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EXACT SOLUTIONS TO NONLINEAR DIFFUSION-CONVECTION PROBLEMS ON FINITE DOMAINS

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Abstract

New exact solutions are presented for nonlinear diffusion and convection on a finite domain $0 \leq z \leq 1$. These solutions are developed for the conditions of constant fluxes at both boundaries $z = 0$ and $z = 1$. In particular, solutions for the flux Q_L at the lower boundary $z = 1$, being a multiple of the flux Q_s at the surface $z = 0$, (that is $Q_L = aQ_s$, where $a =$ constant), are presented. Solutions for any constant, a, are given for an initial condition which is independent of space z. For the special cases (i) $a = 1$, and (ii) $Q_s = 0$ and hence $Q_L = 0$, solutions are given for an initial condition which has an arbitrary dependence on z.

1. Introduction

Consider the conservation equation

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z}, \tag{1.1}$$

where θ is the concentration, q is the flux defined by

$$q = -D(\theta)\frac{\partial \theta}{\partial z} + K(\theta), \tag{1.2}$$

and $D(\theta)$ is the concentration-dependent diffusivity. Equations (1.1) and (1.2) combined, describe nonlinear diffusion-convection processes which arise in many physical contexts, for example, porous media flow (Philip [17]), dopant diffusion in semiconductors (King [12], Tuck [24]), and the evolution of thermal waves in plasma (Grundy [9]). Without restricting the applications of this paper, I shall adopt the terminology corresponding to the flow of

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water through soil, in which case θ represents the moisture content of the soil (appropriately scaled so that $0 \le \theta \le 1$), and $K(\theta)$ is the hydraulic conductivity representing the influence of the gravitational gradient. For soil moisture flow, generally both $dK/d\theta$ and $d^2K/d\theta^2$ are positive. A value of $\theta = 0$ corresponds to a dry medium, while $\theta = 1$ corresponds to a saturated medium. For negligible gravity effects, (1.1) and (1.2) reduce to the standard nonlinear concentration-dependent diffusion equation.

In deriving exact solutions to nonlinear diffusion problems ($K = 0$), the general approach has been to seek similarity solutions. In fact, Hill [10] notes that most known exact solutions turn out to be of this type. The functional forms of diffusivities for which similarity solutions have been found include the exponential $D(\theta) = ae^{b\theta}$ (Hill, [10]), general power law $D(\theta) = a(b+\theta)^{\lambda}$, (Parlange et al, [16]) where a, b and λ are constants, the Fujita diffusivities for $\lambda = -1$ and -2 (Fujita [7], [8]), plus the recently derived exact solution for $\lambda = -4/3$ in Hill [10]. Presently, there do not appear to be any exact similarity solutions to (1.1) and (1.2) which include the nonlinear conductivity function $K(\theta)$. However, Hogarth et al [11] have used similarity methods to reduce (1.1) and (1.2) combined with the boundary and initial conditions,

$$t \ge 0, \qquad z = 0, \qquad -D(\theta)\frac{\partial \theta}{\partial z} + K(\theta) = Q_s(t), \tag{1.3a}$$

$$z \to \infty, \qquad \theta = 0, \tag{1.3b}$$

$$t = 0, \qquad z \ge 0, \qquad \theta = 0, \tag{1.3c}$$

to a two-point boundary-value problem for a power-law dependence of both D and K on θ, when the surface flux $Q_s = \alpha t^{\beta}$, where α and β are constants.

Storm [23] and Knight and Philip [15] have shown that for the Fujita diffusivity ($\lambda = -2$), exact solutions may be found without recourse to similarity methods only if the boundary conditions specified are of the flux type. Additionally, Bluman and Kumei [2] prove that the nonlinear diffusion equation is invariant under Lie-Bäcklund transformations, and that it can therefore be associated with some corresponding linear partial differential equation, if and only if the diffusivity is of the Fujita type,

$$D(\theta) = D_0/(1-\nu\theta)^2 \tag{1.4}$$

where D_0 and ν are constants with $0 \le \nu \le 1$.

Usually the exact solutions derived from Lie-Bäcklund transformations are obtained in parametric form, thus making it difficult to observe their general features. However, a considerable variety of flux boundary conditions on both semi-infinite and finite flow domains, with or without nonlinear convective terms in the flow equation, can be handled by this method, whereas these types of problems do not appear to be amenable to similarity methods. Broadbridge [3] notes that the integrability of partial differential equations requires that they must possess Lie-Bäcklund symmetries of arbitrarily high order, and shows that, while this property holds for (1.1) and (1.2) in one spatial dimension, it does not hold for even two spatial dimensions. Thus for the rest of this paper, I shall only be considering exact solutions derived for the Fujita diffusivity (1.4) by Lie-Bäcklund transformations in one spatial dimension. The most general nonlinear form of the conductivity K which in combination with (1.4) allows linearisation through the Lie-Bäcklund transformation is

$$K(\theta) = (K_1 + K_2\theta + K_3\theta^2)/(1 - \nu\theta). \tag{1.5}$$

On the semi-infinite-domain flow problem given by (1.1) and (1.3), solutions have been obtained by Broadbridge and White [5] and Sander et al [21] for a constant flux Q_s and a conductivity given by (1.5). Warrick et al [25] have looked at the same problem but with an evaporative surface flux ($Q_s < 0$) for both the initial condition of (1.3c) and a step-function initial condition. These solutions stem directly from Fokas and Yortsos [6] and Rogers et al [19] where the hydraulic functions (1.4) and (1.5) were used to describe the combined two-phase flow of oil and water subject to a constant-flux boundary condition (Sander et al. [22]). Barry and Sander [1] have recently extended these solutions to include an arbitrary time-dependent surface flux. In particular, they have presented concentration (moisture) profiles for a periodic flux behaviour at the surface boundary.

