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A SET OF PLANE MEASURE ZERO CONTAINING ALL FINITE POLYGONAL ARCS

D. J. WARD

1. Introduction. We say a (plane) set A contains all sets of some type $\mathscr{B}$ if, for each B of type $\mathscr{B}$, there is a subset of A that is congruent to B. Recently, Besicovitch and Rado [**3**] and independently, Kinney [**5**] have constructed sets of plane measure zero containing all circles. In these papers it is pointed out that the set of all similar rectangles, some sets of confocal conics and other such classes of sets can be contained in sets of plane measure zero, but all these generalizations rely in some way on the symmetry, or similarity of the sets within the given type.

In this paper we construct a set of plane measure zero containing all finite polygonal arcs (i.e., the one-dimensional boundaries of all polygons with a finite number of sides) with slightly stronger results if we restrict our attention to k-gons for some fixed k. One of our aims in this work was to throw more light on the conjecture that there is a set of plane measure zero containing all rectifiable curves. All that can be said, however, is that these constructions keep the conjecture alive. To go from polygonal arcs to rectifiable curves would require compactness of the containing set and this may well be too much to ask.

Notation and definitions. Given a plane set E, $U(E, \rho)$ is an enumerable set of convex sets u, each of diameter $du < \rho$ containing E. $\Lambda_\rho^s(E)$ is the lower bound of $\sum_{U(E,\rho)} (du)^s$, taken over all $U(E, \rho)$. The *Hausdorff s-dimensional measure* of E is given by

$$\Lambda^s(E) = \lim_{\rho \to 0} \Lambda_\rho^s(E).$$

Λ^s is an outer measure and all the sets we will deal with will be measurable with respect to this measure for all s. The *dimension* of a set E is that unique number t such that $\Lambda^s(E) = 0$ for $s > t$ and $\Lambda^s(E) = \infty$ for $s < t$. $\Lambda^t(E)$ may be zero, finite or infinite.

If we have some class $\mathscr{U}$ of point sets, u, the notation $E = \bigcup_{u \in \mathscr{U}} u$ will refer both to the set of points in at least one u and to the set whose members are elements of $\mathscr{U}$. Throughout this work we will assume that k is an integer greater than two.

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2. Construction of the set A. We require a set A with the following properties:

(P1) *A is a closed subset of the real line with* $\dim(A) = 0$*;*

(P2) *Given any k real numbers $\{a_i\}$, $1 \leqq i \leqq k$, there exists a real number t such that $t + a_i \in A$ for all i;*

(P3) *If $0 \leqq a_i \leqq 1$ for all i, then we may suppose A to be compact.*

For the existence and construction of A we appeal to [**4**], where this and many other related results are given. Indeed the authors of [**4**] remove the dependence of A on a particular integer k and, further, construct a bounded F_σ set containing all uniformly bounded sequences of real numbers. However we also need a set, of dimension less than one, satisfying conditions similar to (P2) and (P3) but having different density properties from the sets constructed in [**4**]. For this set, the set B of the next section, the dependence on k seems essential and so we prefer to show the integer k in the properties of the set A.

3. Construction of the set B. Let B be the set of all real numbers x, $0 \leqq x \leqq 1$, such that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{(4k+1)^n}, \qquad \text{where } 0 \leqq x_n \leqq 4k \quad \text{and} \quad x_n \neq 3k.$$

(B is "decimals" to base $4k + 1$ without the digit $3k$, excepting possibly as the final digit of a terminating "decimal".)

LEMMA 3.1. *Let $\{\alpha_i\}$, $0 \leqq \alpha_i \leqq 4k$, $i = 1, \ldots, k$, be k given integers. Then there is a non-negative integer $p \leqq 4k - 3$ such that $\alpha_i \neq p, p+1, p+2$ for all i.*

Proof. If the k numbers α_i can take any of the $4k + 1$ places $0 \leqq \alpha_i \leqq 4k$, then either there must be four adjacent places not containing an α_i or two sets of three adjacent places not containing an α_i. In other words, either

(i) there is a p_1 such that $\alpha_i \neq p_1, p_1 + 1, p_1 + 2$, or, $p_1 + 3$ where $p_1 + 3 \leqq 4k$

or

(ii) there is a p_2 and p_3 such that $\alpha_i \neq p_2, p_2 + 1, p_2 + 2$, where $p_2 + 4 < p_3 + 2 \leqq 4k$.

