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ON THE IRREDUCIBILITY OF A CLASS OF EULER FROBENIUS POLYNOMIALS

BY
A. SHARMA AND E. G. STRAUS*

In [1, 2] the sequence of polynomials

(1) $\Pi_{n,r}(x)$

$$=\begin{vmatrix} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} & 1-x & 0 & \cdots & 0 \\ 1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & 1-x \quad 0 & \cdots & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & 0 \\ 1 & \binom{n-r}{1} & & \cdots & & & \binom{n-r}{n-r-1} & 1-x \\ 1 & \binom{n-r+1}{1} & & \cdots & & & & \binom{n-r+1}{n-r} \\ & \cdot & & & & & & \\ & \cdot & & & & & & \\ & \cdot & & & & & & \\ 1 & \binom{n}{1} & & \cdots & & & & \binom{n}{n-r} \end{vmatrix}$$

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which we shall call Euler-Frobenius polynomials were considered and it was conjectured that these polynomials are irreducible in $Q[x]$ for all odd values of n. Since $\Pi_{n,r}(x)$ is a monic reciprocal polynomial and $\deg \Pi_{n,r}(x)=n-2r+1$ it is clear that for even values of n one of zeros must be $(-1)^r$ and thus $\Pi_{n,r}(x)$ must have a factor of first degree, $x+(-1)^{r+1}$. Since all roots of $\Pi_{n,r}(x)$ have sign $(-1)^r$ and all roots are simple it follows that there can be only one integral zero so that for even n we get

$$\Pi_{n,r}(x) = (x+(-1)^{r+1})\Pi^*_{n,r}(x)$$

where $\Pi^*_{n,r}(x)$ is a reciprocal monic polynomial without rational roots and it might be reasonable to conjecture that $\Pi^*_{n,r}(x)$ is also irreducible.

Eisenstein's criterion. *The polynomial*

$$P(x) = a_0x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n$$

with integral a_i is irreducible in $Q[x]$ if there exists a prime p so that

$$a_0 \not\equiv 0 \pmod p), \qquad a_1 \equiv a_2 \equiv \cdots \equiv a_{n-1} \equiv a_n \equiv 0 \pmod p) \tag{2}$$

$$a_n \not\equiv 0 \pmod{p^2}. \tag{3}$$

With its help we can prove the two cited conjectures in a number of cases.

We first set $y=1-x$ and use the recursion relation for binomial coefficients to transform the last $r-1$ rows in (1)

$$\Pi_{n,r}(x) = P_{n,r}(y) \tag{4}$$

$$= \begin{vmatrix} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} & y & 0 & \cdots & 0 \\ \cdot & & & & & & \cdot & \cdot \\ \cdot & & & & & & \cdot & \cdot \\ \cdot & & & & & & \cdot & \cdot \\ 1 & \binom{n-r}{1} & \cdots & & & & \binom{n-r}{n-r-1} & y \\ 1 & \binom{n-r+1}{1} & & & \cdots & & & \binom{n-r+1}{n-r} \\ 0 & 1 & & & \cdots & & & \binom{n-r+1}{n-r-1} \\ \cdot & \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & \cdot & & & & & \cdot \\ 0 & 0 & 1 & \binom{n-r+1}{1} & & & \cdots & \binom{n-r+1}{n-2r+1} \end{vmatrix}$$

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Thus if $n-r+1=p$, a prime, then

(5)
$$P_{n,r}(y) \equiv \left|\begin{array}{ccc|ccc} & & & y & & 0 \\ & & & & \ddots & \\ & & & & & y \\ \hline 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \end{array}\right|$$
$$\equiv \pm y^{n-2r+1} \pmod{p},$$

so that condition (2) of Eisenstein's criterion is satisfied. To check condition (3) we set $y=0$ in (5) and factor out a factor p from last column. The terms in the last column are

$$\binom{p}{s} = \frac{p}{s}\,\frac{(p-1)\cdots(p-s+1)}{(s-1)\cdots 1} \equiv (-1)^{s-1}\frac{p}{s} \pmod{p^2}$$

with $s=1, 2, \ldots, r$. Thus

$$\frac{1}{p}P_{n,r}(0) \equiv \left|\begin{array}{cccc|cccc|c} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} & 0 & & \cdots & & 0 \\ 1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & 0 & \cdots & & 0 \\ \vdots & & & & & & & & \\ 1 & \binom{p-1}{1} & & \cdots & \cdot & & & \binom{p-1}{p-2} & 0 \\ \hline 1 & & & 0 & & & & & 1 \\ & \ddots & & & & & & & -\frac{1}{2} \\ & & \ddots & & & 0 & & & \frac{1}{3} \\ & & & \ddots & & & & & \vdots \\ 0 & & & 1 & & & & & \frac{(-1)^{r-1}}{r} \end{array}\right|$$