Very few solutions have been developed with respect to the finite-domain flow problems, i.e. (1.1) and (1.2) subject to the boundary and initial conditions

$$t > 0, \quad z = 0, \qquad -D(\theta)\frac{\partial\theta}{\partial z} + K(\theta) = Q_s(t), \tag{1.6a}$$

$$z = 1, \qquad -D(\theta)\frac{\partial\theta}{\partial z} + K(\theta) = Q_L(t), \tag{1.6b}$$

$$t = 0, \quad 0 \le z \le 1, \qquad \theta = \theta_i(z). \tag{1.6c}$$

Note that, without loss of generality, only the region $0 \le z \le 1$ is considered, thus making all variables and functions in (1.1), (1.2) and (1.6) dimensionless. The corresponding relationships to the dimensioned variables (starred quantities) and functions are $z = z^*/L$, $t = D_0 t^*/L^2$, $Q_s = LQ_s^*/D_0$, $Q_L = LQ_L^*/D_0$, $D(\theta) = D^*(\theta)/D_0$ and $K(\theta) = LK^*(\theta)/D_0$. In dimensionless form then, the Fujita diffusivity $D(\theta)$ will now be given by (1.4) with $D_0 = 1$ and $K(\theta)$ will still be given by (1.5) but with K_1, K_2, and K_3 taken as dimensionless.

When $K(\theta) = 0$ and $Q_s = Q_L = 0$, Knight and Philip [15] solved (1.1), (1.2) and (1.6) for the redistribution of some initial distribution of water $\theta_i(z)$. This solution was subsequently extended to include nonlinear convection by Sander et al [20] for $K_1 = K_3 = 0$ and $K_2 = K_s(1-\nu)$ in (1.5) where K_s is the saturated conductivity defined by $K(\theta = 1) = K_s$. The only other solution I am aware of that includes nonlinear convection on a finite domain is given in Broadbridge et al [4] $(K_1 = K_2 = 0, K_3 = K_s(1-\nu))$ being solutions of (1.1) and (1.2) subject to (1.6) for Q_s constant and $Q_L = 0$ with $\theta_i(z) = 0$.

This paper will centre around two problems. Firstly, when the initial water content is constant and provided Q_s and Q_L obey the relationship $Q_L = aQ_s$, it will be shown how the solution of Broadbridge et al, [4] $(a = 0)$, can be extended to include any nonzero constant flux boundary conditions. The constant a can be either positive or negative, hence water can either be draining from or entering across either boundary. For example, if we consider evaporation at the surface through $Q_s < 0$, and take $a < 0$, then water is being removed from the medium at both boundaries. Secondly, for the special case of $a = 1 (Q_s = Q_L)$, a new solution will be obtained which places no restrictions on the initial water content distribution; that is, $\theta_i(z)$ is completely arbitrary. Additionally, for Q_s and Q_L zero, and $K_1 = K_3 = 0$, the redistribution solution of Sander et al [20] is recovered.

2. Theory

Following Rogers et al [19] and Kingston and Rogers [14], the Bäcklund transformation

$$\frac{\partial \Theta}{\partial z'} = \frac{\nu}{(1-\nu\theta)^3}\frac{\partial \theta}{\partial z}, \tag{2.1a}$$

$$\frac{\partial \Theta}{\partial t'} = \frac{\nu}{(1-\nu\theta)^2}\left[\frac{\partial \theta}{\partial t} + \frac{\nu}{1-\nu\theta}\left(D\frac{\partial \theta}{\partial z} - K\right)\frac{\partial \theta}{\partial z}\right], \tag{2.1b}$$

$$dz' = (1-\nu\theta)\,dz - \nu\left(D\frac{\partial \theta}{\partial z} - K\right)dt, \tag{2.1c}$$

$$t' = t, \tag{2.1d}$$

maps (1.1), (1.2) to

$$\frac{\partial \Theta}{\partial t'} = \frac{\partial}{\partial z'}\left(D'(\Theta)\frac{\partial \Theta}{\partial z'}\right) - \frac{\partial K'(\Theta)}{\partial z'}, \tag{2.2}$$

where

$$\Theta = 1/(1-\nu\theta), \tag{2.3}$$

$$D'(\Theta) = (1-\nu\theta)^2 D(\theta), \tag{2.4a}$$

and

$$K'(\Theta) = \nu K(\theta)/(1-\nu\theta). \tag{2.4b}$$

The boundary and initial conditions (1.6) become

$$-D'(\Theta)\frac{\partial\Theta}{\partial z'} + K'(\Theta) = \nu\Theta q, \tag{2.5}$$

