Proceedings of the Edinburgh Mathematical Society (2000) **43**, 139–147 ©

LINEAR ISOMETRIES ON SUBALGEBRAS OF UNIFORMLY CONTINUOUS FUNCTIONS

JESÚS ARAUJO[1] AND JUAN J. FONT[2]

[1] *Departamento de Matemáticas, Estadistica y Computación, Facultad de Ciencias, Universidad de Cantabria, Avenida de los Castros s.n., E-39071, Santander, Spain* (araujo@matesco.unican.es)

[2] *Departamento de Matemáticas, Universitat Jaume I, Campus Riu Sec, E-12071, Castellón, Spain* (font@mat.uji.es)

(Received 2 April 1998)

Abstract We describe the linear surjective isometries between various subalgebras of uniformly continuous bounded functions defined on closed subsets of Banach spaces.

Keywords: linear isometries; Banach–Stone theorem; uniformly continuous functions

AMS 1991 *Mathematics subject classification:* Primary 47B38
Secondary 46E25

1. Introduction

Let $(X, \|\cdot\|)$ be a (real or complex) Banach space and let K_X be a closed subset of X. Let $C_u(K_X)$ denote the Banach algebra of all real or complex-valued, uniformly continuous bounded functions defined on K_X endowed with the usual supremum norm: given $f \in C_u(K_X)$, $\|f\|_\infty := \sup\{|f(x)| : x \in K_X\}$. A closed subalgebra $A_u(K_X)$ of $C_u(K_X)$ is said to be *weakly normal* if, given any subsets A and B of K_X with a positive distance $d(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}$, there is an $f \in A_u(K_X)$ such that $|f(x)| \geqslant 1$ for every $x \in A$, and $|f(y)| \leqslant \frac{1}{2}$ for every $y \in B$.

In [**5**], Lacruz and Llavona characterized uniform continuity for maps between the unit balls of two Banach spaces, K_X and K_Y, in terms of composition operators of $C_u(K_X)$ into $C_u(K_Y)$. In this paper, we obtain similar results for weakly normal subalgebras of $C_u(K_X)$ and $C_u(K_Y)$, and use them to describe the linear isometries T between such subalgebras. It is shown that T can be written as a weighted composition map. Thus T induces a uniform homeomorphism h of K_Y onto K_X; that is, h and its inverse are uniformly continuous bijections. We would like to remark that these results, though similar to the Banach–Stone theorem and many of its generalizations (see [**4**] for a thorough survey on this subject), are obtained in spite of the absence of any kind of local compactness in K_X.

We shall also deal with certain subalgebras of $A_u(K_X)$: namely, those whose elements vanish at some fixed point $x_0 \in K_X$. With no loss of generality, we shall assume that $x_0 = 0$. Such subalgebras will be denoted by a superscript 0; that is,

$$A_u^0(K_X) := \{f \in A_u(K_X) : f(0) = 0\}.$$

We shall show that for $A_u^0(K_X)$, characterizations similar to those for $A_u(K_X)$ can be obtained.

2. Preliminaries

Let $\boldsymbol{N}$ (respectively $\boldsymbol{K}$) denote the set of positive integers (respectively the field of real or complex numbers). Given a compact space $\mathcal{X}$, we denote by $C(\mathcal{X})$ the Banach algebra of all $\boldsymbol{K}$-valued continuous functions defined on $\mathcal{X}$ endowed with its usual supremum norm. If $f \in C(\mathcal{X})$ and U is a subset of $\mathcal{X}$, then $f_{|U}$ stands for the restriction of f to U and $\mathrm{cl}_{\mathcal{X}}\, U$ for the closure of U in $\mathcal{X}$. By 1 we will denote the function such that $1(x) = 1$ for every $x \in \mathcal{X}$.

We will say that a linear subspace A of $C(\mathcal{X})$ *separates strongly* two elements x_1 and x_2 of $\mathcal{X}$ if there exists an $f \in A$ such that $|f(x_1)| \neq |f(x_2)|$.

Let A be a linear subspace of $C(\mathcal{X})$. We will denote by ∂A the *Shilov boundary* for A, that is, the *unique* minimal closed boundary for A. Let us recall that a subset U of $\mathcal{X}$ is a *boundary* for A if each function in A attains its maximum on U.

