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ANDRÉ–OORT CONJECTURE AND NONVANISHING OF CENTRAL L-VALUES OVER HILBERT CLASS FIELDS

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Abstract

Let $F/\mathbf{Q}$ be a totally real field and K/F a complex multiplication (CM) quadratic extension. Let f be a cuspidal Hilbert modular new form over F. Let λ be a Hecke character over K such that the Rankin–Selberg convolution f with the θ-series associated with λ is self-dual with root number 1. We consider the nonvanishing of the family of central-critical Rankin–Selberg L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$, as χ varies over the class group characters of K. Our approach is geometric, relying on the Zariski density of CM points in self-products of a Hilbert modular Shimura variety. We show that the number of class group characters χ such that $L(\frac{1}{2}, f \otimes \lambda\chi) \neq 0$ increases with the absolute value of the discriminant of K. We crucially rely on the André–Oort conjecture for arbitrary self-product of the Hilbert modular Shimura variety. In view of the recent results of Tsimerman, Yuan–Zhang and Andreatta–Goren–Howard–Pera, the results are now unconditional. We also consider a quaternionic version. Our approach is geometric, relying on the general theory of Shimura varieties and the geometric definition of nearly holomorphic modular forms. In particular, the approach avoids any use of a subconvex bound for the Rankin–Selberg L-values. The Waldspurger formula plays an underlying role.

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1. Introduction

For a self-dual family of L-functions with root number 1, the central-critical L-values are generically believed to be nontrivial. An L-value can often be expressed as a period of a modular form over an algebraic cycle in a

Shimura variety. The nontriviality is thus typically related to an appropriate density of such cycles.

An instructive setup arises from a self-dual Rankin–Selberg convolution of a cuspidal Hilbert modular new form and a Hecke character over a complex multiplication (CM) quadratic extension of the totally real field with root number 1. Twists of the Hecke character by the class group characters of the CM extension keep the root number unchanged and give rise to a family alluded to. The number of class group characters with the corresponding central-critical L-value being nonzero is expected to grow with the absolute value of the discriminant of the CM extension. The Waldspurger formula expresses the central-critical L-values as a twisted toric period of a nearly holomorphic modular form associated with the Jacquet–Langlands transfer of the Hilbert modular form to a well-chosen quaternionic Shimura variety. The twisted toric period is in fact a twisted finite sum of the evaluation of the nearly holomorphic form at CM points arising from the ideal classes in the class group of the CM extension. Based on the expression and the geometric definition of nearly holomorphic modular forms, we relate the nonvanishing to the Zariski density of CM points in a self-product of the quaternionic Shimura variety. The Brauer–Siegel lower bound on the size of the class groups also plays a key role. The CM points in consideration are the images of CM points arising from the ideal classes in the CM quadratic extensions by a skewed diagonal map. Based on the general theory of Shimura varieties, we show that the density follows from the André–Oort conjecture for the self-product. The Brauer–Siegel bounds on the size of the class groups again play a key role. Conditional on the density, our result on the nonvanishing states that the number of class group characters such that the central-critical L-values of the corresponding twists of the Rankin–Selberg convolution are nonzero increases with the absolute value of the discriminant of the CM extension. Here a twist of the Rankin–Selberg convolution refers to the convolution of the new form with a twist of the Hecke character by a class group character. In view of the recent progress on the André–Oort conjecture, the nonvanishing now holds unconditionally. For a prime p, the p-adic Bloch–Kato Selmer group is naturally associated with the convolution. The Bloch–Kato conjecture implies that the nonvanishing of the central-critical L-value is equivalent to the Selmer group having rank zero in an appropriate sense. Combined with a recent result on the Bloch–Kato conjecture for the convolution over the rationals, our result imply that as the number of class group characters such that the p-adic Bloch–Kato Selmer groups associated with twists of the convolution have rank zero increases with the absolute value of the discriminant of the imaginary quadratic extension. Here p is a prime unramified in the imaginary quadratic extensions.

In the introduction, for simplicity we mostly restrict to the Hilbert modular case.

Let $\iota_\infty : \overline{\mathbf{Q}} \to \mathbf{C}$ be an embedding. For a prime p, let $\iota_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$ be an embedding.

Let F be a totally real field of degree d and O the integer ring. Let $\widehat{O} = O \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$. Let I be the set of infinite places of F.

Let K/F be a CM quadratic extension and Σ a CM type. We often identify Σ with I. Let $\mathcal{K}$ be the set of CM quadratic extensions. Let c denote the complex conjugation on $\mathbf{C}$, which induces the unique nontrivial element of $\mathrm{Gal}(K/F)$ via the embedding ι_∞. Let D_K be the discriminant of $K/\mathbf{Q}$. Let Cl_K be the ideal class group of the CM field K and $\widehat{\mathrm{Cl}_K}$ the character group of Cl_K with values in $\mathbf{C}^\times$.

For an ideal $\mathfrak{a}$ of O, we fix a decomposition $\mathfrak{a} = \mathfrak{a}^+\mathfrak{a}^-$, where $\mathfrak{a}^+$ (respectively $\mathfrak{a}^-$) is divisible only by split (respectively ramified or inert) primes in the extension K/F.

For an ideal $\mathfrak{n} \subset O$, let

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\widehat{O}) | c \equiv 0 \bmod \mathfrak{n}\widehat{O} \right\}$$

and

$$\Gamma_1(\mathfrak{n}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(\mathfrak{n}) | d \equiv 1 \bmod \mathfrak{n}\widehat{O} \right\}.$$

Let $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ be the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let f be a cuspidal Hilbert modular new form over F with unitary central character ω. Let $k = \sum k_\sigma \sigma \in \mathbf{Z}_{>0}[\Sigma]$ be the weight and $\Gamma_0(\mathfrak{n})$ the level.

Let λ be a unitary Hecke character over K of infinity type $m \in \mathbf{Z}[\Sigma \cup \Sigma c]$ such that:

(C1) $\lambda_{|\mathbf{A}_F^\times} = \omega^{-1}$; and

(C2) $m = k + \kappa(1 - c)$ for some $\kappa \in \mathbf{Z}_{\geqslant 0}[\Sigma]$ independent of K.

In other words, we choose for each K a CM type Σ and a Hecke character λ satisfying (C1) and (C2). But k and κ regarded as elements of $\mathbf{Z}_{\geqslant 0}[I]$ are fixed independent of K. Condition (C1) implies independence of k_σ on $\sigma \in \Sigma$ (a parallel weight), as Hecke character ω of the totally real field can only have parallel weight k.

In this article, we later consider quaternionic cases for the sake of completeness.

For a place v of F, let $\epsilon_v(f \otimes \lambda)$ be the normalized local root number of the Rankin–Selberg convolution of f with the θ-series associated with λ. In what follows, we suppose that

(RN) $\epsilon_v(f \otimes \lambda) = 1$, for all $v | \mathfrak{n}^-$.

Let $L(s, f \otimes \lambda)$ be the automorphic L-function associated with the Rankin–Selberg convolution. In view of hypotheses (C1) and (RN), it follows that the Rankin–Selberg convolution $L(s, f \otimes \lambda)$ is self-dual with root number 1 (cf. [30, 35, 39]).

For $\chi \in \widehat{\mathrm{Cl}_K}$, the Rankin–Selberg convolution $L(s, f \otimes \lambda\chi)$ is again self-dual with root number 1. As the discriminant of the CM extension K becomes large, the Brauer–Siegel bound implies that the size of the class group Cl_K becomes large. We consider the nonvanishing of central-critical L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$ as $K \in \mathcal{K}$ varies.

Our result is the following.