Either of p_1 and p_2 will satisfy the conditions of the lemma.

Definition 3.1. *We say that a, $0 < a < 1$, is of length n if its representation in base $4k + 1$ has non-zero digits in only its first n places. (Note that if a is of length n_1, it is also of length n_2 for $n_2 > n_1$.)*

LEMMA 3.2. *If $0 \leqq a_i \leqq \frac{1}{2}$, $0 \leqq i \leqq k$, and they are all of length n, then there is a $t > 0$, also of length n, such that $a_i + t \in B$ for all i and none of $a_i + t$, $i = 1, \ldots, k$, terminate in the digits $3k - 1$, $3k$, or $3k + 1$.*

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Proof. By induction. If $n = 1$, since $0 \leqq a_i \leqq \frac{1}{2}$,

$$a_i = \frac{\alpha_i}{4k+1}, \qquad 0 \leqq \alpha_i \leqq 2k, \quad \text{and} \quad t = \frac{1}{4k+1}$$

will satisfy the conditions of the lemma. Suppose that the lemma is true for some n. Suppose that $a_1, \ldots, a_k$ are given and of length $n + 1$. Write

$$a_i = \overline{a_i} + \frac{\alpha_i}{(4k+1)^{n+1}}, \qquad 0 \leqq \alpha_i \leqq 4k,$$

where $\overline{a_i}$ is of length n. By the hypothesis, there is a $\bar{t} > 0$ of length n such that $\overline{a_i} + \bar{t} \in B$ for all i and $\overline{a_i} + \bar{t}$ does not terminate in $3k - 1$, $3k$ or $3k + 1$. By the previous lemma, there is a $p \geqq 0$ such that $\alpha_i \neq p$, $p + 1$ or $p + 2$ and $p \leqq 4k - 3$. Set

$$t = \bar{t} + \frac{3k - (p+1)}{(4k+1)^{n+1}}.$$

First we note that, although $3k - (p + 1)$ may be negative, $\bar{t}$ is positive and of length n, and hence $t > 0$. Write

$$t + a_i = \bar{t} + \overline{a_i} + \frac{3k - (p+1) + \alpha_i}{(4k+1)^{n+1}}$$

$$= \bar{t} + \overline{a_i} + \frac{\mu_i}{(4k+1)^{n+1}}, \quad \text{say.}$$

There are three possibilities:

(a) $-4k \leqq \mu_i < 0$,

(b) $0 \leqq \mu_i \leqq 4k$,

(c) $\mu_i > 4k$.

Case (a): Since $p \leqq 4k - 3$ and $\alpha_i \geqq 0$,

$$p \neq 4k - 1 + \alpha_i,\ 4k + \alpha_i \text{ or } 4k + 1 + \alpha_i$$

or

$$4k + 1 + \mu_i \neq 3k - 1,\ 3k \text{ or } 3k + 1.$$

Case (b). $0 \leqq \mu_i \leqq 4k$. Then if $\mu_i = 3k - 1, 3k$ or $3k + 1$, then $\alpha_i = p, p + 1$ or $p + 2$, a contradiction.

Case (c). $\mu_i > 4k$. Since $\alpha_i \leqq 4k$ and $p \geqq 0$,

$$\alpha_i - p \neq 4k + 1,\ 4k + 2 \text{ or } 4k + 3,$$

i.e. $\mu_i - (4k + 1) \neq 3k - 1$, $3k$ or $3k + 1$. Thus in each case $t + a_i \in B$ and does not terminate in $3k - 1$, $3k$ or $3k + 1$.