It is said that $x_0 \in \mathcal{X}$ is a *peak point* for A if there is a function f in A such that $|f(x_0)| = \|f\|_\infty$ and $|f(x)| < \|f\|_\infty$ for all $x \in \mathcal{X} \setminus \{x_0\}$.

In the sequel, given a Banach space X, K_X will stand for a closed subset of X and $A_u(K_X)$ for a *weakly normal* subalgebra of $C_u(K_X)$. Let βK_X be the Stone–Cech compactification of K_X. We define the quotient space $\gamma X := \beta K_X/\!\sim$, where $\sim$ is the equivalence relation, defined as $x_1 \sim x_2$ if $f(x_1) = f(x_2)$ for every $f \in A_u(K_X)$. It is straightforward to verify that γX is a compactification of K_X and that every function in $A_u(K_X)$ can be continuously extended to γX. Then we can identify $A_u(K_X)$ with a closed subalgebra $A(X)$ of $C(\gamma X)$. Likewise, we can identify $A_u^0(K_X)$ with a closed subalgebra $A_0(X)$ of $C(\gamma X)$.

3. Some previous results on algebras of uniformly continuous functions

Lemma 3.1. *Let X be a Banach space. Then $A(X)$ and $A_0(X)$ separate strongly the points of γX.*

Proof. From § 2, we know that $A(X)$ and $A_0(X)$ are subalgebras of $C(\gamma X)$ that separate the points of γX.

https://doi.org/10.1017/S0013091500020757 Published online by Cambridge University Press

Now take $x, y \in \gamma X$. If f in any of these subalgebras satisfies $f(x) \neq f(y)$ but $|f(x)| = |f(y)|$, then, for some scalar α, we have $|\alpha f(x) + (f(x))^2| \neq |\alpha f(y) + (f(y))^2|$. This implies that both subalgebras strongly separate the points of γX. $\square$

Remark 3.2. Notice that $A(X)$ is a uniform algebra, and, consequently, the Shilov boundary for $A(X)$ exists (see, for example, [**3**]). A similar argument works for $A_0(X)$ (see also [**2**, Theorem 1]).

Lemma 3.3. *Let X be a Banach space.*

(1) *Every point of K_X belongs to $\partial A(X)$. Moreover, $\partial A(X) = \gamma X$.*

(2) *Every point of $K_X \setminus \{0\}$ belongs to $\partial A_0(X)$. Moreover, $\partial A_0(X) \setminus \{0\} = \gamma X \setminus \{0\}$.*

Proof. (1) By a well-known characterization of the elements of a Shilov boundary (see, for example, [**1**]), it is enough to prove that given any $x_0 \in K_X$ and any open neighbourhood U of x_0 in γX, there exists an $f \in A_u(K_X)$ such that $\sup_{x \in K_X \setminus U} |f(x)| < \|f\|_\infty/2$. We have $d(\{x_0\}, K_X \setminus U) > 0$, and, since $A_u(K_X)$ is weakly normal, the assertion follows.

(2) Let $x_0 \in K_X \setminus \{0\}$. Take any open neighbourhood U of x_0 in γX, and suppose, without loss of generality, that $0 \notin \mathrm{cl}_{\gamma X} U$. Let us define f as in part (1). It is clear that if $f(0) = 0$ since $f \in A_0(X)$, then we are done. So we assume that $f(0) = \alpha \neq 0$. Then, as above, we can find a function $g \in A_u(K_X)$ such that $g(0) = 1$ and $|g(x)| \leqslant \frac{1}{2}$ for every $x \in K_X \cap U$. Hence, it is clear that $(f - \alpha g)(0) = 0$ and $|(f - \alpha g)(x)| \geqslant \frac{1}{2}$ for all $x \in K_X \cap U$. It follows that there exists an $n \in \boldsymbol{N}$ such that $|f^n(f - \alpha g)(x)| < \|f^n(f - \alpha g)\|_\infty/2$ for every $x \in K_X \setminus U$. Since $f^n(f - \alpha g)(0) = 0$, we have the result. $\square$