THEOREM A. *Let f be a cuspidal Hilbert modular new form over a totally real field F of level $\Gamma_0(\mathfrak{n})$ and $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let Θ be an infinite set of CM quadratic extensions of the totally real field. For $K \in \Theta$, let λ be a Hecke character over K such that hypotheses (C1), (C2) and (RN) hold. Moreover, suppose that the André–Oort conjecture holds for any self-product of the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. Then, we have*

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : L(\tfrac{1}{2}, f \otimes \lambda\chi) \neq 0\}| = \infty.$$

By the work on the André–Oort conjecture prior to the aforementioned progress (cf. [21, 27, 28, 36]), we have the following version.

COROLLARY A. *Let f be a cuspidal Hilbert modular new form over a totally real field F of level $\Gamma_0(\mathfrak{n})$ and $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let Θ be an infinite set of CM quadratic extensions of the totally real field. For $K \in \Theta$, let λ be a Hecke character over K such that the hypotheses (C1), (C2) and (RN) hold. Moreover, suppose that either:*

(1) *the generalized Riemann hypothesis (GRH) holds for CM fields; or*

(2) $[F : \mathbf{Q}] \leqslant 6$.

Then, we have

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : L(\tfrac{1}{2}, f \otimes \lambda\chi) \neq 0\}| = \infty.$$

In view of the recent results of Andreatta *et al.*, Tsimerman and Yuan and Zhang (cf. [1, 34, 40]), the André–Oort conjecture now holds for Abelian type Shimura varieties. As an arbitrary self-product of a Hilbert modular Shimura variety is of Abelian type, Theorem A is now unconditional. The result of Tsimerman builds

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on a strategy of Pila–Zanier and previous work of Tsimerman–Pila. We refer to the introduction of [34] for an overview.

For the case of quaternionic Rankin–Selberg convolution, we refer to Section 4.

In view of the Bloch–Kato conjecture, the nonvanishing of a central-critical Rankin–Selberg L-value implies that the p-adic Bloch–Kato Selmer groups associated with the convolution have rank zero in an appropriate sense (cf. [2]). The conjecture has been recently proven over the rationals under mild hypotheses (cf. [5, Theorem A]). We refer to the references in [5] for the preceding results. Associated with the pair $(f, \lambda\chi)$ and a prime p, we have the corresponding p-adic Bloch–Kato Selmer group $\mathrm{Sel}(K, V_{f,\lambda\chi}/T_{f,\lambda\chi})$. Here $V_{f,\lambda\chi}$ is the p-adic Galois representation associated with the convolution and $T_{f,\lambda\chi} \subset V_{f,\lambda\chi}$ a lattice. We consider the rank of the Bloch–Kato Selmer groups $\mathrm{Sel}(K, V_{f,\lambda\chi}/T_{f,\lambda\chi})$ as $K \in \mathcal{K}$ varies.

Our result is the following.

COROLLARY B. *Let $p > 5$ be a prime, f a p-ordinary elliptic modular new form of weight $k \geqslant 2$ with $k \equiv 2 \bmod p-1$ and level $\Gamma_0(N)$ with N prime to p. Let L be the p-adic Hecke field of f, $\rho_f : G_{\mathbf{Q}} \to \mathrm{Aut}_L(V_f)$ the p-adic Galois representation associated with f and $\overline{\rho}_f$ the mod p reduction of ρ_f. Let Θ be an infinite set of imaginary quadratic extensions K with integer ring $\mathcal{O}_K$ and odd discriminant D_K satisfying the following hypotheses:*

(1) *The prime p splits in K and $p \nmid h_K$.*

(2) *There exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$.*

(3) *$\overline{\rho}_f|_{G_K}$ is absolutely irreducible.*

(4) *$\overline{\rho}_f$ is ramified at all primes dividing the greatest common divisor (D_K, N).*

Let λ be an unramified Hecke character over K as in Theorem A and $\widehat{\lambda}$ the p-adic avatar. Let $V_{f,\lambda} = V_f|_{G_K} \otimes \widehat{\lambda}$ and $\mathrm{Sel}(K, V_{f,\lambda})$ the corresponding Bloch–Kato Selmer group. Then, we have

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : \mathrm{rank}\, \mathrm{Sel}(K, V_{f,\lambda\chi}/T_{f,\lambda\chi}) = 0\}| = \infty.$$

For the definition of the Bloch–Kato Selmer group $\mathrm{Sel}(K, V_{f,\lambda}/T_{f,\lambda\chi})$, we refer to [5, Definition 5.1]. The existence of infinitely many imaginary quadratic extensions satisfying hypotheses (1)–(4) is well known, for example [29, Lemma 5.1]. We would like to emphasize that the corollary involves the use of an unconditional proof of the André–Oort conjecture for arbitrary self-products of the modular curve $X_1(N)$. The case is originally due to Pila (cf. [27]).

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We now describe the strategy of the proof. In view of the Waldspurger formula, the nonvanishing of the Rankin–Selberg L-values is equivalent to the nonvanishing of twisted toric periods of a nearly holomorphic Hilbert modular form associated with f. Here the torus arises from the CM quadratic extension K/F. An auxiliary argument based on the Brauer–Siegel lower bound on the size of the class groups then reduces the theorem to an Ax–Lindemann type functional independence for functions induced by a nearly holomorphic Hilbert modular form associated with f on the class group Cl_K. We recall that the Ax–Lindemann independence is with regard to an independence of a class of exponential functions. More precisely, it states that for finitely many linearly independent algebraic numbers over the rationals, the corresponding exponentials are algebraically independent over the rationals. Based on the geometric interpretation of nearly holomorphic Hilbert modular forms (cf. [37, Section 2.3]), the independence in our setup is essentially equivalent to the Zariski density of well-chosen CM points on a self-product of the Hilbert modular Shimura variety. Here we need to consider arbitrary self-products of the Hilbert modular Shimura variety. We formulate a conjecture regarding such a Zariski density (cf. Conjecture A). The conjecture is later shown to follow from the recent progress on the André–Oort conjecture. The CM points in consideration are the images of CM points arising from the ideal classes in the CM quadratic extensions by a skewed diagonal map and we consider the density as the CM quadratic extensions vary in the infinite subset Θ. It seems delicate to directly study the density. As indicated before, we show that it follows from the André–Oort conjecture for the self-product of the Hilbert modular Shimura variety. The proof is based on the general theory of Shimura varieties, in particular on the CM theory of Shimura–Taniyama–Weil and an explicit classification of a class of special subvarieties of the self-product. A key role is also played by the observation that an ideal class in a CM quadratic extension with a sufficiently large discriminant does not typically arise from an ideal class of a proper CM subfield of the CM extension. The observation is readily implied by the Brauer–Siegel bounds on the size of the class groups as the class number of a CM extension grows faster than the class number of its proper CM subfields. Strictly speaking, a slight variant of Conjecture A is used to prove Theorem A. The variant also follows from the André–Oort conjecture for the self-product and is thus unconditional.

The strategy seems to be suggestive of horizontal nonvanishing in other situations; for example, analogous nonvanishing of central L-values modulo p or that of central L-values in characteristic zero for automorphic forms on higher rank groups. In the former case, the nonvanishing is perhaps closely related to the analog of Conjecture A for mod p reduction of the CM points on the corresponding self-product. We recall that there is no analog of the

André–Oort conjecture for mod p Shimura varieties as of now. In the later case, the nonvanishing is perhaps closely related to the analog of Conjecture A for CM points on a relevant Shimura variety. The Shimura variety need not be of Abelian type. Even though the main results of the article are unconditional, we still keep the structure of the article conditional in appearance to evoke the possibilities of a mod p or a higher rank analog.

