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A residue theorem for rational functions on star-shaped domains

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Abstract. M. Heins demonstrated that any finite Blaschke product defined on the open unit disc, provided it has at least one finite pole, possesses a nonzero residue. In this work, we extend Heins' result by generalizing the class of functions under consideration. Specifically, we prove that a broader class of rational functions, defined on certain star-shaped domains in the complex plane, also exhibits this nonzero residue property. This class includes, as a special case, the family of finite Blaschke products. Our findings contribute to a better understanding of the analytic behavior of rational functions on more complex domains, opening new avenues for exploration in this area.

1 Introduction

A domain $\Omega \subset \mathbb{C}$ is said to be *star-shaped* with respect to a point $\zeta \in \Omega$, referred to as a *center* of Ω, if for every point $z \in \Omega$, the line segment

$$[\zeta, z] = \{\lambda\zeta + (1-\lambda)z : 0 \le \lambda \le 1\}$$

lies entirely within Ω. This is a weaker form of convexity, where the difference lies in the fact that the center ζ is fixed, whereas for a convex set, the line segment must remain inside the set for any pair of points $z_1, z_2 \in \Omega$.

Consider a rational function

$$R = \frac{P}{Q},$$

where P and Q are polynomials of the same degree. The sets

$$\Gamma_c = \{z \in \mathbb{C} : |R(z)| = c\},$$

where c is a positive constant, are called the *level curves* of R. Since R is a rational function, Γ_c is a finite disjoint union of $\mathcal{C}^\infty$ curves (for a finite number of constants c, Γ_c may contain branch points and thus is not a Jordan curve). The rational function R is called *zero-pole separable* if there exists a Jordan level curve Γ_c such that all zeros of R are inside Γ_c and all poles of R are outside Γ_c. This specific Γ_c is referred to as a *separating level curve*.

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Two important classes of zero-pole separable rational functions merit mention. Let $\{z_k\}_{1\le k\le N}$ be a finite sequence of nonzero complex numbers inside the open unit disc $\mathbb{D}$, and let $m_k \ge 1$ and $m \ge 0$. Then, the rational function

$$B(z) = z^m \prod_{k=1}^{N} \left(\frac{z - z_k}{1 - \bar{z}_k z} \right)^{m_k}$$

is called a finite Blaschke product for the open unit disc $\mathbb{D}$ [8, p. 158]. While finite Blaschke products have long been a classical subject, they have recently gained renewed attention, particularly in relation to the Crouzeix conjecture. The role of finite Blaschke products in this context can be found in [1] and the references therein. For a detailed treatment of finite Blaschke products, see [3].

A fundamental property of finite Blaschke products is that

$$\Gamma_1 = \mathbb{T},$$

where $\mathbb{T}$ is the unit circle in the complex plane. Therefore, $\mathbb{T}$ is a separating level curve, and every point inside $\mathbb{D}$ can be regarded as a center.

Similarly, if we take $\{z_k\}_{1\le k\le N}$ to be a finite sequence in the upper half-plane $\mathbb{C}_+$, and let $m_k \ge 1$, then the rational function

$$B(z) = \prod_{k=1}^{N} \left(\frac{z - z_k}{z - \bar{z}_k} \right)^{m_k}$$

is called a finite Blaschke product for the upper half-plane. In this case, the level curve

$$\Gamma_1 = \mathbb{R},$$

where $\mathbb{R}$ is the real line, forms a degenerate level curve. However, for $c < 1$ sufficiently close to one, Γ_c is a $\mathcal{C}^\infty$ curve with a convex interior containing all the zeros of B.

M. Heins [6] showed that every finite Blaschke product for the unit disc, which has at least one finite pole, has a nonzero residue. Recent developments on the residue theorem can be found in [2, 4, 5, 9, 10]. It is important to emphasize the necessity of having at least one finite pole, as the function $f(z) = z^n$, for $n \ge 1$, is a well-defined finite Blaschke product for the unit disc, yet it has no finite poles.

In [7], J. Mashreghi extended this result to the Blaschke products in the upper half-plane, although the proofs for the two cases are entirely different. In this article, we generalize Heins' method using the concept of zero-pole separable rational functions. This provides a unified approach that applies to both the unit disc and the upper half-plane cases.

2 The main result

In the following, a global primitive of a function f refers to a function F such that $F' = f$, except possibly at the poles of f.