Lemma 3.4. *Let X be a Banach space.*

(1) *Every $x \in K_X$ is a G_δ-set in γX.*

(2) *If $x \in \gamma X \setminus K_X$, then x cannot be a G_δ-set in γX.*

Proof. (1) Fix $x_0 \in K_X$. For each $n \in \boldsymbol{N}$, let us take a function $g_n \in A_u(K_X)$ such that $g_n(x_0) \geqslant 1$ and $|g_n(x)| \leqslant \frac{1}{2}$ for every $x \in K_X$ with $\|x - x_0\| \geqslant 1/n$. Now we consider $U_n := \{z \in \gamma X : |g_n(z)| > \frac{1}{2}\}$. It is clear that $x_0 \in \bigcap_{n \in \boldsymbol{N}} U_n$. It suffices to check that there is no other point in this intersection. So, suppose that $z_0 \in \gamma X$, $z_0 \neq x_0$. Then there exist open subsets U and V of γX with $x_0 \in U$, $z_0 \in V$, and $U \cap V = \emptyset$. Now, if $B(x_0, 1/n)$ stands for the open ball with centre x_0 and radius $1/n$ in K_X, it is clear that there exists $n' \in \boldsymbol{N}$ such that $B(x_0, 1/n') \subset K_X \cap U$. Thus we infer that $(K_X \cap U_{n'}) \cap V \subset B(x_0, 1/n') \cap V = \emptyset$, and then $U_n \cap V = \emptyset$, which implies that $z_0 \notin \bigcap_{n \in \boldsymbol{N}} U_n$. As a consequence, x_0 is a G_δ-set in γX.

(2) Assume that there is an $x_0 \in \gamma X \setminus K_X$ which is a G_δ-set in γX. Then there exists a countable family of open subsets of γX, say $\{B_n\}_{n \in \boldsymbol{N}}$, such that $\{x_0\} = \bigcap_{n \in \boldsymbol{N}} B_n$. With no loss of generality, we can assume that $\mathrm{cl}\, B_{n+1} \subseteq B_n$. Now we construct a sequence (x_n) in K_X such that $x_n \in B_n \cap K_X$ for every $n \in \boldsymbol{N}$ and claim that (x_n) converges to

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x_0. Indeed, if we consider an open neighbourhood U of x_0, then $\gamma X \setminus U$ is a compact set that does not meet $\bigcap_{n\in \boldsymbol{N}} \operatorname{cl} B_n$. Hence, there is an $n' \in \boldsymbol{N}$ such that $\operatorname{cl} B_{n'} \subset U$, and, consequently, if $n > n'$, then $x_n \in U$.

Next, we deduce that no subsequence of (x_n) is Cauchy. Otherwise, since K_X is a closed subset of a Banach space, such subsequence would converge to some $x_1 \in K_X$ and then $x_1 = x_0 \notin K_X$. As a consequence, since (x_n) is not relatively compact, we can choose $\epsilon > 0$ and a subsequence (x_{n_m}) of (x_n) such that $d(x_{n_m}, x_{n_k}) > \epsilon$ for every $m, k \in \boldsymbol{N}$. Thus, as $A_u(K_X)$ is weakly normal, there is an $f \in A_u(K_X)$ such that $|f(x_{n_{2m+1}})| \geqslant 1$ and $|f(x_{n_{2m}})| \leqslant \frac{1}{2}$ for every $m \in \boldsymbol{N}$. This implies that we cannot extend f to γX, which is absurd. □

4. Linear surjective isometries

Lemma 4.1. *Let (x_n), (x'_n) be sequences in K_X with the property that*

$$\lim |f(x_n) - f(x'_n)| = 0,$$

for every $f \in A_u(K_X)$. Then $\lim \|x_n - x'_n\| = 0$.