The nonvanishing over the family of twists by class group characters has been considered in various settings in the literature. Here we only mention [22–24] and refer to the references in them. The approach in these articles is perhaps more analytic/ergodic. Typically, the first step is to obtain a mean-value theorem for the Rankin–Selberg L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$ over $\chi \in \widehat{\mathrm{Cl}_K}$. The Waldspurger formula again plays a key role in practice. The equidistribution of CM points on the Hilbert modular Shimura variety then often implies that the mean value is positive, for $K \in \mathcal{K}$ with a large discriminant. Note that the mean-value result alone together with the positivity produces some nonvanishing twists but not a growing number of them. As is evident, the equidistribution is finer than analytic density and does not follow from the André–Oort conjecture. The next step is to invoke a subconvex bound for the Rankin–Selberg L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$. The previous step then allows to deduce a quantitative version of the nonvanishing, that is, a quantitative version of growing nonvanishing twists. We would like to emphasize that the approach in these articles involves the use of equidistribution of CM points only in a single copy of the Hilbert modular Shimura variety. For a related consideration, we also refer to [33]. Here we only mention that the results of [33] in particular improve the results of [23] and also recover the results of [22]. As is clear from the sketch, our approach is intrinsically geometric and avoids the use of equidistribution and a subconvex bound. It seems instructive to view the Zariski density in an arbitrary self-product of the Hilbert modular Shimura variety as a replacement of the subconvex bound. Our approach produces a growing number of nonvanishing twists. However, we are unable to deduce a quantitative version of the nonvanishing as of now. Our result is thus somewhere between what one can get from a pure mean-value result and a subconvexity result. Moreover, our approach treats the case of general number fields more easily than classical techniques in analytic number theory. As far as we know, the nonvanishing in the setting of the theorem is not considered in the literature. A detailed comparison of our approach with the analytic one will appear in a survey.

Chai and Oort found an application of the André–Oort conjecture to show the existence of many Abelian varieties not isogenous to Jacobians (cf. [7]). Pila recently found an application to modular Fermat equations (cf. [26]). Cornut and Cornut–Vatsal found an application of the conjecture to Mazur's conjecture on the nontriviality of Heegner points over anticyclotomic towers in Iwasawa theoretic situations (cf. [9–11]). As far as we know, the theorem is the first application of the André–Oort conjecture, which avoids any use of a subconvex bound to the nonvanishing of a class of L-values.

As is clear from the sketch, the perspective on the nonvanishing based on the André–Oort conjecture is likely to admit several variants. We hope to investigate them in the near future. The Gan–Gross–Prasad conjecture often relates nontriviality of a central L-value to the nontriviality of an appropriate period of an automorphic form. As mentioned before, the André–Oort conjecture is also now known for Abelian type Shimura varieties. It seems tempting that a naive analog of the Zariski density holds for the Hilbert modular Shimura variety modulo p (cf. remark (3) following Theorem 2.10). It would be worthwhile to explore the analog as it is closely related to the nonvanishing of L-values modulo p. Even in the case of modular curves, a conjecture along these lines does not seem to be considered in the literature.

A closely related instructive setup arises from a self-dual Rankin–Selberg convolution of a Hilbert modular new form of parallel weight two and a Hecke character over a CM quadratic extension of the totally real field with root number -1. Twists of the Hecke character by the class group characters of the CM extension keep the root number unchanged and give rise to a family alluded to. The Gross–Zagier formula and its generalizations due to Zhang express the central-critical derivative of the L-function as the Néron–Tate height of a relevant Heegner point (cf. [41]). In an ongoing joint work with Ye Tian, we plan to show that the number of class group characters such that the corresponding central-critical derivative is nonvanishing increases with the absolute value of the discriminant of the CM extension. Somewhat surprisingly, the approach again relies on an appropriate Zariski density of CM points on a self-product of a relevant Shimura curve. This seems to be analogous to the underlying principle being the same in [4, 17].

We note that our perspective on the nonvanishing follows a general principle outlined in the introduction of [18]. Our motivation partly came from [15, 23]. We refer to these articles for a general introduction.

The article is organized as follows. In Section 2, we consider the Zariski density of well-chosen generic CM points on self-products of the Hilbert modular Shimura variety. In Section 2.1, we recall basic setup regarding the Hilbert modular Shimura variety. In Section 2.2, we give an explicit description of a class of special subvarieties of the self-product. In Section 2.3, we consider CM points on the Hilbert modular Shimura variety. In Section 2.4, we describe the conjecture and results regarding the density. We also consider a slight variant. The variant plays an underlying role in the nonvanishing. We would like to emphasize that the conjecture and its variant are unconditional based on the recent progress on

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the André–Oort conjecture. In Section 3, we prove Theorem A. In Section 3.1, we prove nonvanishing of toric periods of a nearly holomorphic Hilbert modular form based on the Zariski density. In Section 3.2, the theorem is proven. In Section 4, we consider the case of quaternionic Rankin–Selberg convolution for the sake of completeness.

Notation. We use the following notation unless otherwise stated.

For sets S and T, let T^S denote the product of copies of T indexed by S.

For a number field L, let $\mathcal{O}_L$ be the integer ring. Let h_L be the class number, D_L the discriminant and R_L the regulator. Let $\mathbf{A}_L$ be the adele ring and $\mathbf{A}_{L,f}$ the finite adeles of L.

2. Zariski density of CM points

In this section, we consider the Zariski density of well-chosen generic CM points on a self-product of a Hilbert modular Shimura variety. In Section 2.1, we recall basic setup regarding the Hilbert modular Shimura variety. In Section 2.2, we give an explicit description of a class of special subvarieties of the self-product. In Section 2.3, we consider CM points on the Hilbert modular Shimura variety. In Section 2.4, we describe the conjecture and results regarding the density. We also consider a slight variant. We would like to emphasize that the conjecture and its variant are unconditional based on the recent progress on the André–Oort conjecture.

2.1. Hilbert modular Shimura variety. In this subsection, we recall basic setup regarding the Hilbert modular Shimura variety. We follow [16].

Let $G = \mathrm{Res}_{F/\mathbf{Q}}\, GL_2$ and $h_0 : \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\, \mathbb{G}_m \to G_{/\mathbf{R}}$ be the morphism of real group schemes arising from

$$a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a + bi \in \mathbf{C}^\times$. Let X be the set of $G(\mathbf{R})$-conjugacy classes of h_0. We have a canonical isomorphism $X \simeq (\mathbf{C} - \mathbf{R})^I$. The pair (G, X) satisfies Deligne's axioms for a Shimura variety. It gives rise to a tower $(\mathrm{Sh}_K = \mathrm{Sh}_K(G, X))_K$ of quasiprojective smooth varieties over $\mathbf{Q}$ indexed by open compact subgroups K of $G(\mathbf{A}_{\mathbf{Q},f})$. The complex points of these varieties are given as follows

$$\mathrm{Sh}_K(\mathbf{C}) = G(\mathbf{Q})\backslash X \times G(\mathbf{A}_{\mathbf{Q},f})/K. \tag{2.1}$$

From (2.1), it follows that the tower $(\mathrm{Sh}_{K/\mathbf{Q}})_K$ is endowed with an action of $G(\mathbf{A}_{\mathbf{Q},f})$ (cf. [16, Section 4.2]). This gives rise to the Hecke action.

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In what follows, we consider the case when K arises from the congruence subgroup $\Gamma_1(\mathfrak{n})$ for an ideal $\mathfrak{n} \subset O$ (cf. Section 1).

2.2. Special subvarieties. In this subsection, we give an explicit description of a class of special subvarieties of a self-product of the Hilbert modular Shimura variety.

Let S be a finite set. We recall the definition of a special subvariety of the self-product.