Theorem 2.1 *Let R be a zero-pole separable rational function. Suppose R has a separating level curve Γ_c, whose interior is star-shaped, with a center ζ in the interior of*

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Γ_c. *Then, for any* $n \geq 0$, *the rational function*

$$(z-\zeta)^n R(z)$$

has a nonzero residue.

We emphasize that Theorem 2.1 asserts, in particular, that a zero-pole separable rational function R, with a separating level curve whose interior is star-shaped, has a nonzero residue. This corresponds to the special case where $n = 0$.

To illustrate, in the examples of zero-pole separable rational functions mentioned earlier, the finite Blaschke product

$$R(z) = \prod_{k=1}^{N} \left(\frac{z - z_k}{1 - \bar{z}_k z} \right)^{m_k}$$

is zero-pole separable with $\Gamma_1 = \mathbb{T}$ as the separating level curve, and $\zeta = 0$ as its center. Therefore, for each $n \geq 0$,

$$B(z) = z^n \prod_{k=1}^{N} \left(\frac{z - z_k}{1 - \bar{z}_k z} \right)^{m_k}$$

has a nonzero residue. This is precisely Heins' theorem [6]. Similarly, in the upper half-plane case,

$$R(z) = (z-\zeta)^n \prod_{k=1}^{N} \left(\frac{z - z_k}{z - \bar{z}_k} \right)^{m_k}, \quad (n \geq 0),$$

has a nonzero residue for all $\zeta \in \mathbb{C}_+$ and for all $n \geq 0$.

3 Proof of Theorem 2.1

Let $R = \dfrac{P}{Q}$ where

$$Q(z) = \prod_{k=1}^{N} (z - p_k)^{m_k}.$$

Then, by the Partial Fraction Expansion theorem, $(z-\zeta)^n R(z)$ has the unique decomposition

$$(3.1) \qquad (z-\zeta)^n R(z) = \frac{(z-\zeta)^n P(z)}{\prod_{k=1}^{N} (z-p_k)^{m_k}} = \sum_{k=0}^{n} \alpha_k z^k + \sum_{k=1}^{N} \sum_{\ell=1}^{m_k} \frac{\beta_{k,\ell}}{(z-p_k)^{\ell}},$$

where α_k and $\beta_{k,\ell}$ are numerical constants; $\beta_{k,1}$ is the residue of $(z-\zeta)^n R(z)$ at the pole p_k.

We now appeal to an elementary, but very important fact from complex analysis: the function $(z-\zeta)^n R(z)$ has a global primitive if and only if $\beta_{k,1} = 0$ for each k. In other words, $(z-\zeta)^n R(z)$ has a global primitive if and only if all its residues are zero.

To proceed, suppose that

$$(3.2) \qquad \beta_{k,1} = 0$$

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for *all* k. We then seek a contradiction. The assumption ensures that $(z-\zeta)^n R(z)$ has a global primitive F. Since

$$F'(z) = (z-\zeta)^n R(z),$$

we have, for each z inside Γ_c,

$$F(z) = F(\zeta) + \int_\gamma F'(w)\, dw = F(\zeta) + \int_\gamma (w-\zeta)^n R(w)\, dw, \tag{3.3}$$

where γ is any rectifiable curve inside Γ_c from ζ to z. Remember that the poles of R are outside Γ_c. By the Maximum Modulus principle [8, p. 129], for each z inside Γ_c, we have $|R(z)| \le c$. Hence,

$$\begin{aligned} |F(z) - F(\zeta)| &= \left| \int_\gamma (w-\zeta)^n R(w)\, dw \right| \\ &\le c \int_\gamma |w-\zeta|^n\, |dw| = c\, \frac{|z-\zeta|^{n+1}}{n+1}, \end{aligned} \tag{3.4}$$

if γ is the line segment $[\zeta, z]$. Here, we used the fact that the interior of Γ_c is star-shaped. Otherwise, we cannot connect ζ and z by a line segment and thus such a crucial estimation is not valid. Since F is continuous inside and on the curve Γ_c, the inequality (3.4) is also valid for all points of Γ_c. Up to here, we have not profoundly used the fact that R is a rational function. The estimation (3.4) is valid for any analytic function F defined on the interior of Γ_c, provided that Γ_c is a level curve of R where R is given by

$$R(z) = \frac{F'(z)}{(z-\zeta)^n}.$$

Now, we dig further to detect the implications of the rational function R.