LEMMA 3.3. *B is a compact nowhere dense set,*

$$\dim(B) = \frac{\log(4k)}{\log(4k+1)}$$

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and if $0 \leqq a_i \leqq \frac{1}{2}$, $i = 1, \ldots, k$, *then there is a* $t \geqq 0$ *such that* $t + a_i \in B$ *for all* i.

Proof. B can be contained in $(4k)^n$ intervals, each of length $(4k+1)^{-n}$, and so $\dim(B) = \beta$, where β is such that

$$\lim_{n\to\infty} \frac{(4k)^n}{(4k+1)^{n\beta}} = 1,$$

i.e. $\beta = \log(4k)/\log(4k+1)$. The complement of B is a set of open intervals no two of which have a common end point and so B is perfect. The previous lemma and the compactness of B yield the remaining requirements of this lemma.

Although the sets A and B are similar in a set-theoretic sense, their measure properties are different. B is, in quite a strong sense, uniformly distributed along the interval $[0, 1]$, whereas A is not. This property of B can best be expressed in terms of the density of linear sets as developed by Besicovitch in [**1**].

LEMMA 3.4. *The Besicovitch lower two-sided density of* B *is positive at all its points.*

Proof. Let $x \in B$ and consider

$$\frac{\Lambda^\beta(B \cap (x-\delta, x+\delta))}{\delta^\beta} \qquad \text{for varying } \delta.$$

This expression takes its smallest values when x is near the end point of some complementary interval of B, I say, and when δ is such that $(x - \frac{1}{2}\delta, x + \frac{1}{2}\delta)$ just contains the next largest complementary interval of B that is adjacent to I. This is when

$$\frac{\delta}{2} = \frac{k+1}{4k+1}|I| = \frac{k+1}{4k+1}\left(\frac{1}{4k+1}\right)^n,$$

for some n. Hence

$$\frac{\Lambda^\beta(B \cap (x - \frac{1}{2}\delta, x + \frac{1}{2}\delta))}{\delta^\beta} \geqq \frac{k}{4k+1}\left(\frac{1}{4k}\right)^n \left(\frac{2(k+1)}{(4k+1)^n} + 1\right)^{-\beta} \geqq \lambda > 0$$

for some λ and all sufficiently large n. Hence $\Lambda^\beta(B \cap (x - \frac{1}{2}\delta, x + \frac{1}{2}\delta)) \geqq \lambda\delta^{-\beta}$ for some λ and all $\delta \leqq \delta_0$ for some δ_0 independent of x. This uniform behaviour is rather more than we need for the lemma but is, in fact, the condition we need for the property (P5) following.

By multiplying by a scale factor of 2π, we can construct a compact linear set θ with the following properties.

(P4) $\theta \subset [0, 2\pi]$, $0 < \Lambda^\beta(\theta) < \infty$, $\beta = \log(4k)/\log(4k+1)$.

(P5) *There is a* $\lambda > 0$ *and a* $\delta_0 > 0$ *such that if* $\delta \leqq \delta_0$ *and* $\theta \in \theta$, *then*

$$\Lambda^\beta(\theta \cap (\theta - \tfrac{1}{2}\delta, \theta + \tfrac{1}{2}\delta)) \geqq \lambda\delta^\beta.$$

(P6) *Given* $0 \leqq \theta_i \leqq \pi$ $(i = 1, 2, \ldots, k)$, *there is a* $\psi \geqq 0$ *such that* $\psi + \theta_i \subset \theta$ *for all* i.

4. A set of dimension less than two that contains all polygons with k sides.

Definition 4.1. (a, r, θ) *is the line, of length* $2r$, *centre at the point* a *making an angle* θ *with some fixed base line in the Euclidean plane.* (a, θ) *is the infinite line containing* (a, r, θ).

Definition 4.2.

$$E^k(a, r) = \bigcup_{\theta \in \theta} (a, r, \theta), \qquad E^k(a) = \bigcup_{\theta \in \theta} (a, \theta).$$

Because θ *is compact, the complement of* $E^k(a)$ *is a pencil of open double cones, through* a.

This leads to the following.

