m$ may be reduced to $b_1^{-1} \ldots b_{m-1}^{-1} b_1 \ldots b_{m-1}$.

Theorem 4. *$t \in C$ if and only if $t = y^{-1}x$ and $y \sim_k x$ for some k.*

Proof. If $t \in C$, then $t = a_1^{-1} \ldots a_{k+1}^{-1} a_1 \ldots a_{k+1}$ for some k (Theorem 3). Let $x = a_1 \ldots a_{k+1}$ and $y = a_{k+1} \ldots a_1$. By Theorem 2, $y \sim_k x$. Conversely, assume that $y \sim_k x$. Then $x = a_1 \ldots a_{k+1}$ and $y = a_{k+1} \ldots a_1$ (Theorem 2), and $y^{-1}x \in C$ (Theorem 3).

As a generalisation of the fact that $1 \sim_2 x$ if and only if x is a commutator, we obtain the following:

Theorem 5. *If $y \sim_k x$, then x equals y times $\frac{1}{2}k$ or $\frac{1}{2}(k+1)$ commutators according as k is even or odd. Conversely, if x equals y times m commutators, then $y \sim_{2m} x$.*

Proof. By Theorem 2, if $y \sim_k x$, then $y^{-1}x = a_1^{-1} \ldots a_{k+1}^{-1} a_1 \ldots a_{k+1}$, which may be written as a product of $\frac{1}{2}k$ or $\frac{1}{2}(k+1)$ commutators, according as k is even or odd (Theorem 3). This proves the first part.

Now suppose that $x = yc_1 \ldots c_m$, each c_i being a commutator. If $m = 1$, $x = ya^{-1}b^{-1}ab = (yay^{-1})^{-1}y(yb)^{-1}(yay^{-1})(yb) \sim_2 y$. Assume that $y \sim_{2r} x$ when $m = r$. If $m = r+1$, then $x = yc_1(c_2 \ldots c_{r+1})$, so $x \sim_{2r} yc_1$. Therefore $x = u^{-1}(yc_1)v$, in which $v \sim_{2r-1} u$. Then since $yc_1 \sim_2 y$, we have

$$x = u^{-1}(s^{-1}yt^{-1}st)v = (su)^{-1}yt^{-1}(su)(u^{-1}tv).$$

Now $u^{-1}tv \sim_{2r} t$, so $t^{-1}(su)(u^{-1}tv) \sim_{2r+1} su$, and $x \sim_{2r+2} y$.

A variation of this proof may be obtained by using an alternative definition of k-conjugacy; namely, that $y \sim_k x$ if there exist $a_1, \ldots, a_k$ such that

$$y = x(x^{-1}a_1^{-1}xa_1)(a_1^{-1}a_2^{-1}a_1a_2)\ldots(a_{k-1}^{-1}a_k^{-1}\, a_{k-1}a_k).$$

This definition may be shown without difficulty to be equivalent to the one we have used. The number of commutators may then be reduced by noting that a product $(a^{-1}b^{-1}ab)(b^{-1}c^{-1}bc)$ reduces to a single commutator [(**1**), p. 37, Ex. 11], for example, $(a^{-1}ba)^{-1}(a^{-1}c)^{-1}(a^{-1}ba)(a^{-1}c)$. This process may be reversed to prove the converse.

Theorem 6. *Let there be a fixed k such that $t \in C$ implies $t = a_1^{-1} \ldots a_k^{-1} a_1 \ldots a_k$. (By the remark after Theorem 3, we may assume k to be even.) Then k-conjugacy is an equivalence relation separating G into the cosets of C. Furthermore, each element of C is expressible as a product of $\frac{1}{2}k$ commutators.*

Proof. For any k, if $y \sim_k x$, then $y^{-1}x \in C$ (Theorem 5), so x and y are in the same coset of C. Conversely, assume $y^{-1}x \in C$. By hypothesis,

$$y^{-1}x = a_1^{-1} \ldots a_k^{-1} a_1 \ldots a_k$$

and therefore is a product of $\frac{1}{2}k$ commutators (Theorem 3). By Theorem 5, it follows that $y \sim_k x$, and this establishes k-conjugacy as an equivalence relation.

REFERENCES

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