DEFINITION 2.1. A closed irreducible subvariety $Z \subset \mathrm{Sh}^S_{\Gamma_1(\mathfrak{n})}$ is said to be a special subvariety if there exists a morphism $\varphi : (H, Y) \to (G^S, X^S)$ of Shimura data and an element $g \in G^S(\mathbf{A}_{\mathbf{Q},f})$ such that Z is an irreducible component of the g-translate of the image of the Shimura variety associated with (H, Y) arising from φ.

The following proposition is well known (for example, [20, Section 1.1]).

PROPOSITION 2.2. *Let the notation and assumptions be as above. Then, the special subvarieties of the Hilbert modular variety* $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ *arise from* $\mathrm{Res}_{E/\mathbf{Q}} D^\times$ *for a subfield E of F and a quaternion algebra D over E with an embedding into* $M_2(F)$.

Proof. Let $x \in \mathrm{Sh}_{\Gamma_1(\mathfrak{n})}(\mathbf{C})$ be a closed point. Then, the minimal Shimura subvariety containing x arises from the Mumford–Tate group of x. In view of the classification of the Mumford–Tate groups, the proposition follows. □

We now consider the case of arbitrary self-product. For $s \in S$, let π_s be the projection to the s-component of the self-product $\mathrm{Sh}^S_{\Gamma_1(\mathfrak{n})}$. Following [12, Section 2], an explicit description of a class of special subvarieties of the self-product is given as follows.

PROPOSITION 2.3. *Let* $Z \subset \mathrm{Sh}^S_{\Gamma_1(\mathfrak{n})}$ *be a special subvariety with dominant projections* π_s *onto an irreducible component of* $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ *for all* $s \in S$. *Then, S has a partition* $(S_1, \ldots, S_r)$ *such that Z is a product of subvarieties Z_i of* $X_i = \mathrm{Sh}^{S_i}_{\Gamma_1(\mathfrak{n})}$, *which are the image of*

$$X \to (\Gamma_1(\mathfrak{n}))\backslash X)^{S_i}$$

under the map

$$\tau \mapsto ([g_\sigma(\tau)])_{\sigma \in S_i}.$$

Here $g_\sigma \in G(\mathbf{Q})$ *for* $\sigma \in S_i$.

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Proof. Suppose that Z arises from a couple (H, Y) as in Definition 2.1. Recall that H is a reductive subgroup of G^S. Without loss of generality, we suppose that H is connected and $|S| \geqslant 2$ (cf. Proposition 2.2). For $s, t \in S$, let p_s (respectively $p_{s,t}$) be the projection to the s-component (respectively (s, t)-component) of the self-product G^S.

In view of the hypothesis on π_s, it follows that

$$p_s H = G.$$

Suppose that there exists a pair (s, t) with $s \neq t$ such that

$$p_{s,t} H \neq G^2.$$

Recall that Goursat's lemma states that the subgroups of a product $A \times B$ are the inverse images of the graphs of isomorphisms from subquotients of A to subquotients of B. In our setup, the lemma accordingly implies that

$$p_{s,t} H = G.$$

Here we regard G being embedded inside G^2 via the map $x \mapsto (g_s x g_s^{-1}, g_t x g_t^{-1})$ for some $g_s, g_t \in G(\mathbf{Q})$. It follows that $\pi_{s,t} Z$ is a special subvariety of Sh^2 of the form described in the proposition.

Now, suppose that for all pairs (s, t) with $s \neq t$

$$p_{s,t} H = G^2.$$

From Goursat's lemma and induction on the number of elements in S, it follows that

$$H = G^S.$$

In view of the above consideration, the induction on the number of elements in S now finishes the proof. $\square$

Remark. Let $W = \varprojlim_\Gamma W_\Gamma$ be a proper special subvariety of $\varprojlim_\Gamma \mathrm{Sh}^2_\Gamma$ with W_Γ satisfying the hypothesis in the proposition. Here the projective limit is over the congruence subgroups of $\Gamma_1(\mathfrak{n})$ for an ideal $\mathfrak{n} \subset O$. The proposition implies that W is a correspondence of bounded degree to the left and the right. For our applications, this consequence is crucial. When $F \neq \mathbf{Q}$, here is an alternate argument. Since the universal cover of W_Γ is X, we have $W_\Gamma = X/\Delta_\Gamma$ for an arithmetic subgroup Δ_Γ of $GL_2(F_\infty)$ (which may not be a congruence subgroup). Here F_∞ denotes the infinite part of the adele ring $\mathbf{A}_F$ over F. Moreover, the projection Δ_L (respectively Δ_R) to the left (respectively right) factor of Δ_Γ is a subgroup of finite index of $\Gamma_1(\mathfrak{n})$. Adding $\widehat{\cdot}$ (respectively $\bar{\cdot}$), we denote the

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full profinite completion (respectively congruence subgroup completion) of the discrete subgroups of $GL_2(F_\infty)$. If $\widehat{\Delta}_L$ and $\widehat{\Delta}_R$ are subgroups of $\overline{\Gamma_1(\mathfrak{n})}$, after taking the projective limit, it follows that W is a graph of a morphism of $\varprojlim_\Gamma \mathrm{Sh}_\Gamma$. When $F \neq \mathbf{Q}$, Serre has shown that the congruence kernel $C_{\Gamma_1(\mathfrak{n})}$ is finite (cf. [31]). Thus, W is a correspondence of bounded degree to the left and the right. An analogous argument applies in the case of an arbitrary self-product.

2.3. CM points. In this subsection, we consider generalities regarding CM points on the Hilbert modular Shimura variety.

Let the notation and assumptions be as in Section 1. In particular, K/F is a CM quadratic extension and Σ a CM type. Let $N_{K/F}$ be the norm map

$$N_{K/F} : \mathrm{Cl}_K \to \mathrm{Cl}_F^+$$

induced by the norm map of ideal groups. Here Cl_F^+ denotes the strict class group of F.

For an ideal class $[\mathfrak{a}] \in \mathrm{Cl}_K$, let $x(\mathfrak{a})$ be the corresponding CM point on the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ (cf. [17, Section 4.4]). The underlying Abelian variety is of the CM type (K, Σ) for the fixed CM type Σ. Group theoretically, they arise from the embedding of the torus associated with the extension K/F into G (cf. (2.1)).

Based on Shimura's global reciprocity law, we have the following useful proposition.

PROPOSITION 2.4. *Let the notation and assumptions be as above. For K with a sufficiently large discriminant, each irreducible component of the Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ contains precisely $|\ker(N_{K/F})|$-many CM points arising from the ideal classes in Cl_K.*

Proof. The Brauer–Siegel lower bound implies that the class number of the CM quadratic extensions of a fixed totally real field grows with the discriminant (cf. Theorem 2.8). As the CM extensions K/F are of fixed degree, from class field theory it is readily seen that the norm map $N_{K/F}$ is surjective for K/F with a sufficiently large discriminant. In fact, the discriminant being nontrivial suffices for the surjectivity in the case of CM quadratic extensions. In what follows, we only consider CM quadratic extensions K/F with a surjective norm map. Let $e_K \in \mathrm{Cl}_K$ be an ideal class mapping to the identity class in Cl_F^+ under the map of the connected components of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ to Cl_F^+.

We now recall that the geometrically irreducible components of the finite level Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ are indexed by the strict class group Cl_F^+ (cf. [17, (3.6)]).

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In terms of the moduli interpretation of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$, the indexing corresponds to a choice of a polarization class on an underlying Abelian variety with real multiplication.