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$$\equiv \left|\begin{array}{cccc|ccc} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} & 0 & \cdots & \\ & & & & \binom{r+1}{r} & & \\ 1 & & & & & & \binom{p-1}{p-2} \\ \hline \frac{1}{2} & 1 & & 0 & & & \\ -\frac{1}{3} & & \cdot & & & & \\ \cdot & & & \cdot & & 0 & \\ \cdot & & & \cdot & & & \\ \frac{(-1)^r}{r} & & & 1 & & & \end{array}\right|$$

$$\equiv \pm \frac{(p-1)!}{r!}\left|\begin{array}{c|ccc} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} \\ \hline \frac{1}{2} & 1 & & 0 \\ -\frac{1}{3} & & \cdot & \\ \cdot & & & \cdot \\ \cdot & & & \\ \frac{(-1)^r}{r} & 0 & & 1 \end{array}\right|$$

$$\equiv \pm\frac{1}{r!}\left(1-\frac{1}{2}\binom{r}{1}+\frac{1}{3}\binom{r}{2}+\cdots+(-1)^{r+1}\frac{1}{r}\binom{r}{r-1}\right) \pmod p$$

Now

$$1-\frac{1}{2}\binom{r}{1}+\cdots+(-1)^{r+1}\frac{1}{r}\binom{r}{r-1} = \int_0^1((1-x)^r-(-x)^r)\,dx$$

$$= \frac{1}{r+1}(1+(-1)^{r+1})$$

so that

$$\frac{1}{p}P_{n,r}(0) \equiv \pm\frac{2}{r+1} \not\equiv 0 \pmod p$$

for odd r, that is for even degree $p-r$. We have thus proved:

6. THEOREM. *If r is an odd integer and p a prime greater than r then $\Pi_{p+r-1,r}(x)$ is irreducible over $Q[x]$.*

If r is even and $n=p+r-1$ then

$$\Pi_{n,r}(1) = P_{n,r}(0) = 0$$

and

$$\Pi_{n,r}(x) = (1-x)\Pi^*_{n,r}(x) = yP^*_{n,r}(y)$$

Obviously $P^*_{n,r}(y) \equiv \pm y^{n-2r} (\text{mod } p)$ so that condition (2) of Eisenstein's Criterion is satisfied. In order to verify condition (3) we check the coefficient of y in $P_{n,r}(y)$.

We get this term by setting all but one of the y's in (4) equal to 0 and summing the $p-r$ determinants obtained in this manner. All terms, except those in which the y is in the $(1, r+1)$ or in the $(p-r, p)$ position, are $\equiv 0(\text{mod } p^2)$ by the same argument as that showing that $P_{n,r}(0) \equiv 0(\text{mod } p^2)$. Thus

$$\frac{1}{p}P'_{n,r}(0) \equiv \left|\begin{array}{cccc|ccc|c}
1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-1} & 0 & \cdots & & 0 \\
1 & \binom{r+2}{1} & \cdots & \binom{r+2}{r-1} & \binom{r+2}{r+1} & & & \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
1 & \binom{p-1}{1} & & & & & \binom{p-1}{p-2} & 0 \\
\hline
1 & & & 0 & & & & 1 \\
& \cdot & & & & & & -\frac{1}{2} \\
& & \cdot & & & 0 & & \cdot \\
& & & \cdot & & & & \cdot \\
0 & & & 1 & & & & \dfrac{(-1)^{r+1}}{r}
\end{array}\right|$$

$$+\left|\begin{array}{ccc|ccc|c}
1 & \binom{r}{1} & \binom{r}{r-1} & 0 & \cdots & \cdot & 0 \\
\cdot & & & & \cdot & & \cdot \\
\cdot & & & \cdot & & \cdot & \cdot \\
\cdot & & & & \cdot & \cdot & \cdot \\
1 & \binom{p-2}{1} & & & & \binom{p-2}{p-3} & 0 \\
\hline
1 & & 0 & & & & -\frac{1}{2} \\
& \cdot & & & & & \frac{1}{3} \\
& & \cdot & & 0 & & \cdot \\
& & & & & & \cdot \\
0 & & 1 & & & & \dfrac{(-1)^r}{r+1}
\end{array}\right| (\text{mod } p)$$