Proof. The idea of the proof of [**5**, Lemma 2.2] remains valid for this theorem. Just the first step of that proof needs some changes. So we assume that the result is not true, and suppose that there is an $\epsilon > 0$ and an increasing sequence of natural numbers (n_j) such that $\|x_{n_j} - x'_{n_j}\| \geqslant \epsilon$ for every j. The first step consists of proving that the sequence (x_{n_j}) does not have any convergent subsequence in K_X. So we suppose that there exists a subsequence $(x_{n_{j_k}})$ converging to x_0. Then, for each $f \in A_u(K_X)$, we have that $\lim f(x'_{n_{j_k}}) = f(x_0)$. Now suppose that there exists an accumulation point z_0 of $\{x'_{n_{j_k}} : k \in \boldsymbol{N}\}$ in γX, $z_0 \neq x_0$. By the definition of γX, we have that there exists an $f_0 \in A_u(K_X)$ such that $f_0(x_0) \neq f_0(z_0)$. This implies clearly that $\lim f_0(x'_{n_{j_k}}) \neq f_0(x_0)$, which is impossible. Consequently, x_0 is the only accumulation point of $\{x'_{n_{j_k}} : k \in \boldsymbol{N}\}$ in γX. This contradicts our hypothesis on (x'_{n_j}), and the first step is proved.

The rest of the proof, which we now sketch, follows as in that of [**5**, Lemma 2.2], with slight changes. Thus, as a consequence of the first step, there is an η with $0 < \eta < \epsilon$ and an increasing sequence of positive integers $(j_k)_k$ such that $\|x_{n_{j_r}} - x_{n_{j_s}}\| \geqslant \eta$ and $\|x'_{n_{j_r}} - x'_{n_{j_s}}\| \geqslant \eta$ for all $r \neq s$.

The second step consists of checking the existence of another increasing sequence of positive integers $(k_l)_l$ with the property that

$$\|x_{n_{j_{k_r}}} - x'_{n_{j_{k_s}}}\| \geqslant \eta/2, \quad \text{for all } r \neq s.$$

Finally, we deduce from the second step that the distance between the sets

$$A = \{x_{n_{j_{k_l}}} : l \geqslant 1\} \quad \text{and} \quad A' = \{x'_{n_{j_{k_l}}} : l \geqslant 1\}$$

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is strictly positive. Hence, there exists $f \in A_u(K_X)$, such that $|f(x_{n_{j_{k_l}}})| \geqslant 1$ and $|f(x'_{n_{j_{k_l}}})| \leqslant \frac{1}{2}$ for every $l \geqslant 1$. This fact contradicts the hypothesis and we are done. □

Also with the same proof as in [**5**, Theorem 2.3], but using Lemma 4.1 instead, we have the following result.

Theorem 4.2. *Let $h : K_Y \to K_X$ be a map with the property that $f \circ h \in C_u(K_Y)$ for every $f \in A_u(K_X)$. Then h is uniformly continuous.*

Theorem 4.3. *Let X and Y be Banach spaces and let $T : A_u(K_X) \to A_u(K_Y)$ be a linear surjective isometry. Then there exists a uniform homeomorphism h of K_Y onto K_X and a function $a \in C_u(K_Y)$, such that $|a(y)| = 1$ for all $y \in K_Y$, and $(Tf)(y) = a(y)f(h(y))$ for all $y \in K_Y$ and all $f \in A_u(K_X)$.*

Proof. By Lemma 3.1 and [**1**, Theorem 4.1], there exists a homeomorphism h' of $\partial A(Y) = \gamma Y$ onto $\partial A(X) = \gamma X$ and a continuous map $a' : \gamma Y \to \boldsymbol{K}$, such that $|a'(y)| = 1$ for all $y \in \gamma Y$, and $(Tf)(y) = a'(y)f(h'(y))$ for all $y \in \gamma Y$ and all $f \in A(X)$. We shall define $a := a'_{|K_Y}$.

Claim 4.4. $h'(K_Y) = K_X$.

Since h' is a homeomorphism, it suffices to check that $h'(K_Y) \subseteq K_X$. Suppose that there is a $y \in K_Y$ such that $h'(y) \in \gamma X \setminus K_X$. By Lemma 3.4 (2), $h'(y)$ cannot be a G_δ-set in γX, whereas, from Lemma 3.4 (1), we know that every $y \in K_Y$ is a G_δ-set in γY. This contradicts the fact that h' preserves G_δ-sets. As a consequence, $h := h'_{|K_Y}$ is a homeomorphism of K_Y onto K_X.