By the construction, the CM point $x(e_K)$ lies on the neutral component of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. Let $[\mathfrak{a}]$ and $[\mathfrak{b}]$ be a pair of ideal classes in Cl_K. In view of Shimura's local and global reciprocity law (cf. [17, Proposition 3.2] and [16, Theorem 4.14]), the CM points $x(\mathfrak{a})$ and $x(\mathfrak{b})$ lie on the same component of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ as long as $N_{K/F}(\mathfrak{a})$ and $N_{K/F}(\mathfrak{b})$ are in the same ideal class in Cl_F^+. It follows that the CM points arising from the ideal classes in $e_K \ker(N_{K/F})$ lie on the neutral component of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$.

Moreover, once an irreducible component contains the CM point arising from an ideal class $\mathfrak{b}$ then the component precisely contains the CM points arising from the ideal classes in the coset $\mathfrak{b}\ker(N_{K/F})$. The surjectivity of the norm map thus finishes the proof. □

We consider the following notion.

DEFINITION 2.5. A CM point on the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ is said to be generic if does not arise from a CM point on a proper positive dimensional special subvariety of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$.

We further consider the following notion.

DEFINITION 2.6. A CM quadratic extension K/F is said to be generic if it is not of the form MF, where M/E is a CM quadratic extension and E a proper subfield of F.

Note that there are infinitely many generic CM extensions of the totally real field F.

We have the following proposition regarding the determination of generic CM points.

PROPOSITION 2.7. (1) *The tori associated with generic CM extensions of a totally real field give rise to generic CM points on the corresponding Hilbert modular Shimura variety.*

(2) *Let E be a proper subfield of F and M/E a CM quadratic extension. If a CM point arises from the torus associated with the CM extension MF such that the corresponding CM type is not an inflation from a CM type of M, then the CM point does not arise from a special subvariety.*

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Proof. The first part immediately follows from Proposition 2.2. The second part follows from the CM theory of Shimura–Taniyama–Weil (cf. [32]). □

Remark. For M with a sufficiently large discriminant, typically an ideal class of the CM extension MF is not the inflation of an ideal class of M (cf. [proof, Theorem 2.8]). Thus, there always exists a generic CM point arising from a nongeneric CM quadratic extension with a sufficiently large discriminant. We still use the above terminology for simplicity.

2.4. Zariski density. In this subsection, we consider the Zariski density of well-chosen CM points in self-products of the Hilbert modular Shimura variety. The density is shown to be an unconditional result based on the recent progress on the André–Oort conjecture.

We begin with the following.

THEOREM 2.8 (Brauer–Siegel). *Let F be a totally real field and Ξ an infinite set consisting of CM quadratic extensions of F. Then,*

$$\lim_{K\in\Xi} h_K = \infty.$$

Proof. For $\epsilon > 0$, the Brauer–Siegel lower bound states that

$$h_K R_K \gg |D_K|^{1/2-\epsilon} \tag{2.2}$$

(cf. [3]). An elementary argument shows that

$$R_K = 2^s \cdot R_F \tag{2.3}$$

for an integer s with $d-1 \leqslant s \leqslant d$ (cf. [25, Proposition 3.7]).

In view of (2.2), this finishes the proof. □

Let the notation and assumptions be as in Section 2.3.

Let n be a positive integer. Let d_n be a positive integer such that for any CM quadratic extension K/F with $|D_K| \geqslant d_n$, we have

$$\ker(N_{K/F}) \geqslant n. \tag{2.4}$$

The existence follows from Theorem 2.8.

In what follows, we also suppose that the discriminant is large enough so that Proposition 2.4 holds.

Let Θ be an infinite subset of CM extensions K/F as above and $e_K \in \mathrm{Cl}_K$ as in the proof of Proposition 2.4. We now choose n distinct ideal classes $[\mathfrak{a}_i]_{1\leqslant i\leqslant n} \in e_K \ker(N_{K/F})$ such that $[\mathfrak{a}_1] = e_K$ and the following hypothesis holds.

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(N) The class $[\mathfrak{a}_i\mathfrak{a}_j^{-1}]$ cannot be represented by an integral ideal of bounded norm for all distinct i and j with $1 \leqslant i, j \leqslant n$ as $K \in \Theta$ varies.

Recall that the number of integral ideals with norm at most a given integer is bounded by a constant only dependent on the integer and the degree of the extension. The existence of the ideal classes satisfying (N) thus follows from Theorem 2.8. We also recall that the subset Θ is actually an infinite set of K and its CM type by our choice.

Let

$$\Xi_n = \{(x(\mathfrak{a}), \ldots, x(\mathfrak{a}\mathfrak{a}_n)) | [\mathfrak{a}] \in \mathrm{Cl}_K, K \in \Theta\} \tag{2.5}$$

be the subset of CM points in the n-fold self-product $\mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})/\overline{\mathbf{Q}}}$. In particular, the CM points are well-chosen skewed diagonal images of CM points arising from the ideal classes in the CM quadratic extensions. We recall that the CM points $x(\mathfrak{a})$ are of CM type (K, Σ) for the fixed CM type Σ.

The following is an analog of the mixing conjecture [**23**, Conjecture 2] for the Zariski topology.

CONJECTURE A. *Let the notation and assumptions be as above. Then, the subset Ξ_n of CM points is Zariski dense in the n-fold self-product* $\mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})}$.

As we will shortly see, the case when n equals one follows from the André–Oort conjecture for the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. The case is in fact known unconditionally. The following is an immediate consequence of the equidistribution result [**38**, Theorem 7.1].

THEOREM 2.9 (Venkatesh). *Let the notation and other assumptions be as above. Let Θ be an infinite set of CM quadratic extensions of the totally real field F. Then, the subset $\{x(\mathfrak{a}) | [\mathfrak{a}] \in \mathrm{Cl}_K, K \in \Theta\}$ of CM points is Zariski dense in the Hilbert modular Shimura variety* $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$.

Conditional on a subconvex bound, the case also follows from the equidistribution results in [**8**, **9**, **42**].

For the case of arbitrary self-products, we have the following result. We also refer to the remark following Proposition 2.12.

THEOREM 2.10. *Let the notation and assumptions be as above. Moreover, suppose that the André–Oort conjecture holds for the Shimura variety* $\mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})}$. *Then, the subset Ξ_n of CM points is Zariski dense in the Shimura variety* $\mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})}$.

Proof. We first recall that the André–Oort conjecture for the Shimura variety $\mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})}$ implies the conjecture for the Shimura variety $\mathrm{Sh}^k_{\Gamma_1(\mathfrak{n})}$ for $1 \leqslant k \leqslant n$. We proceed via induction on n.

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The case when n equals one follows from the André–Oort conjecture for $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ as follows. Let $X \subset \mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ be the Zariski closure of Ξ_1. In view of Proposition 2.4 and Theorem 2.8, each irreducible component of X contains infinitely many CM points. Let Y be an irreducible component of X. The Andre–Oort conjecture implies that Y is a special subvariety of the Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. In view of Proposition 2.4, it suffices to show that the subvariety is nothing but a Hecke translate of an irreducible component of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. Suppose that the subvariety is proper. It thus arises from a quaternion algebra over a proper subfield E of F (cf. Proposition 2.2). In particular, the CM extensions in consideration are not generic (cf. part (1) of Proposition 2.7). In view of the definition of the CM points and the subset Ξ_1, it suffices to show that

$$\liminf_{M} \frac{h_{MF}}{h_M} = \infty. \tag{2.6}$$

Here M varies over CM quadratic extensions of E. Indeed, $|\ker(N_{M/E})|$-many (respectively $|\ker(N_{M/F})|$-many) CM points arise from the ideal classes in M (respectively MF) on the special subvariety (respectively Y) (cf. Proposition 2.4). For M with a sufficiently large discriminant, typically an ideal class of the CM extension MF is thus not the inflation of an ideal class of M. Accordingly (2.6) would imply that the CM points on Y corresponding to the ideal classes eventually do not arise from the special subvariety (cf. part (2) of Proposition 2.7).