Since $F'(z) = (z-\zeta)^n R(z)$, if we directly integrate (3.1), we get

$$F(z) = \alpha + \sum_{k=0}^{n} \frac{\alpha_k}{k+1} z^{k+1} + \sum_{k=1}^{N} \sum_{\ell=2}^{m_k} \frac{\frac{-\beta_{k,\ell}}{(\ell-1)}}{(z-p_k)^{(\ell-1)}}, \tag{3.5}$$

where α is an arbitrary constant. Note that, by assumption, the index ℓ starts from 2. We choose the free parameter α such that $F(\zeta) = 0$. Since $F(\zeta) = 0$ and

$$F'(z) = (z-\zeta)^n R(z),$$

F has a zero of order at least $n+1$ at ζ. Here, by taking the common denominator in the equation (3.5), we should get

$$F(z) = \frac{(z-\zeta)^{n+1} S(z)}{\prod_{k=1}^{N} (z-p_k)^{(m_k-1)}}, \tag{3.6}$$

where S is a polynomial of degree $\sum_{k=1}^{N} (m_k - 1)$. The function

$$G(z) = \frac{(n+1)\, F(z)}{(z-\zeta)^{n+1} R(z)} \tag{3.7}$$

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will lead us to a contradiction. According to (3.1) and (3.6),

$$G(z) = \frac{(n+1)\, S(z)\, \prod_{k=1}^{N}(z - p_k)}{P(z)},$$

and thus the poles of G are the zeros of P which are all inside Γ_c. Hence, G is analytic outside Γ_c and has at least a simple zero at each p_k.

In the first place, by (3.1),

$$(z-\zeta)^n R(z) = \alpha_n z^n + O(z^{n-1}),$$

and by (3.5),

$$F(z) = \frac{\alpha_n z^{n+1}}{n+1} + O(z^n).$$

Therefore, according to the definition (3.7),

$$\lim_{z\to\infty} G(z) = 1. \tag{3.8}$$

In other words, G is also analytic at infinity with $G(\infty) = 1$. Furthermore, according to (3.4),

$$|\,G(z)\,| = \left| \frac{(n+1)\, F(z)}{(z-\zeta)^{n+1}\, R(z)} \right| = \left| \frac{(n+1)\, F(z)}{c\,(z-\zeta)^{n+1}} \right| \le 1$$

for each z on Γ_c. Therefore, by the Maximum Modulus principle, G is a unimodular constant outside Γ_c. But, G has some zeros in that domain, which is absurd.

In short, the identity (3.2) never happens for all values of k. This means that

$$(z-\zeta)^n\, R(z)$$

always has a nonzero residue.

4 Concluding remarks

We conclude with two intriguing open questions, followed by a comment, that arise from the above work.

(1) Does every infinite Blaschke product possess a nonzero residue? This remains an open problem and a natural extension of Heins' result on finite Blaschke products. The behavior of residues for infinite Blaschke products is far less understood, and further investigation could yield deeper insights into the underlying structure of these functions.

(2) Can the assumption of the domain being *star-shaped* in Theorem 2.1 be removed? The requirement of a star-shaped domain plays a crucial role in our proof. However, it is unclear whether this condition is essential or if the result can be generalized to broader classes of domains. Relaxing this assumption could lead to a more comprehensive understanding of zero-pole separable rational functions.

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(3) The following example is verified numerically. However, it lacks a rigorous proof. For $n \geq 1$ and $1 \leq k \leq n$,

$$z_k = \frac{k}{n+1} + i\sqrt{1 - \left(\frac{k}{n+1}\right)^2},$$

and

$$R(z) = \frac{(z+1)^n (z+i)^n \prod_{k=1}^{n}(z - z_k)(z - \bar{z}_k)(z - iz_k)(z - i\bar{z}_k)}{z^{6n}}.$$

The nominator is so chosen that the coefficient of z^{6n-1} is zero and besides the zeros of R are placed on the arc $-\pi/2 \leq \arg z \leq \pi$ of the unit circle $\mathbb{T}$. Hence R has no nonzero residue. However, it seems that the level curves Γ_c, for some values of $c < 1$, are separating the zeros and poles of R.

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