Claim 4.5. *The functions a and $1/a$ belong to $C_u(K_Y)$.*

First we prove that a is uniformly continuous. Suppose this is not the case. Then there exist an $\epsilon > 0$ and two sequences (y_n) and (y'_n) in K_Y such that $\lim \|y_n - y'_n\| = 0$ and $|a(y_n) - a(y'_n)| \geqslant \epsilon$ for every $n \in \boldsymbol{N}$. It is easy to see that we may assume without loss of generality that the set $\{y_n : n \in \boldsymbol{N}\} \cup \{y'_n : n \in \boldsymbol{N}\}$ is not dense in K_Y. Consequently, $\{h(y_n) : n \in \boldsymbol{N}\} \cup \{h(y'_n) : n \in \boldsymbol{N}\}$ is not dense in K_X, and, since $A_u(K_X)$ is weakly normal, we can find an $f \in A_u(K_X)$ such that $|f(h(y_n))|, |f(h(y'_n))| \geqslant 1$ for every $n \in \boldsymbol{N}$. Taking into account that Tf is uniformly continuous, we have that $\lim(a(y_n)f(h(y_n)) - a(y'_n)f(h(y'_n))) = 0$, and, for the same reason, $\lim(a(y_n)f^2(h(y_n)) - a(y'_n)f^2(h(y'_n))) = 0$. Thus, since $\lim(a(y_n)f^2(h(y_n)) - a(y'_n)f(h(y_n))f(h(y'_n))) = 0$, we deduce that

$$\lim(a(y'_n)f(h(y_n))f(h(y'_n)) - a(y'_n)f^2(h(y'_n))) = 0,$$

that is,

$$0 = \lim a(y'_n)f(h(y'_n))(f(h(y_n)) - f(h(y'_n))).$$

Then it is clear that $\lim(f(h(y_n)) - f(h(y'_n))) = 0$. On the other hand, since

$$\lim a(y_n)(f(h(y_n)) - f(h(y'_n))) = \lim(f(h(y_n)) - f(h(y'_n))) = 0,$$

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we conclude, as $0 = \lim(a(y_n)f(h(y_n)) - a(y'_n)f(h(y'_n)))$, that

$$\lim(a(y_n) - a(y'_n))f(h(y'_n)) = 0,$$

which contradicts our assumption above. Thus a is uniformly continuous.

Next, we are going to see that $1/a$ is uniformly continuous. Choose two sequences (y_n) and (y'_n) in K_Y such that $\lim \|y_n - y'_n\| = 0$. Then

$$\lim\left|\frac{1}{a(y_n)} - \frac{1}{a(y'_n)}\right| = \lim\left|\frac{a(y'_n) - a(y_n)}{a(y'_n)\cdot a(y_n)}\right| = 0,$$

since $a \in C_u(K_Y)$ and $|a(y'_n) \cdot a(y_n)| = 1$ for every $n \in \boldsymbol{N}$.

Claim 4.6. *The mappings h and its inverse are uniformly continuous.*

For every $f \in A_u(K_X)$, the function $(Tf)(y) = a(y)f(h(y))$ belongs to $C_u(K_Y)$. Hence, from the above claim, we have that the function

$$(1/a)(y) \cdot a(y)f(h(y)) = f(h(y))$$

also belongs to $C_u(K_Y)$ for all $f \in A_u(K_X)$. Hence, by Theorem 4.2, h is uniformly continuous.

It is a routine matter to verify that the inverse of T, T^{-1}, can be written as $(T^{-1}g)(x) = b(x)g(h^{-1}(x))$, where $b \in C_u(K_X)$, for all $x \in X$ and all $g \in A_u(K_Y)$. Hence, an analogous argument shows that the inverse of h, h^{-1}, is uniformly continuous. □

The following straightforward consequence of the above theorem does not yet seem to have made its way into the literature (see, for example, [**4**]).