This follows from the Brauer–Siegel bounds. For $\epsilon > 0$, the Brauer–Siegel upper bound states that

$$|D_M|^{1/2+\epsilon} \gg h_M R_M$$

and the lower bound states that

$$h_{MF} R_{MF} \gg |D_{MF}|^{1/2-\epsilon}.$$

As

$$|D_{MF}| = N_{M/\mathbf{Q}}(\Delta_{MF/M})|D_M|^{[MF:M]},$$

the bounds and (2.3) readily imply (2.6). Here $\Delta_{MF/M}$ is the relative discriminant ideal of the extension MF/M.

We now suppose that $n \geqslant 2$.

Let $X \subset \mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})}$ again denote Zariski closure of the CM points Ξ_n. Let I be an irreducible component of X. It evidently contains an infinite subset T_n of Ξ_n. The André–Oort conjecture implies that I is a special subvariety of the self-product $\mathrm{Sh}^n_{\Gamma_1(\mathfrak{n})}$. In view of Proposition 2.3, we have an explicit list of the possibilities for I. Suppose that the subvariety is a proper subvariety corresponding to a partition, that is, partition in the proposition is nontrivial. In particular, there exist $s \neq t$ and i such that

$$\pi_{s,t} T_n \subset Z_i$$

with $|S_i| = 2$. The description of Z_i implies that the sth and tth components of I are isogenous by an isogeny of a fixed degree. Here isogeny refers to a finite morphism of bounded degree. For $[\mathfrak{a}] \in \mathrm{Cl}_K$ with $K \in \Theta$, the CM points $x(\mathfrak{a}\mathfrak{a}_s)$ and $x(\mathfrak{a}\mathfrak{a}_t)$ are isogenous by construction. The corresponding isogeny degree is however not independent of K (cf. (N)).

The contradiction finishes the proof. □

Remark. (1) The proof shows that the André–Oort conjecture implies the Zariski density of a thin subset $\widetilde{\Xi}_n$ of the neutral component of $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ given by

$$\widetilde{\Xi}_n = \{(x(e_K), \ldots, x(\mathfrak{a}_n)) | K \in \Theta\}$$

and we only consider generic CM extensions. For $n = 1$ and $F \neq \mathbf{Q}$, the density does not seem to follow from the equidistribution result in [38].

(2) In view of the proof, the theorem can also be proven based on the remark following Proposition 2.3. We also note that the use of the Brauer–Siegel upper bound can be avoided in the argument when Θ consists of infinitely many generic CM extensions.

(3) For a prime p, it seems tempting to conjecture an analog of Conjecture A for the Hilbert modular Shimura variety of modulo p. Even for $n = 1$, the analog seems to be open in general. A naive mod p analog of the André–Oort conjecture is the Chai–Oort rigidity principle that a Hecke stable subvariety of a mod p Shimura variety is a Shimura subvariety (cf. [6]). The principle does not directly imply Conjecture A as the set Ξ_n of CM points is not Hecke stable.

(4) There seems to be much room to consider variants of Conjecture A. For example, the proof also shows that conditional on the André–Oort conjecture an analog of Conjecture A holds for a broad class of Shimura varieties. We plan to consider the analog in the near future. As indicated, a slight variant is considered shortly.

We have an immediate application of the Zariski density for functions on Ξ_1 induced by Hilbert modular functions.

Let us first introduce some notation. Let $\mathcal{F}$ be the $\overline{\mathbf{Q}}$-algebra of functions on Ξ_1 with values in $\mathbb{P}^1(\overline{\mathbf{Q}})$.

Let

$$\phi : \mathcal{O}_{\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}} \rightarrow \mathcal{F}$$

be the morphism sending f to an element in $\mathcal{F}$ given by

$$x(\mathfrak{a}) \mapsto f(x(\mathfrak{a})).$$

Here $f \in \mathcal{O}_{\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}}$ and $\mathfrak{a} \in \mathrm{Cl}_K$ for $K \in \Theta$.

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COROLLARY 2.11. *Let the notation and assumptions be as above. Moreover, suppose that Conjecture A holds for the case $n = 1$. Then, we have an embedding of the algebra $\overline{\mathbf{Q}}(\mathrm{Sh}_{\Gamma_1(\mathfrak{n})})$ of the rational functions on $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ into $\mathcal{F}$.*

Here $\overline{\mathbf{Q}}(\mathrm{Sh}_{\Gamma_1(\mathfrak{n})})$ is the algebra of $\overline{\mathbf{Q}}$-rational functions on the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$.

For later application, we now consider a variant of the Zariski density. Let

$$\Xi'_n = \{(x(\mathfrak{b}_i\mathfrak{c}_j))_{1\leqslant i,j\leqslant n} | [\mathfrak{b}_i], [\mathfrak{c}_j] \in \mathrm{Cl}_K, K \in \Theta\} \tag{2.7}$$

be the subset of CM points in the n^2-fold self-product $\mathrm{Sh}^{n^2}_{\Gamma_1(\mathfrak{n})/\overline{\mathbf{Q}}}$. As the notation might be misleading, we would like to emphasize that $[\mathfrak{b}_i]$ and $[\mathfrak{c}_j]$ are a set of representatives for Cl_K. Our variant of Conjecture A is the following.

PROPOSITION 2.12. *Let the notation and assumptions be as above. Moreover, suppose that the André–Oort conjecture holds for the Shimura variety $\mathrm{Sh}^{n^2}_{\Gamma_1(\mathfrak{n})}$. Then, the subset Ξ'_n of CM points is Zariski dense in the Shimura variety $\mathrm{Sh}^{n^2}_{\Gamma_1(\mathfrak{n})}$.*

Proof. The argument is very similar to the proof of Theorem 2.10.

For $n = 1$, the density is nothing but Theorem 2.10 for $n = 1$. We now suppose that $n \geqslant 2$. Let $X' \subset \mathrm{Sh}^{n^2}_{\Gamma_1(\mathfrak{n})}$ denote the Zariski closure of the CM points Ξ'_n. In view of Theorem 2.10 for $n = 2$, the projection of X' to any two factors of $\mathrm{Sh}^{n^2}_{\Gamma_1(\mathfrak{n})}$ is dominant.

Let I' be an irreducible component of X'. It evidently contains an infinite subset T'_n of Ξ'_n. The André–Oort conjecture implies that I' is a special subvariety of the self-product $\mathrm{Sh}^{n^2}_{\Gamma_1(\mathfrak{n})}$. In view of Proposition 2.3, we have an explicit list of the possibilities for I'. Suppose that the subvariety is proper corresponding, that is, partition in the proposition is nontrivial. In particular, there exist $s \neq t$ and m such that

$$\pi_{s,t} T'_n \subset Z_m$$

with $|S_m| = 2$. This contradicts the dominance of $\pi_{s,t}$ and finishes the proof. □

Remark. In view of the recent results of Andreatta *et al.*, Tsimerman and Yuan and Zhang (cf. [**1**, **34**, **40**]), the André–Oort conjecture is now proven for Abelian type Shimura varieties. In particular, Theorem 2.10 and Proposition 2.12 are unconditional. As indicated in Section 1, we kept the structure of the section conditional in appearance to evoke the possibilities of an analog in other situations.

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The proposition plays a key role in establishing an Ax–Lindemann type functional independence implicit in the proof of Theorem 3.2. The functional independence underlies the nonvanishing of the Rankin–Selberg L-values (cf. Theorem A).