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Hence

$$\frac{1}{p}P'_{n,r}(0) \equiv \frac{(p-1)!}{(r+1)!}\begin{vmatrix} 1 & \binom{r+1}{1} & \cdots & \binom{r+1}{r-1} \\ \frac{1}{2} & 1 & & 0 \\ -\frac{1}{3} & & \cdot & \\ \cdot & & & \\ \cdot & & \cdot & \\ \cdot & & & \\ \frac{(-1)^r}{r} & 0 & \cdot & 1 \end{vmatrix}$$

$$+\frac{1}{2}\frac{(p-2)!}{r!}\begin{vmatrix} 1 & \binom{r}{1} & \cdots & \binom{r}{r-1} \\ \frac{2}{3} & 1 & & 0 \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & & \\ \frac{(-1)^r 2}{r+1} & 0 & \cdot & 1 \end{vmatrix}$$

$$\equiv -\frac{1}{(r+1)!}\left(1-\frac{1}{2}\binom{r+1}{1}+\cdots+\frac{(-1)^{r+1}}{r}\binom{r+1}{r-1}\right)$$

$$-\frac{1}{r!}\left(\frac{1}{2}-\frac{1}{3}\binom{r}{1}+\cdots+\frac{(-1)^{r+1}}{r+1}\binom{r}{r-1}\right)$$

$$= -\frac{1}{(r+1)!}\int_0^1((1-x)^{r+1}-(r+1)x^r+x^{r+1})\,dx$$

$$-\frac{1}{r!}\int_0^1(x(1-x)^r-x^{r+1})\,dx$$

$$= -\frac{1}{(r+1)!}\left(\frac{2}{r+2}-1\right)-\frac{1}{r!}\left(\frac{1}{(r+1)(r+2)}-\frac{1}{r+2}\right)$$

$$= \frac{2r}{(r+2)!} \not\equiv 0 \pmod{p}$$

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Thus

7. THEOREM. *If r is an even integer and p a prime greater than r, then*

$$\Pi_{p+r-1,r}(x) = (1-x)\Pi^*_{p+r-1,r}(x)$$

where $\Pi^*_{p+r-1,r}(x)$ is irreducible over $Q[x]$.

Another application of Eisenstein's criterion yields

8. THEOREM. *If p is an odd prime then $\Pi_{3p-2,p}(x)$ is irreducible in $Q[x]$.*

Proof. According to (4) we have

$$\Pi_{3p-2,p}(x) = \begin{vmatrix} 1 & \binom{p}{1} & \cdots & \binom{p}{p-1} & 1-x & 0 & \cdots & 0 \\ \vdots & & & & & \ddots & \ddots & \vdots \\ & & & & & & \ddots & 0 \\ 1 & \binom{2p-2}{1} & & & \cdots & & & 1-x \\ 1 & \binom{2p-1}{1} & & & \cdots & & & \binom{2p-1}{p-2} \\ 1 & \binom{2p}{1} & \cdots & & \binom{2p}{p} & & \cdots & \binom{2p}{p-2} \\ 0 & & & & & & & \\ \vdots & \ddots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \binom{2p}{1} & & \cdots & \binom{2p}{p} \end{vmatrix}$$

$$\equiv \begin{vmatrix} 1 & 0 & \cdots & 0 & 1-x & & 0 & \cdots & & 0 \\ & & & & 1 & & \ddots & & & \\ \vdots & \ddots & & \vdots & \vdots & & & & & \vdots \\ & & & & & & & & & 0 \\ 1 & & & 1 & 0 & 1 & & \cdots & 1 & 1-x \\ 1 & -1 & \cdots & -1 & 1 & 1 & -1 & \cdots & 1 & -1 \\ 1 & & \cdots & & 0 & 2 & & \cdots & & \\ \vdots & \ddots & 0 & \vdots & \vdots & \vdots & & 0 & & \vdots \\ & 0 & \ddots & & & & & & \ddots & \vdots \\ & \cdots & & 1 & 0 & & & \cdots & & 2 \end{vmatrix} \pmod{p}$$