Corollary 4.7. *Let X and Y be Banach spaces and let $T : C_u(K_X) \to C_u(K_Y)$ be a linear surjective isometry. Then there exists a uniform homeomorphism h of K_Y onto K_X and a function $a \in C_u(K_Y)$, such that $|a(y)| = 1$ for all $y \in K_Y$, and $(Tf)(y) = a(y)f(h(y))$ for all $y \in K_Y$ and all $f \in C_u(K_X)$.*

Proof. By [**5**, Lemma 2.1], it is apparent that $C_u(K_X)$ is weakly normal. Thus, the result follows immediately from Theorem 4.3. □

Theorem 4.8. *Let X and Y be Banach spaces and let $T : A_u^0(K_X) \to A_u^0(K_Y)$ be a linear surjective isometry. Then there exists a uniform homeomorphism h of K_Y onto K_X with $h(0) = 0$. Furthermore, there is a function $a \in C(K_Y \setminus \{0\})$ with $|a(y)| = 1$ for all $y \in K_Y \setminus \{0\}$, such that, for all $f \in A_u^0(K_X)$*

$$(Tf)(y) = \begin{cases} a(y)f(h(y)), & y \in K_Y \setminus \{0\}, \\ 0, & y = 0. \end{cases}$$

Proof. By Lemma 3.1 and [**1**, Theorem 4.1], there exists a homeomorphism h' of $\partial A_0(Y) \setminus \{0\} = \gamma Y \setminus \{0\}$ onto $\partial A_0(X) \setminus \{0\} = \gamma X \setminus \{0\}$ and a continuous map $a' : \gamma Y \setminus \{0\} \to \boldsymbol{K}$, such that $|a'(y)| = 1$ for all $y \in \gamma Y \setminus \{0\}$, and $(Tf)(y) = a'(y) f(h'(y))$ for all $y \in \gamma Y \setminus \{0\}$ and all $f \in A_0(X)$. We shall define $a := a'_{|K_Y \setminus \{0\}}$.

Arguments like those in Claim 4.4 show that $h := h'_{|K_Y \setminus \{0\}}$ is a homeomorphism of $K_Y \setminus \{0\}$ onto $K_X \setminus \{0\}$.

Claim 4.9. *The weight function a is uniformly continuous in $B_\alpha := \{y \in K_Y : \|y\| \geqslant \alpha\}$, for every $0 < \alpha < 1$.*

Fix $0 < \alpha < 1$. It is clear that 0 does not belong to the compact set $\mathrm{cl}_{\gamma Y} B_\alpha$. Hence, such a set is compact in $\gamma Y \setminus \{0\}$ and, consequently, $h'(\mathrm{cl}_{\gamma Y} B_\alpha)$ is a compact subset of $\gamma X \setminus \{0\}$. Then $h(B_\alpha)$, which turns out to be $h'(\mathrm{cl}_{\gamma Y} B_\alpha) \bigcap (K_X \setminus \{0\})$, is a closed subset of K_X that does not contain the point 0. By definition, there exists a function $f \in A_u^0(K_X)$ such that $|f(x)| \geqslant 1$ on $h(B_\alpha)$. Thus, if we assume that a is not uniformly continuous on B_α, we can proceed, with slight changes, as in Claim 4.5, and obtain a contradiction. This shows that a is uniformly continuous on B_α.

Claim 4.10. *Let $f \in A_u^0(K_Y)$. Then $(1/a) \cdot f$ belongs to $C_u(K_Y \setminus \{0\})$.*

Choose an $\epsilon > 0$. Since f vanishes at 0, there exists $0 < c < 1$ such that if $\|x\| \leqslant c$, then $|f(x)| < \epsilon/4$.

On the other hand, since f is uniformly continuous, there exists a $\delta' > 0$ such that if $\|x - y\| < \delta'$, then $|f(x) - f(y)| < \epsilon/2$.

By Claim 4.9 and following the details in Claim 4.5, we yield the uniform continuity of $1/a$ on B_c. Hence there exists a $\delta'' > 0$ such that if $\|x - y\| < \delta''$, $x, y \in B_c$, then

$$\left| \frac{1}{a}(x) - \frac{1}{a}(y) \right| < \frac{\epsilon}{2\|f\|_\infty}.$$

Let us define $\delta := \min\{\delta', \delta''\}$. Suppose first that $\|x - y\| < \delta$ and either $\|x\| \leqslant c$ or $\|y\| \leqslant c$. Assume, without loss of generality, that $\|y\| \leqslant c$. Then

$$\begin{aligned} \left| \frac{1}{a}(x) f(x) - \frac{1}{a}(y) f(y) \right| &= \left| \frac{1}{a}(x)(f(x) - f(y)) + \left(\frac{1}{a}(x) - \frac{1}{a}(y) \right) f(y) \right| \\ &\leqslant |f(x) - f(y)| + \left| \frac{1}{a}(x) - \frac{1}{a}(y) \right| |f(y)| \\ &< \tfrac{1}{2}\epsilon + 2\tfrac{1}{4}\epsilon = \epsilon. \end{aligned}$$