3. Nonvanishing of Rankin–Selberg L-values I

In this section, we prove the nonvanishing of a class of Rankin–Selberg L-values over the Hilbert class fields (cf. Theorem A). In Section 3.1, we consider the nonvanishing of toric periods of a nearly holomorphic Hilbert modular form over varying CM quadratic extensions of a fixed totally real field. In Section 3.2, we prove Theorem A. We would like to emphasize that the results are unconditional.

3.1. Nonvanishing of toric periods. In this subsection, we consider the nonvanishing of toric periods of a nearly holomorphic Hilbert modular form over varying CM quadratic extensions of the totally real field.

Let the notation and hypotheses be as in Section 1. In particular, $F/\mathbf{Q}$ is a totally real field and K/F a CM quadratic extension with CM type Σ. Moreover, k and κ are fixed elements of $\mathbf{Z}_{>0}[\Sigma]$ and ω a fixed unitary character over F. For an ideal $\mathfrak{n} \subset O$, let $S_k(\Gamma_0(\mathfrak{n}), \omega)$ denote the space of Hilbert modular forms of weight k, level $\Gamma_0(\mathfrak{n})$ and central character ω.

Let $g \in S_k(\Gamma_0(\mathfrak{n}), \omega)$ be a nonconstant and classical Hilbert modular form defined over a number field. Let $\kappa \in \mathbf{Z}_{\geqslant 0}[\Sigma]$ and $h = d^\kappa g$ a nearly holomorphic Hilbert modular form of weight $k + 2\kappa$. Here d is the Maass–Shimura differential operator. For the geometric definition of classical (respectively nearly holomorphic) Hilbert modular forms, we refer to [**16**, Section 4.2] (respectively [**13**, **14**, **37**, Section 2.2]).

Let λ be a Hecke character over K of infinity type $k + \kappa(1 - c)$ satisfying (C1) and (C2) (cf. Section 1).

We first note that

$$[\mathfrak{a}] \mapsto h(x(\mathfrak{a}))\lambda(\mathfrak{a})$$

is a well-defined function on Cl_K.

For $\chi \in \widehat{\mathrm{Cl}_K}$, let $P_{h,\lambda}(\chi)$ be the toric period given by

$$P_{h,\lambda}(\chi) = \frac{1}{|\mathrm{Cl}_K|} \cdot \sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \chi([\mathfrak{a}]) h(x(\mathfrak{a}))\lambda(\mathfrak{a}). \tag{3.1}$$

For a fixed nearly holomorphic form h, we consider the nonvanishing of the toric periods as K varies over an infinite subset of CM quadratic extensions of the totally real field.

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LEMMA 3.1. *Let h be a nearly holomorphic Hilbert modular form over a totally real field F of level $\Gamma_0(\mathfrak{n})$ as above and $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let Θ be an infinite set of CM quadratic extensions of the totally real field. For $K \in \Theta$, let λ be a Hecke character over K as above. Suppose that the André–Oort conjecture holds for the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. Then, for all but finitely many $K \in \Theta$, there exists $\chi \in \widehat{\mathrm{Cl}_K}$ such that the toric period $P_{h,\lambda}(\chi)$ is nonzero.*

Proof. In view of the Fourier inversion, it suffices to show that the above function

$$[\mathfrak{a}] \mapsto h(x(\mathfrak{a}))\lambda(\mathfrak{a})$$

is not identically zero on Cl_K for all but finitely many $K \in \Theta$. This is precisely Corollary 2.11. Strictly speaking, the corollary directly only applies to the Hilbert modular functions. In view of the sheaf theoretic definition of nearly holomorphic modular forms under Zariski topology (cf. [13, 14, 37, Section 2.2]) and division by another nearly holomorphic form of the same weight, the assertion readily follows from the case of modular functions. □

Remark. In view of Theorem 2.9 or the remark following Proposition 2.12, the lemma holds unconditionally.

We in fact have the nonvanishing of many toric periods.

THEOREM 3.2. *Let h be a nonconstant nearly holomorphic Hilbert modular form over a totally real field F of level $\Gamma_0(\mathfrak{n})$ as above and $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let Θ be an infinite set of CM quadratic extensions of the totally real field. For $K \in \Theta$, let λ be a Hecke character over K as above. Suppose that the André–Oort conjecture holds for arbitrary self-products of the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. Then, we have*

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : P_{h,\lambda}(\chi) \neq 0\}| = \infty.$$

Proof. Suppose that there exists an integer l such that exactly $l-1$ of the toric periods

$$\sum_{[\mathfrak{a}] \in \mathrm{Cl}_K} \chi([\mathfrak{a}]) h(x(\mathfrak{a}))\lambda(\mathfrak{a}) \tag{3.2}$$

are nonzero, for all $K \in \Theta$ with a sufficiently large discriminant.

For $K \in \Theta$, let $[\mathfrak{b}_1], \ldots, [\mathfrak{b}_l] \in \mathrm{Cl}_K$ be l-ideal classes. In view of the assumption (3.2), it follows that the functions given by

$$[\mathfrak{a}] \mapsto h(x(\mathfrak{a}\mathfrak{b}_i))\lambda(\mathfrak{a}\mathfrak{b}_i)$$

viewed as elements in the vector space of maps $\mathrm{Cl}_K \to \overline{\mathbf{Q}}$ are linearly dependent for $1 \leqslant i \leqslant l$. Say,

$$\sum_{i=1}^{l} c_{K,i} h(x(\mathfrak{a}\mathfrak{b}_i))\lambda(\mathfrak{a}\mathfrak{b}_i) = 0 \tag{3.3}$$

for some $c_{K,i} \in \overline{\mathbf{Q}}$ and any $[\mathfrak{a}] \in \mathrm{Cl}_K$.

We now choose l-ideal classes $[\mathfrak{c}_1], \ldots, [\mathfrak{c}_l] \in \mathrm{Cl}_K$ and consider (3.3) for $[\mathfrak{a}] = [\mathfrak{c}_1], \ldots, [\mathfrak{c}_l]$. It follows that

$$\det(h(x(\mathfrak{c}_j\mathfrak{b}_i))\lambda(\mathfrak{c}_j\mathfrak{b}_i))_{1\leqslant i,j\leqslant l} = 0. \tag{3.4}$$

Note that

$$\begin{aligned}(h(x(\mathfrak{c}_j\mathfrak{b}_i))\lambda(\mathfrak{c}_j\mathfrak{b}_i))_{1\leqslant i,j\leqslant l}\\ = \mathrm{diag}(\lambda(\mathfrak{c}_j)))_{1\leqslant j\leqslant l}(h(x(\mathfrak{c}_j\mathfrak{b}_i)))_{1\leqslant i,j\leqslant l}\mathrm{diag}(\lambda(\mathfrak{b}_i)))_{1\leqslant i\leqslant l}.\end{aligned} \tag{3.5}$$

Here $\mathrm{diag}(a_k)_{1\leqslant k\leqslant l}$ denotes the diagonal matrix with diagonal entries $\{a_1, \ldots, a_l\}$. We conclude that

$$\det(h(x(\mathfrak{c}_j\mathfrak{b}_i))))_{1\leqslant i,j\leqslant l} = 0. \tag{3.6}$$

We now consider the function h_l on the self-product Sh^{l^2} given by

$$(x_{i,j})_{1\leqslant i,j\leqslant l} \mapsto \det(h(x_{i,j}))_{1\leqslant i,j\leqslant l}. \tag{3.7}$$

The lower triangular entries of the above matrix can be all arranged to be zero. Moreover, the product of the diagonal entries can be arranged to be nonconstant simultaneously as h is nonconstant. It follows that h_l is a nonconstant function.