and

$$t' = 0, \qquad \Theta = \Theta_i(z') \tag{2.6}$$

where $q = Q_s(t')$ for (1.6a) and $q = Q_L(t')$ for (1.6b). From (1.4), (1.5), (2.3) and (2.4)

$$D'(\Theta) = 1, \tag{2.7a}$$

$$K'(\Theta) = \alpha'\Theta^2 + \beta'\Theta + \gamma', \tag{2.7b}$$

where

$$\alpha' = \nu K_1 + K_2 + K_3/\nu, \tag{2.8a}$$

$$\beta' = -(K_2 + 2K_3/\nu), \tag{2.8b}$$

and

$$\gamma' = K_3/\nu. \tag{2.8c}$$

From (A.3) in the appendix, the relationship between z' and z is given by

$$z' = \int_0^z (1-\nu\theta)\, d\overline{z} + \nu\int_0^{t'} Q_s(t)\, dt, \tag{2.9}$$

so at the boundaries $z = 0$

$$z' = \nu\int_0^{t'} Q_s(t)\, dt, \tag{2.10a}$$

and $z = 1$ (see A.9)

$$z' = \mathscr{L} + \nu\int_0^{t'} Q_L(t)\, dt, \tag{2.10b}$$

where

$$\mathscr{L} = \int_0^1 (1-\nu\theta_i(z))\, dz. \tag{2.10c}$$

From (2.9) define the new independent variables

$$x = z' - \nu\int_0^{t'} Q_s(t)\, dt, \tag{2.11a}$$

$$t = t'. \tag{2.11b}$$

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Then (2.2) subject to (2.5) and (2.6) combined with (2.7), (2.10) and (2.11) becomes

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} + [\nu Q_s(t) - \beta']\frac{\partial \Theta}{\partial x} - 2\alpha'\Theta\frac{\partial \Theta}{\partial x}, \tag{2.12}$$

subject to

$$t > 0: \quad \frac{\partial \Theta}{\partial x} - \alpha'\Theta^2 + [\nu Q_s(t) - \beta']\Theta - \gamma' = 0, \qquad x = 0 \tag{2.13a}$$

$$\frac{\partial \Theta}{\partial x} - \alpha'\Theta^2 + [\nu Q_L(t) - \beta']\Theta - \gamma' = 0, \; x = \mathscr{L} + \nu\int_0^t (Q_L - Q_s)\,d\bar{t} \tag{2.13b}$$

$$t = 0: \quad 0 \le x \le \mathscr{L}, \qquad \Theta = \Theta_i(x) \tag{2.13c}$$

where

$$x = \int_0^z (1 - \nu\theta)\,d\bar{z} \quad \text{or} \quad z = \int_0^x \Theta\,d\bar{x}. \tag{2.14}$$

Since (2.12) is just Burger's equation, then applying the Cole-Hopf transformation

$$\Theta = \frac{1}{\alpha'}\frac{1}{c}\frac{\partial c}{\partial x}, \tag{2.15}$$

and defining the new variables

$$\xi = x/\mathscr{L}, \tag{2.16a}$$

$$\tau = t/\mathscr{L}^2, \tag{2.16b}$$

$$F_s = \mathscr{L}Q_s, \qquad F_L = \mathscr{L}Q_L, \tag{2.16c}$$

$$\beta = \mathscr{L}\beta', \tag{2.16d}$$

$$\alpha = \mathscr{L}\alpha', \tag{2.16e}$$

$$\varepsilon = \mathscr{L}^2\alpha'\gamma', \tag{2.16f}$$

the equations (2.12) and (2.13) become

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \xi^2} + (\nu F_s - \beta)\frac{\partial c}{\partial \xi}, \tag{2.17}$$

subject to

$$\tau > 0: \frac{\partial c}{\partial t} + \varepsilon c = 0, \qquad \xi = 0, \tag{2.18a}$$

$$\frac{\partial c}{\partial \tau} + \nu[F_L - F_s]\frac{\partial c}{\partial \xi} + \varepsilon c = 0, \qquad \xi = 1 + \nu\int_0^\tau (F_L - F_s)\,d\bar{\tau}, \tag{2.18b}$$

$$\tau = 0: c(\xi) = \exp\left(-\alpha\int_0^\xi \Theta_i(\bar{\xi})\,d\bar{\xi}\right), \qquad 0 \le \xi \le 1. \tag{2.18c}$$

The solution of (2.17) and (2.18) for $c(\xi, \tau)$ then allows $\theta(z, \tau)$ to be calculated parametrically from (2.3), (2.14), (2.15) and (2.16) as

$$\theta(\xi, \tau) = \frac{1}{\nu}\left[1 + \alpha c(\xi, \tau)\left(\frac{\partial c}{\partial \xi}\right)^{-1}\right], \tag{2.19a}$$

and

$$z(\xi, \tau) = -\frac{\mathscr{L}}{\alpha} \ln\left[\frac{c(\xi, \tau)}{c(0, \tau)}\right]. \tag{2.19b}$$

Thus the nonlinear diffusion-convection equation, (1.1) and (1.2), subject to the boundary and initial conditions (1.6), has now been transformed to the corresponding *linear* system of (2.17) and (2.18). For the rest of this paper however, we shall consider only constant-flux boundary conditions, so that the position of the moving boundary in (2.18b) is now given by $\xi = 1 + \nu(F_L - F_s)\tau$.