If $\|x - y\| < \delta$ and both $\|x\|, \|y\| > c$, then

$$\begin{aligned} \left| \frac{1}{a}(x) f(x) - \frac{1}{a}(y) f(y) \right| &= \left| \frac{1}{a}(x)(f(x) - f(y)) + \left(\frac{1}{a}(x) - \frac{1}{a}(y) \right) f(y) \right| \\ &\leqslant |f(x) - f(y)| + \left| \frac{1}{a}(x) - \frac{1}{a}(y) \right| |f(y)| \\ &< \tfrac{1}{2}\epsilon + \|f\|_\infty \frac{\epsilon}{2\|f\|_\infty} = \epsilon. \end{aligned}$$

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Consequently, $(1/a) \cdot f$ is uniformly continuous on $K_Y \setminus \{0\}$.

Claim 4.11. *$h : K_Y \setminus \{0\} \to K_X \setminus \{0\}$ and its inverse are uniformly continuous.*

From Claim 4.10, we deduce that $(1/a) \cdot a \cdot (f \circ h) = f \circ h$ is uniformly continuous on $K_Y \setminus \{0\}$ for every $f \in A_u^0(K_X)$. On the other hand, a close review of the proofs of Lemma 4.1, Theorem 4.2 and [**5**, Lemma 2.2, Theorem 2.3 and its remark] shows that they remain true if we replace $A_u(K_X)$ (respectively $A_u(K_Y)$) by $A_u^0(K_X)$ (respectively $A_u^0(K_Y)$). Hence, we infer that h is uniformly continuous on $K_Y \setminus \{0\}$.

The uniform continuity of h^{-1} follows as in the last paragraph of the proof of Theorem 4.3.

Claim 4.12. *$h : K_Y \setminus \{0\} \to K_X \setminus \{0\}$ can be extended to a uniform homeomorphism of K_Y onto K_X.*

Let (y_n) be a sequence in $K_Y \setminus \{0\}$ converging to 0. It is clear, from the uniform continuity of h, that $(h(y_n))$ is a Cauchy sequence. Hence, as X is a Banach space, there exists a point, say $h(0)$, such that $(h(y_n))$ converges to $h(0)$.

The remainder of the proof consists of checking that $h(0) = 0$. Suppose $h(0) = x_0 \neq 0$. Since (y_n) converges to 0, then $((Tf)(y_n)) = (a(y_n)f(h(y_n)))$ converges to $(Tf)(0) = 0$ for every $f \in A_u^0(K_X)$. That is, $(f(h(y_n)))$ converges to 0 for every $f \in A_u^0(K_X)$, since $|a| \equiv 1$ on $K_Y \setminus \{0\}$.

On the other hand, since $A_u^0(K_X)$ is weakly normal, there exists an $f_0 \in A_u^0(K_X)$ such that $|f_0(x)| \geqslant 1$ for every $x \in B_{\|x_0\|/2}$ (see definition in Claim 4.9). Thus, $(f_0(h(y_n)))$ does not converge to 0, which contradicts the above paragraph. □

The following example shows that we cannot strengthen Theorem 4.8 to the effect $a \in C_u(K_Y \setminus \{0\})$.

Example 4.13. Let B be the unit ball of the complex numbers. Let a be a function in $C(B \setminus \{0\})$ defined as

$$a(x) := x/|x|,$$

for every $x \in B \setminus \{0\}$. It is then apparent that

$$(Tf)(x) := \begin{cases} a(x)f(x), & x \in B \setminus \{0\}, \\ 0, & x = 0, \end{cases}$$

is a linear isometry of $A_u^0(B)$ onto itself. However, a is not uniformly continuous on $B \setminus \{0\}$.

Acknowledgements. The authors thank the referee for making several suggestions which improved this paper.

The research of J.A. was partly supported by the Spanish Dirección General de Investigación Científica y Técnica (DGICYT, PB95-0582). That of J.J.F. was partly supported by Spanish DGES grant (PB96-1075).

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