Definition 4.3. *Let* $(a, \theta_0 - \gamma)$ *and* $(a, \theta_0 + \gamma)$ *define the boundary of the largest such cone. Let* $L = (a, \theta_0)$.
We construct A *on* L.

Definition 4.4. $E^k = \bigcup_{x \in A} E^k(x)$, *and if* I *is an interval on* L, *we define*

$$E^k(I, r) = \bigcup_{x \in A \cap I} E^k(x, r).$$

THEOREM 1. E^k *contains all polygons with* k *sides.*

Proof. We define the directions of the sides of a given k-gon as $\theta_1, \theta_2, \ldots, \theta_k$ (relative to some fixed base line) and rotate the k-gon until these sides are parallel to rays of each $E^k(x)$. By taking k sets, $E^k(x_i)$, $i = 1, \ldots, k$, we can duplicate the polygon in E^k with suitable choice of $\{x_i\}_{i=1}{}^k$. Such a choice is always possible by property (P2).

THEOREM 2. *There is a closed interval* I *and a positive* r *such that* $E^k(I, r)$ *contains all* k*-gons of diameter less than one.*

Proof. Let P be such a k-gon. Let P' be a k-gon in E^k that is congruent to P. It is clear that one vertex of P' can be assumed to lie on L. By Definition 4.3, if $r > \operatorname{cosec} \gamma$, then all points of $E^k - E^k(R, r)$ are a distance greater than one from L. (R is the real line.) Hence $P' \subset E^k(R, r)$. Similarly if $|b - a| > 2 \cot \gamma$, no line of $E^k(a)$ can meet a line of $E(b)$ at a distance less than one from L, and so an upper bound can be assigned to the distances between points of A that are required. This in turn ((P3)) gives us a bound on the part of A required. We now find a bound for the dimension of E^k and $E^k(I, r)$.

LEMMA 4.1. *There exists a positive number* k_1 *such that if* $\delta \leqq \delta_0$ (δ_0 *as in* (P5)), *then* θ *can be contained in less than* $k_1\delta^{-\beta}$ *intervals, each of length* δ.

Proof. Let $I_1 = (\delta_1 - \frac{1}{2}\delta, \theta_1 + \frac{1}{2}\delta)$, where θ_1 is some point in θ. Let I_2 be $(\theta_2 - \frac{1}{2}\delta, \theta_2 + \frac{1}{2}\delta)$, where $\theta_2 \in \theta$, but $\theta_2 \notin I_1$. Continuing this way be obtain a set of N intervals, $\{I_i\}, i = 1, \ldots, N$, each of length δ, with I_i not containing the midpoint of any other I_i, and such that

$$\theta \subset \bigcup_{i=1}^{N} I_i.$$

Hence $\frac{1}{2}N$ of the I_is are disjoint, hence (by (P5)),

$$\Lambda^\beta(\theta) \geqq \sum_{i=1}^{\frac{1}{2}N} \Lambda^\beta(\theta \cap I_i) \geqq \tfrac{1}{2}N\lambda\delta^\beta,$$

i.e. $N \leqq (2/\lambda)\Lambda^\beta(\theta)\delta^{-\beta} = k_1\delta^{-\beta}$.

LEMMA 4.2. *$E^k(x, r)$ can be contained in $k_1\delta^{-\beta}$ rectangles, each of length $2r$ and width $2r\delta$.*

Proof. $E^k(x, r)$ can be contained in $k_1\delta^{-\beta}$ double cones, angle δ, radius r (Lemma 4.1). We may certainly suppose that the width of each cone is no more than $2r\delta$.

LEMMA 4.3. *Let I_i be an interval on L, $d(I_i) = 2r\delta$ ($\delta \leqq \delta_0$ (P5)) Then $E^k(I_i, r)$ can be contained in $k_1\delta^{-\beta}$ rectangles each of length $2r$ and width $2r\delta + 2dI_i$.*

Proof. Let $x \in I_i \cap A$ (if $I_i \cap A$ is null, there is nothing to prove). Now each $(a, r, \theta) \in E^k(I_i, r)$ can be obtained from (x, r, θ) by a translation through a distance of at most dI_i. The lemma follows from Lemma 4.2.