3. Equal constant fluxes ($F_s = F_L \neq 0$ or $F_s = F_L = 0$)

By taking

$$F_L = F_s, \tag{3.1}$$

in (2.17) and (2.18), the moving boundary is eliminated and the following simpler linear system is obtained:

$$\frac{\partial c}{\partial \xi} = \frac{\partial^2 c}{\partial \xi^2} + (\nu F_s - \beta)\frac{\partial c}{\partial \xi}, \tag{3.2}$$

$$\tau > 0: \quad c = e^{-\epsilon\tau}, \qquad \xi = 0, \tag{3.3a}$$

$$c = e^{-\alpha/\mathscr{L} - \varepsilon\tau}, \qquad \xi = 1, \tag{3.3b}$$

$$\tau = 0: c(\xi) = \exp\left(-\alpha \int_0^{\xi} \Theta_i(\bar{\xi})d\bar{\xi}\right) = h(\xi), \qquad 0 \le \xi \le 1. \tag{3.3c}$$

Physically, the solution of (3.2) and (3.3) which has the most practical interest occurs for $F_L = F_s = 0$, and describes the subsequent nonhysteretic redistribution of soil moisture following the cessation of rainfall. In general, the redistribution of soil moisture involves capillary hysteresis. To model hysteretic redistribution, numerical schemes must be used, as it is not possible to obtain analytical solutions. However, the size of such hysteresis loops tends to be small for fine-textured as opposed to coarse-textured soils (Watson and Sardana [27]), and for *in situ* field soils as compared to laboratory soil columns (Watson et al [26]). Ideally then, the solution is suitable for fine textured, *in situ*, field soils where hysteresis effects are minimal. Additionally, this solution is of significant value for investigating the accuracy of the numerical schemes designed to model hysteretic redistribution, as in Watson and Sardana [27], when they are run for nonhysteretic redistribution. The solution for $F_L = F_s \neq 0$ though, is of a much more mathematical than practical interest, and is presented more for completeness as it requires no extra effort to obtain. Let

$$\beta^* = \nu F_s - \beta \tag{3.4}$$

and define the new variable

$$w(\xi, \tau) = e^{\varepsilon\tau} c(\xi, \tau), \tag{3.5}$$

so that (3.2) and (3.3) become

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial \xi^2} + \beta^* \frac{\partial w}{\partial \xi} + \varepsilon w, \tag{3.6}$$

$$\tau > 0: \quad w = 1, \qquad \xi = 0, \tag{3.7a}$$

$$w = e^{-\alpha/\mathscr{L}}, \qquad \xi = 1, \text{ and} \tag{3.7b}$$

$$\tau = 0: \quad w = h(\xi), \qquad 0 \le \xi \le 1. \tag{3.7c}$$

The solution of (3.6) and (3.7) can then be written as

$$w(\xi, \tau) = \mu(\xi) + \nu(\xi, \tau), \tag{3.8}$$

where $\mu(\xi)$ satisfies

$$\frac{d^2\mu}{d\xi^2} + \beta^* \frac{d\mu}{d\xi} + \varepsilon\mu = 0, \tag{3.9a}$$

$$\mu = 1, \qquad \xi = 0, \tag{3.9b}$$

$$\mu = e^{-\alpha/\mathscr{L}}, \qquad \xi = 1, \tag{3.9c}$$

and $v(\xi, \tau)$ satisfies

$$\frac{\partial \nu}{\partial \tau} = \frac{\partial^2 \nu}{\partial \xi^2} + \beta^* \frac{\partial \nu}{\partial \tau} + \varepsilon\nu, \tag{3.10}$$

$$\tau > 0: \nu = 0, \qquad \xi = 0, \tag{3.11a}$$

$$\nu = 0, \qquad \xi = 1, \text{ and} \tag{3.11b}$$

$$\tau = 0: \nu = h(\xi) - \mu(\xi), \qquad 0 \le \xi \le 1. \tag{3.11c}$$

The form of the solution of (3.9) depends on the sign of $\beta^{*2}/4 - \varepsilon$ and there are three cases to consider.

(i) $\beta^{*2}/4 - \varepsilon > 0$

$$\mu(\xi) = e^{m_2\xi} + \left(\frac{e^{-\alpha/\mathscr{L}} - e^{m_2}}{e^{m_2} - e^{m_1}}\right)(e^{m_2\xi} - e^{m_1\xi}), \tag{3.12a}$$

where

$$m_1 = \frac{-\beta^*}{2} - \left(\frac{\beta^{*4}}{4} - \varepsilon\right)^{1/2}, \tag{3.12b}$$

$$m_2 = \frac{-\beta^*}{2} + \left(\frac{\beta^{*2}}{4} - \varepsilon\right)^{1/2}. \tag{3.12c}$$

(ii) $\beta^{*2}/4 - \varepsilon = 0$

$$\mu(\xi) = e^{-\frac{\beta^*}{2}\xi}\left[1 - \xi + \xi e^{\left(\frac{\beta^*}{2} - \frac{\alpha}{\mathscr{L}}\right)}\right]. \tag{3.13}$$