In view of (3.6), the function h_l vanishes on the collection Ξ_l' of CM points. This contradicts the density in Proposition 2.12 and finishes the proof. □

Remark. (1) The above argument is based on a suggestion of the referee. Our previous consideration was directly based on the functional independence and the Zariski density in Conjecture A. The referee pointed out an error and instead suggested considering a formulation in terms of determinants.

(2) In view of the remark following Proposition 2.12, the theorem holds unconditionally.

3.2. Nonvanishing of Rankin–Selberg *L*-values I. In this subsection, we prove the nonvanishing of a class of Rankin–Selberg L-values over the Hilbert class fields of CM quadratic extensions of a fixed totally real field.

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Let the notation and hypotheses be as in Section 1. In particular, $f \in S_k(\Gamma_0(\mathfrak{n}), \omega)$ is a Hecke eigen new form. Recall that λ is a Hecke character over K of infinity type $k + \kappa(1 - c)$ for some $\kappa \in \mathbf{Z}_{\geqslant 0}[\Sigma]$. We also recall that hypotheses (C1) and (RN) imply that the Rankin–Selberg convolution $L(s, f \otimes \lambda)$ is self-dual with root number 1.

We have the following result regarding the nontriviality of central-critical L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$ for $\chi \in \widehat{\mathrm{Cl}_K}$.

THEOREM 3.3. *Let f be a cuspidal Hilbert modular new form over a totally real field F of level $\Gamma_0(\mathfrak{n})$ and $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$ the Hilbert modular Shimura variety of level $\Gamma_1(\mathfrak{n})$. Let Θ be an infinite set of CM quadratic extensions of the totally real field. For $K \in \Theta$, let λ be a Hecke character over K such that hypotheses (C1), (C2) and (RN) hold. Suppose that the André–Oort conjecture holds for arbitrary self-products of the Hilbert modular Shimura variety $\mathrm{Sh}_{\Gamma_1(\mathfrak{n})}$. Then, we have*

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : L(\tfrac{1}{2}, f \otimes \lambda\chi) \neq 0\}| = \infty.$$

Proof. In view of our hypotheses and the Waldspurger formula ([30, 35, 39] and [19, Proposition 2.1]), it follows that

$$L(\tfrac{1}{2}, f \otimes \lambda\chi) \neq 0 \iff P_{h,\lambda}(\chi) \neq 0.$$

Here h is a Gross–Prasad test vector/toric form associated with the pair (f, λ). For an infinite subset Θ' of Θ, the toric form turns out to be the nearly holomorphic form $d^\kappa f$ in view of our hypotheses.

The theorem thus readily follows from Theorem 3.2. □

Remark. In view of the second remark following Theorem 3.2, the above theorem holds unconditionally.

4. Nonvanishing of Rankin–Selberg L-values II

In this section, we consider the nonvanishing of quaternionic Rankin–Selberg L-values for the sake of completeness.

Recall that F is a totally real field and I the set of infinite places. Moreover, $\mathcal{K}$ denotes the set of CM quadratic extensions of F.

Let B be an indefinite quaternion algebra over F. Let I^B be the subset of I consisting of infinite ramification places of B. Let $I = I^B \cup I_B$. For $l \in \mathbf{Z}[I]$, let $l_B \in \mathbf{Z}[I_B]$ (respectively I^B) be the projection to the I_B-components (respectively I^B-components).

Let $\mathcal{K}_B$ be the subset of $\mathcal{K}$ consisting of CM quadratic extensions K/F such that there exists an embedding

$$\iota_K : K \hookrightarrow B. \tag{4.1}$$

Let Sh_B be the Shimura variety associated with B of a fixed level.

Let f_B be a new form over Sh_B with a unitary central character ω. Let $k = \sum k_\sigma \sigma \in \mathbf{Z}_{>0}[I_B]$ be the weight.

Let $K \in \mathcal{K}_B$ and Σ be a CM type of K/F. Let λ be a unitary Hecke character over K of infinity type m such that:

(Q1) $\lambda_{|\mathbf{A}_F^\times} = \omega^{-1}$; and

(Q2) $m_B = k + \kappa(1-c)$ for some $\kappa \in \mathbf{Z}_{\geqslant 0}[I_B]$ independent of K.

In other words, we choose for each K a CM type Σ and a Hecke character λ satisfying (Q1) and (Q2). But k and κ regarded as elements of $\mathbf{Z}_{\geqslant 0}[I]$ are fixed independent of K. Again condition (Q1) implies independence of k_σ on $\sigma \in \Sigma$ (a parallel weight), once we write $m = k + \kappa(1-c)$ extending κ by 0 outside I_B (and denote it by the same symbol κ).

Let $L(s, f_B \otimes \lambda)$ be the automorphic L-function associated with the Rankin–Selberg convolution. In view of hypothesis (Q1), it follows that the Rankin–Selberg convolution $L(s, f_B \otimes \lambda)$ is self-dual. In what follows, we suppose that

(QRN) the root number of the self-dual Rankin–Selberg convolution $L(s, f_B \otimes \lambda)$ equals 1.

For $\chi \in \widehat{\mathrm{Cl}_K}$, the Rankin–Selberg convolution $L(s, f \otimes \lambda\chi)$ is again self-dual with root number 1. As the discriminant of the CM extension K becomes large, the Brauer–Siegel bound implies that the size of the class group Cl_K becomes large (cf. Theorem 2.8). As $K \in \mathcal{K}_B$ varies, we consider the nonvanishing of the central-critical L-values $L(\frac{1}{2}, f \otimes \lambda\chi)$.

Our result is the following.

THEOREM 4.1. *Let B be an indefinite quaternion algebra over a totally real field F and Sh_B the corresponding Shimura variety of a fixed level. Let $\mathcal{K}_B$ be the set of CM quadratic extensions of F as above and Θ an infinite subset of $\mathcal{K}_B$. Let f_B be a new form over Sh_B. For $K \in \Theta$, let λ be a Hecke character over K such that hypotheses (Q1), (Q2) and (QRN) hold. Moreover, suppose that the André–Oort conjecture holds for any self-product of the Shimura variety Sh_B. Then, we have*

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : L(\tfrac{1}{2}, f_B \otimes \lambda\chi) \neq 0\}| = \infty.$$

By the work on the André–Oort conjecture prior to the aforementioned progress (cf. [21, 27, 28, 36]), we have the following version.

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COROLLARY 4.2. *Let B be an indefinite quaternion algebra over a totally real field F and Sh_B the corresponding Shimura variety of a fixed level. Let $\mathcal{K}_B$ be the set of CM quadratic extensions of F as above and Θ an infinite subset of $\mathcal{K}_B$. Let f_B be a new form over Sh_B. For $K \in \Theta$, let λ be a Hecke character over K such that hypotheses (Q1), (Q2) and (QRN) hold. Moreover, suppose that either:*

(1) *GRH holds for CM fields; or*

(2) $[F : \mathbf{Q}] \leqslant 6$.

Then, we have

$$\liminf_{K \in \Theta} |\{\chi \in \widehat{\mathrm{Cl}_K} : L(\tfrac{1}{2}, f_B \otimes \lambda\chi) \neq 0\}| = \infty.$$

In view of the recent progress on the André–Oort conjecture, Theorem 4.1 is unconditional in general. Here an additional point is to use the well-known comparison between Sh_B and an appropriate unitary Shimura variety.

Note that Theorem 4.1 allows the weight I^B-component of the weight of λ to be less than the weight of the corresponding component of the Jacquet–Langlands transfer of f_B to G. In particular, the theorem is more general than Theorem A.

The proof is similar to the proof of Theorem A. The details and generalization to another class of L-values will appear in another article.

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