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If we subtract 2 times the i-th column from the $(p+i)$-th column, $i=1, 2, \ldots, p-1$ we get

$$\Pi_{3p-2,p}(x) \equiv \left|\begin{array}{cccc|c|cccc} 1 & 0 & \cdots & 0 & 0 & -1-x & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & & & & \ddots & \vdots \\ & & \cdot & 0 & & & & \cdot & 0 \\ 1 & & \cdots & 1 & 0 & \cdot & \cdot & \cdot & -1-x \\ \hline 1 & -1 & \cdots & -1 & 1 & -1 & 1 & \cdots & 1 \\ \hline 1 & & & & 0 & & & & \\ & \ddots & 0 & & \vdots & & & & \\ & & \ddots & & \vdots & & 0 & & \\ & 0 & & \ddots & \vdots & & & & \\ & & & 1 & 0 & & & & \end{array}\right|$$

$$\equiv (1+x)^{p-1} \pmod{p}.$$

Thus condition (2) of Eisenstein's criterion is satisfied if we set $\Pi_{3p-2,p}(x)=P(z)$ where $z=1+x$. In order to check condition (3) we have to evaluate $P(0)=\Pi_{3p-2,p}(-1) \pmod{p^2}$. Now

$$(9) \qquad \Pi_{3p-2,p}(-1) = \left|\begin{array}{cccccccc} 1 & \binom{p}{1} & \cdots & \binom{p}{p-1} & 2 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 1 & \binom{2p}{1} & & \cdots & \binom{2p}{p} & & \cdots & \binom{2p}{2p-2} \end{array}\right|$$

so that by subtracting the first row from the $(p+1)$st row and using the fact that

$$\binom{2p}{i}-\binom{p}{i} = \frac{2p}{i}\binom{2p-1}{i-1}-\frac{p}{i}\binom{p-1}{i-1}$$

$$\equiv (-1)^{i-1}\frac{p}{i} \pmod{p^2}; \qquad i=1, 2, \ldots, p-1$$

$$\binom{2p}{p}-2 = 2\left(\binom{2p-1}{p-1}-1\right) = 2\,\frac{(1+p)(2+p)\cdots(p-1+p)-(p-1)!}{(p-1)!}$$

$$\equiv 2p\left(1+\frac{1}{2}+\ldots+\frac{1}{p-1}\right) \equiv 0 \pmod{p^2}$$

$$\binom{2p}{j} = \frac{2p}{j}\binom{2p-1}{j-1} \equiv (-1)^{j-1}\frac{2p}{j} \pmod{p^2}$$

$$j=p+1, \ldots, 2p-2;$$

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we get from (9)

$$
(10)\quad \Pi_{3p-2,p}(-1) \equiv p
\left|\begin{array}{cccc|c|cccc}
1 & & & 0 & 0 & 2 & & & 0 \\
 & \cdot & & & \cdot & & \cdot & & \\
 & & \cdot & & \cdot & & & \cdot & \\
 & & & \cdot & \cdot & & & & \cdot \\
 & & & 1 & 0 & & & & 2 \\
\hline
1 & -1 & \cdots & -1 & 1 & 1 & & \cdots & -1 \\
\hline
0 & 1 & \cdots & \dfrac{1}{p-2} & \dfrac{-1}{p-1} & 0 & 2 & \cdots & \dfrac{2}{p-2} \\
\hline
0 & 1 & & 0 & 0 & 0 & 2 & & 0 \\
\cdot & & \cdot & & \cdot & \cdot & & \cdot & \\
\cdot & & & \cdot & \cdot & \cdot & & & \cdot \\
\cdot & & & \cdot & \cdot & \cdot & & & \cdot \\
0 & 0 & & 1 & 0 & 0 & & & 2
\end{array}\right| \pmod{p^2}.
$$

If we subtract twice the i-th column from the $(p+i)$th column $i=1, 2, \ldots, p-1$ and then expand according to the 1st, $(p+2)$nd, $\ldots$, last rows we get

$$
\frac{1}{p}\Pi_{3p-2,p}(-1) \equiv
\left|\begin{array}{cccccc}
0 & -1 & & 0 & \cdots & 0 \\
0 & & -2 & & \cdot & \\
\cdot & & & \cdot & \cdot & \cdot \\
\cdot & & & \cdot & \cdot & \cdot \\
\cdot & & & \cdot & & \cdot \\
0 & & & & -(p-2) & 0 \\
1 & & & & & 1 \\
1 & & & \cdots & & 0
\end{array}\right| \equiv 1 \pmod{p}
$$

Thus condition (3) is satisfied and the theorem is proved.

References

1. P. R. Lipow and I. J. Schoenberg, *Cardinal interpolation and spline functions III. Cardinal hermite interpolation, linear algebra and applications.*

2. I. J. Schoenberg and A. Sharma, *Cardinal interpolation and spline functions V. The B-splines for cardinal hermite interpolation.* Mathematics Research Center Technical Report 1150 (1971).

University of Alberta, Edmonton
University of California, Los Angeles