(iii) $\beta^{*2}/4 - \varepsilon < 0$

$$\mu(\xi) = e^{-\frac{\beta^*}{2}\xi}\{\cos(r\xi) + A\sin(r\xi)\}, \tag{3.14a}$$

where

$$r = \left(\varepsilon - \frac{\beta^{*2}}{4}\right)^{1/2}, \tag{3.14b}$$

$$A = \frac{\left[e^{\beta^*/2-\alpha/\mathscr{L}} - \cos(r)\right]}{\sin(r)}. \tag{3.14c}$$

Without loss of generality $K(\theta = 0) = K_1 = 0$, then from (2.8), (2.16) and (3.4)

$$\frac{\beta^{*2}}{4} - \varepsilon = \frac{\nu^2 F_s^2}{4} + \frac{K_2^2}{4} + \frac{\nu\mathscr{L}}{2}F_s\left(K_2 + \frac{2K_3}{\nu}\right). \tag{3.15}$$

For $F_s = F_L = 0$, $\beta^{*2}/4 - \varepsilon$ is always positive and $\mu(\xi)$ will be given by (3.12), however if $F_s < 0$ then it is possible that $\beta^{*2}/4 - \varepsilon \leq 0$ and the solution for $\mu(\xi)$ will be given by either (3.13) or (3.14) accordingly.

Using separation of variables, the solution of (3.10) and (3.11) for $v(\xi, \tau)$ is found to be

$$\nu(\xi, \tau) = \sum_{n=1}^{\infty} b_n \exp\left[-\frac{\beta^*\xi}{2} + \left(\varepsilon - \frac{\beta^{*2}}{4} - n^2\pi^2\right)\tau\right]\sin(n\pi\xi), \tag{3.16}$$

where the Fourier coefficients b_n are given by

$$b_n = 2\int_0^1 [h(\xi) - \mu(\xi)]e^{\beta^*\xi/2}\sin(n\pi\xi)d\xi. \tag{3.17}$$

From (2.19) and (3.5), $\theta(z, \tau)$ is given parametrically by

$$\theta(\xi, \tau) = \frac{1}{\nu}\left[1 + \alpha w(\xi, \tau)\left(\frac{\partial w}{\partial \xi}\right)^{-1}\right], \tag{3.18}$$

$$z(\xi, \tau) = -\frac{\mathscr{L}}{\alpha}\ln[w(\xi, \tau)], \tag{3.19}$$

where

$$w(\xi, \tau) = \mu(\xi) + \nu(\xi, \tau), \tag{3.20}$$

with $\mu(\xi)$ given by either (3.12), (3.13) or (3.14) and $v(\xi, \tau)$ given by (3.16) and (3.17) above.

Firstly consider the redistribution solution which occurs for $F_L = F_s = 0$ (i.e. $\beta^* = -\beta$ from (3.4)) and represents the extension of the solution of Sander et al [20] ($\varepsilon = 0$) to include both K_2 and K_3 nonzero. The redistribution solution of Knight and Philip [15] (no convection) is obtained by taking the limit $\beta^* = \alpha = \varepsilon = 0$ in (3.16) to (3.20) along with (3.12). To demonstrate the solution take an initial profile $\theta_i(z)$ as follows (Sander et al [20])

$$z = \frac{0.285(1-\theta)}{(1-0.95\theta)} - 0.015\ln\left(\frac{0.05\theta}{1-0.95\theta}\right), \tag{3.21}$$

then the subsequent redistribution of this profile with $\nu = 0.85$, and for simplicity $K_s = 20$, $K_3 = 0$ and $K_2 = K_s(1-\nu) = 3$ is shown in Figure 1. The value of β^* is given from (2.8b), (2.16d) and (3.4) as $\beta^* = \mathscr{L}K_2 = 3\mathscr{L}$. From (2.10c) and (3.21) above $\mathscr{L} = 0.745$ thus $\beta^* = 2.235$. Lastly from (2.8a) and (2.16e) $\alpha = \mathscr{L}K_2 = \beta^* = 2.235$ also.

The numbers on the curves in Figure 1 correspond to the various times (t) at which these profiles are realised, with $t = \infty$ being the steady-state profile. For this same initial moisture distribution, the effects of the parameters β^*, $\alpha = \beta^*$, and ν are as follows. Increasing β^* is equivalent to increasing the saturated conductivity K_s, if ν is kept fixed, and results in water draining from the surface much more quickly, with more water reaching deeper into the soil profile. When K_s is fixed, then as ν increases then so too does nonlinearity of $D(\theta)$ and $K(\theta)$ given by (1.4) and (1.5). This reduces the influence of the gravity gradients, resulting in increased capillary effects and more water remaining near the surface.