LEMMA 4.4. *If $d(I_i) = 2r\delta$, then $E^k(I_i, r)$ can be contained in*

$$\frac{2rk_1}{3d(I_i)}\left(\frac{d(I_i)}{2r}\right)^{-\beta}$$

convex sets, each of diameter $\leqq 5d(I_i)$.

Proof. Divide each of the rectangles of Lemma 4.3 into square pieces.

THEOREM 3. $\dim(E^k) = \dim(E^k(I, r)) = 1 + \beta = 1 + \log(4k)/\log(4k + 1)$.

Proof. Let $t = 1 + \beta + \alpha$, where α is any positive number. We first show that $\Lambda^t(E^k(I, r)) = 0$. Consider $A \cap I$. Since $\Lambda^\alpha(A \cap I) = 0$ (P1), given any $\epsilon, \rho > 0$, we can find an open covering of $A \cap I$, $\{I_i\}$, $i = 1, \ldots, \infty$, say, such that $dI_i \leqq \rho$ and

$$\sum_{i=1}^{\infty} (d(I_i))^\alpha < \epsilon. \tag{4.1}$$

Put

$$\rho = 2r\delta_0 \qquad (\delta_0 \text{ as in (P5)}) \tag{4.2}$$

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and consider $E^k(I, r) = \bigcup_{i=1}^{\infty} E^k(I_i, r)$. Now, by (4.2), $d(I_i) \leqq 2r\delta_0$; thus *define* δ by the equation

$$d(I_i) = 2r\delta. \tag{4.3}$$

We can now apply Lemma 4.4 and let $E^k(I_i, r)$ be contained in

$$\frac{2rk_1}{3d(I_i)}\left(\frac{d(I_i)}{2r}\right)^{-\beta}$$

convex sets, each of diameter $\leqq 5d(I_i)$. Put $k_2 = \frac{1}{3}k_1(2r)^{\beta+1}$ and we have

$$\Lambda_{5\rho}{}^t(E^k(I_i, r)) \leqq k_2(d(I_i))^{-(1+\beta)}(5d(I_i))^t \leqq k_3(d(I_i))^{t-(1+\beta)} = k_3(d(I_i))^\alpha,$$

$$\Lambda_{5\rho}{}^t(E^k(I, r)) \leqq k_3 \sum_{i=1}^{\infty} (d(I_i))^\alpha < k_3\epsilon.$$

Hence, since ρ and ϵ are arbitrary positive numbers,

$$\Lambda^t(E^k(I, r)) = 0.$$

Thus $\dim(E^k(I, r)) \leqq 1 + \beta$ and since $\dim(E) = \dim(E^k(I, r))$, we have only to prove that $\dim(E^k(I, r)) \geqq 1 + \beta$. But $E^k \supset E^k(I, r) \supset E^k(a, r)$ for any point $a \in A \cap I$. Now results of Besicovitch and Moran [**2**] show that $\dim(E^k(a, r)) \geqq 1 + \beta$, which completes the proof.

THEOREM 4. *Let* $E = \bigcup_{k=1}^{\infty} E^k$. *Then* $\Lambda^2 E = 0$, $\dim(E) = 2$, *and* E *is an* F_σ *set containing all finite polygonal arcs.*

Proof. We have shown in Theorem 3 that $\dim(E^k) = 2 - \eta(k)$, where $\eta(k) > 0$, $\eta(k) \to 0$ as $k \to \infty$. Hence $\dim(E) = 2$ and $\Lambda^2(E) = 0$. Theorem 1 and the closure of each E^k complete the proof.

Added in proof. It has now been shown (by Roy O. Davies (*Some remarks on the Kakeya problem*, to appear) and independently by J. M. Marstrand) that all polygons can be *translated* into a set of plane measure zero: a substantial improvement on my Theorem 4. However, work by them indicates that results similar to Theorems 1, 2, and 3 are not possible if one allows only translations.

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The University of Toronto,
Toronto, Ontario;
The University of Sussex,
Brighton, England