The solution of (3.16) to (3.20), when $F_L = F_s \neq 0$ (i.e. $\beta^* = \nu F_s - \beta$ in (3.4)) will only apply as long as the boundary conditions can be maintained physically, that is, until one of the boundaries either dries $(\theta = 0)$ or saturates $(\theta = 1)$ depending on the sign of F_L or F_s. For example if $F_s > 0$ then the surface $z = 0$ is wetting while the boundary $z = 1$ is drying. Thus the solution applies as long as $\theta(z = 0, \tau) < 1$ and $\theta(z = 1, \tau) > 0$, with

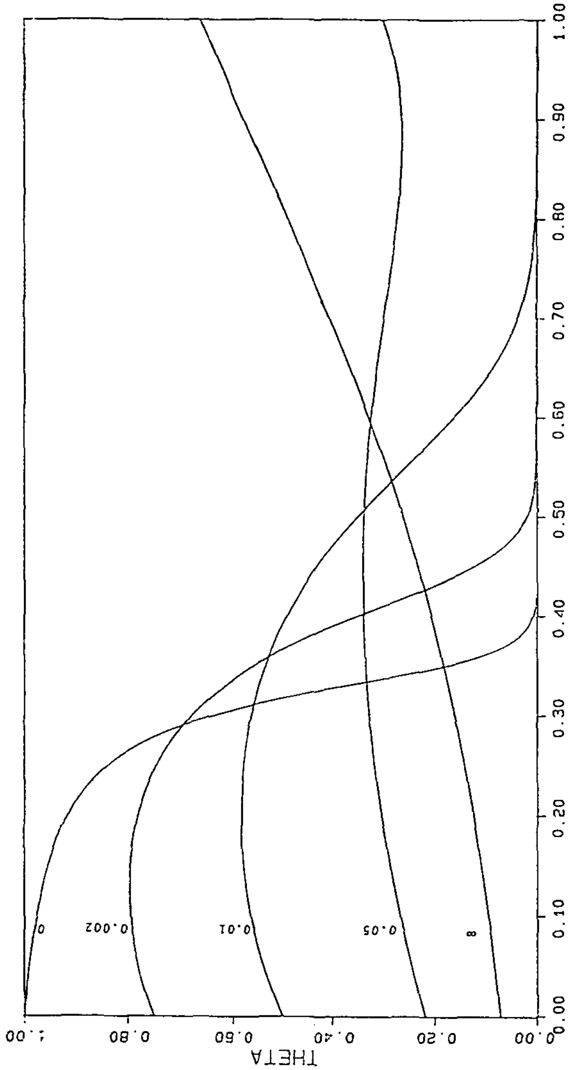

FIGURE 1. Redistribution of water within the soil layer from an initial concentration profile $\Theta_i(z)$, $(t = \mathcal{L}^2\tau)$. Numbers on the curves represent the times when the profiles are realised.

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the precise times at which either of these conditions are violated being found iteratively from (3.16) to (3.20)).

4. Asymptotic solution for redistribution $(F_s = F_L = 0)$

Consider the asymptotic limit $\tau \to \infty$ for the redistribution solution ($F_s = F_L = 0$, $\beta^* = -\beta$) of the previous section. As $\tau \to \infty$, $v \to 0$ and $w \to u(\xi)$ with $\mu(\xi)$ given by (3.12). It has already been noted before, that for this steady-state limit to hold for soil moisture flow, the water content at $z = 1$ must remain less than or equal to one. Thus from (3.12) and (3.18)

$$1 + \alpha\mu\left(\frac{\partial \mu}{\partial \xi}\right)^{-1}\Bigg|_{\xi=1} \le \nu\,, \tag{4.1}$$

or

$$\alpha e^{-\alpha/\mathscr{L}}(e^{m_2} - e^{m_1}) \le (\nu - 1)\left[(m_1 - m_2)e^{(m_1+m_2)} + e^{-\alpha/\mathscr{L}}(m_2 e^{m_2} - m_1 e^{m_1})\right]. \tag{4.2}$$

Whether or not (4.2) will hold depends on both the volume of water in the initial profile, expressed through $\mathscr{L}$ in (2.10c), and the saturated conductivity $K_s = (K_2 + K_3)/(1-\nu)$, expressed through α, β and ε, being a measure of capacity of soil to conduct water to the lower boundary. For the example given in the previous section, (4.2) holds and the steady-state solution is shown labelled as $\tau = \infty$. If (4.2) does not hold, then (3.16) to (3.20) only apply until the time taken for the lower boundary to saturate, defined as $\theta(z = 1, \tau = \tau_s) = 1$. The value of τ_s is found iteratively from (3.16) to (3.20). Even in the steady-state limit, ξ cannot be eliminated between (3.18) and (3.19) to obtain $\theta(z)$ or $z(\theta)$ except when $\varepsilon = 0$ as in Sander et al. [20]. For $\varepsilon = 0$, $K_3 = 0$, $K_2 = K_2(1-\nu)$, $m_2 = 0$, $m_1 = -\beta^* = \beta = -K_2$, $\alpha = -\beta$ and (3.12a), (3.18) and (3.19) give $z(\theta)$ as

$$z = \frac{\mathscr{L}}{\beta}\ln\left\{\frac{(1-\nu\theta)\left(e^{\beta} - e^{\beta/\mathscr{L}}\right)}{\nu\theta(1 - e^{\beta})}\right\}, \tag{4.3}$$

and (4.2) simplifies to

$$(1 - \nu e^{\beta})e^{\beta/\mathscr{L}} \le (1 - \nu)e^{\beta}\,, \tag{4.4}$$

which are equivalent to (37) and (38) of Sander et al [20].

5. Unequal constant boundary fluxes $(F_L = aF_s)$

Consider a two-layer soil system, where both layers are initially dry. The top layer represents a ploughed soil which consists of large pore spaces, is coarse textured and has a high saturated conductivity. The lower layer is then the plough pan which is tightly compacted, fine textured and has a very low saturated conductivity. If this field is irrigated until the top layer is saturated and water begins to penetrate the bottom layer, the flux at the boundary between the two layers will quickly approach a constant value given by the saturated conductivity of the lower fine textured layer. Because there is now plenty of water available in the coarse layer, then a constant drainage approximation into the fine lower layer is reasonable. After the cessation of irrigation, water can then also evaporate from the surface of the top layer at the constant potential evaporation rate governed by the prevailing atmospheric conditions. Therefore under this physical situation the fluxes at both boundaries differ only by a constant value.

To solve this problem we first consider the solution of Broadbridge et al [4] where (2.17) and (2.18) were solved for $F_L = 0$ $(a = 0)$, F_s constant and the soil is initially dry, i.e. $\theta_i = 0$ $(\Theta_i = 1)$. Under these conditions the moving boundary remains so that (2.18b) and (2.18c) become

$$\tau > 0: \quad \frac{\partial c}{\partial \tau} - \nu F_s \frac{\partial c}{\partial \xi} + \varepsilon c = 0, \qquad \xi = 1 - \nu F_s \tau, \tag{5.1a}$$

$$\tau = 0: \quad c(\xi) = e^{-\alpha\xi}, \qquad 0 \le \xi \le 1. \tag{5.1b}$$

The solution of (5.1a) is given by

$$c = A \exp(-[\alpha\xi + (\varepsilon + \alpha\nu F_s)\tau]), \tag{5.2}$$

where $A = 1$ to satisfy the initial condition (5.1b). By applying (5.2) at the boundary $\xi = 1 - \nu F_s \tau$, (5.1a) is replaced by

$$\tau > 0, \qquad c = e^{-(\alpha + \varepsilon\tau)}, \qquad \xi = 1 - \nu F_s \tau. \tag{5.3a}$$

Similarly, (2.18a) can be integrated and replaced by

$$\tau > 0, \qquad c = e^{-\varepsilon\tau}, \qquad \xi = 0. \tag{5.3b}$$

The set of equations (2.17), (5.1b) and (5.3) is the problem solved by Broadbridge et al [4] using the Laplace-transform boost method of King [13].
It is a fairly simple procedure now to extend the solution of the above problem to the desired nonzero constant lower boundary fluxes F_L, provided that F_L and F_s are related through

$$F_L = aF_s \tag{5.4}$$

where a is some constant being either positive or negative. Taking $F_L = aF_s$ in (2.18b) yields

$$\frac{\partial c}{\partial \tau} - \nu(1-a) F_s \frac{\partial c}{\partial \xi} + \varepsilon c = 0, \qquad \xi = 1 - \nu(1-a) F_s \tau. \tag{5.5}$$

For a nonzero flux to exist at the lower boundary, then a nonzero initial moisture distribution is required, hence taking $\theta_i = \theta_i^* =$ constant in (2.3) ($\Theta_i = 1/(1-\nu\theta_i^*)$), (2.18c) yields

$$\tau = 0, \qquad c(\xi) = \exp\left(\frac{-\alpha\xi}{1-\nu\theta_i^*}\right), \qquad 0 \le \xi \le 1. \tag{5.6}$$

By defining

$$F_s^* = (1-a)F_s, \tag{5.7a}$$

$$\beta^* = \beta - \nu a F_s, \tag{5.7b}$$

and

$$\alpha^* = \alpha/(1-\nu\theta_i^*), \tag{5.7c}$$

(2.17), (2.18a), (5.4), (5.5) and (5.6) become

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \xi^2} + (\nu F_s^* - \beta^*)\frac{\partial c}{\partial \xi}, \tag{5.8}$$

$$\tau > 0: \quad c = e^{-\varepsilon\tau}, \qquad \xi = 0, \tag{5.9a}$$

$$c = e^{-\alpha^* - \varepsilon\tau}, \qquad \xi = 1 - \nu F_s^* \tau, \tag{5.9b}$$

and

$$\tau = 0: \quad c(\xi) = e^{-\alpha^*\xi}, \qquad 0 \le \xi \le 1, \tag{5.9c}$$

which is again exactly the same set of equations solved by Broadbridge et al [4]. Since the analytic solution of (5.8) and (5.9) is very involved and complex, it is not repeated here, but given in detail in Broadbridge et al [4]. The special case of $a = 0$ and $\theta_i = 0$ returns the original results of Broadbridge et al [4]. These new solutions will again only apply as long as the boundary conditions can be maintained physically, that is, until θ reaches either 0 or 1 at either boundary.

For the situation described at the beginning of this section where $F_s < 0$ and given by the potential evaporation rate, and F_L equals the saturated conductivity of the lower layer, then $a < 0$. The solution of (5.8) and (5.9) for this problem will apply until the time taken for the soil surface ($z = 0$) to dry and is given by τ_d. That is until $\theta(\xi = 0, \tau = \tau_d) = 0$ is satisfied. Since the lower layer is compacted, it is extremely unlikely that the boundary $z = 1$ will dry before the soil surface does. However this will not be true under all conditions.

For example, consider the following situation, which could be set up in the laboratory by applying a suction across the lower boundary. Take F_s and F_L positive (i.e. $a > 0$); then water enters the soil at the surface $z = 0$ and

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drains from the soil at $z = 1$. Initially, $\partial\theta/\partial z = 0$ and the flux of water in the soil is $K(\theta_i)$ from (1.2). Equation (1.2) also shows that the greatest flux which the soil can accommodate without the help of a concentration gradient is $K(\theta = 1) = K_s$, the saturated conductivity, thus if $F_s > K_s$ then the soil surface must saturate $(\theta = 1)$ in finite time τ_p. If $F_L > K_s$, the lower boundary will be extracting water at a greater rate than it receives water and must therefore dry at some time τ_d. The solution of (5.8) and (5.9) only applies until $\tau = \tau_p$ if $\tau_p < \tau_d$ or $\tau = \tau_d$ if $\tau_d < \tau_p$. For $F_s < K_s$ and $F_L < K(\theta_i)$, the surface will never saturate but the lower boundary will, since water is draining at a slower rate than which it is arriving. The solution of (5.8) and (5.9) under these circumstances will only apply until the time of saturation of the lower boundary. Similar comparisons between the relevant times of drying or saturation of either boundary would need to be considered for any other combination of boundary fluxes.

While physically it is true that time-varying lower boundary fluxes are more prevalent, it is not possible to obtain analytical solutions for these types of boundary conditions and numerical techniques are required. Thus the analytical solution for F_L constant then provides an exact nonlinear solution which can be used to determine the accuracy of any numerical technique developed for time-dependent boundary fluxes F_L.

6. Conclusion and future research

This paper has been concerned with the development of new exact solutions using Lie-Bäcklund transformations for nonlinear diffusion and convection. In particular, flow in finite domains and constant flux boundary conditions have been considered. These solutions are derived especially for the flux at the lower boundary F_L being any multiple of the surface flux F_s and a uniform initial moisture distribution, and for equal boundary fluxes with an arbitrary space-dependent initial moisture distribution. A variety of physical situations can now be modelled with these new solutions; for example, water drainage into a two-layer coarse over fine soil profile while water is evaporating at the surface. Secondly, if the soil layer is overlaying impermeable bedrock, then it would be possible to model the movement of water under constant-rate rainfall and the subsequent redistribution following the cessation of rain. Obviously, there are many more useful situations which can be modelled by simply varying the sign of the fluxes at either boundary.

In nature, rainfall rates are not always constant, and one area of future research is to look for solutions which allow surface fluxes which depend on time. This has already been achieved quite recently on the semi-infinite domain (i.e. for (1.3)) by Barry and Sander [1], but as yet there are no solutions to the finite-domain problem of (1.6) for time-dependent fluxes. Solutions to this problem will of course be beneficial in irrigation scheduling for ensuring adequate water availability for crop growth.

Appendix

Following Rogers [18], the relationship between z and z' is obtained from (2.1c) as

$$\frac{\partial z'}{\partial t} = -\nu\left(D\frac{\partial c}{\partial z} - K\right) = -\nu\int_0^z \frac{\partial}{\partial \bar{z}}\left(D\frac{\partial c}{\partial \bar{z}} - K\right) d\bar{z} + \nu Q_s(t), \tag{A.1}$$

or using (1.1) and (1.2) in the integrand of (A.1)

$$\frac{\partial z'}{\partial t} = \int_0^z \frac{\partial}{\partial t}(1-\nu\theta)\, d\bar{z} + \nu Q_s(t), \tag{A.2}$$

which becomes, after integrating with $z'(0, 0) = 0$,

$$z' = \int_0^z (1-\nu\theta)\, d\bar{z} + \nu\int_0^t Q_s(\bar{t})\, d\bar{t}. \tag{A.3}$$

Hence at the boundaries $z = 0$ and $z = 1$

$$z' = \nu\int_0^t Q_s(\bar{t})\, d\bar{t}, \tag{A.4}$$

and

$$z' = \int_0^1 (1-\nu\theta)\, dz + \nu\int_0^t Q_s(\bar{t})\, d\bar{t}, \tag{A.5}$$

respectively. Since the z-dependent integral in (A.5) is a difficult quantity to evaluate in its present form, we seek an alternative expression. To do this, we apply (A.1) at both boundaries

$$\left[\nu\left(-D\frac{\partial\theta}{\partial x} + K\right)\right]_0^1 = \left[\int_0^z \frac{\partial}{\partial t}(1-\nu\theta)d\bar{z} + \nu Q_s(t)\right]_0^1, \tag{A.6}$$

and combining with (1.6) gives

$$\nu[Q_L(t) - Q_s(t)] = \frac{\partial}{\partial t}\int_0^1 (1-\nu\theta)\, dz. \tag{A.7}$$

Integrating (A.7) from zero to t and using the initial condition $\theta = \theta_i(z)$ at $t = 0$ yields the desired new expression

$$\int_0^1 (1-\nu\theta)\, dz = \int_0^1 (1-\nu\theta_i(z))\, dz + \nu\int_0^t Q_L(\bar{t}) - Q_s(\bar{t})\, d\bar{t}, \tag{A.8}$$

and (A.5) can be written as

$$z'(z=1) = \int_0^1 (1 - \nu\theta_i)\, dz + \nu \int_0^t Q_L(\bar{t})\, d\bar{t}. \tag{A.9}$$

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