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ON UNIFORMLY CONTRACTIVE SYSTEMS AND QUADRATIC EQUATIONS IN BANACH SPACE

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The solution of quadratic equations using the contraction mapping principle is considered. A uniqueness result extending that given by Argyros is proved. Uniformly contractive systems theory is used to find approximate solutions and convergence criteria are given. In particular, only pointwise convergence of approximating operators is required to guarantee convergence of the approximate solutions. A theorem and algorithm for a continuation method are presented, and illustrated on Chandrasekhar's equation.

1. INTRODUCTION

We are interested in solving the quadratic equation:

$$x = y + B(x, x) \tag{1.1}$$

for $x \in X$, where X is a Banach space, $y \in X$ is fixed, and $B : X \times X \to X$ is a bounded bilinear operator. Equations of this form appear frequently in applications, such as scattering theory [**6**], elasticity theory [**1**] and the study of radiative transfer [**5, 2**]. They are of particular interest in systems theory, where so-called "multi-power" equations can be analyzed using properties of multilinear operators [**8, 12**].

Methods for solving equation (1.1) include series solutions (see [**9, 3**] and iterative schemes. McFarland [**7**] obtained convergence criteria for the iterative scheme

$$x_{n+1} = (I - Bx_n)^{-1} y. \tag{1.2}$$

In [**12**], the author and Van Fleet used a similar routine to solve a broader class of equations. We also introduced *uniformly contractive systems* as a framework for guaranteeing that certain approximate solutions in finite-dimensional subspaces would converge to the solution of (1.1).

Another iterative scheme for solving (1.1) is

$$x_{n+1} = B(x_n, x_n) + y. \tag{1.3}$$

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Convergence, existence, and uniqueness results for this approach have been obtained elsewhere (for example, [**1**, **2**]) using the contraction mapping principle. We extend a uniqueness result given in [**2**], but are mainly interested in a variation on the scheme (1.3) that will produce good approximate solutions and avoid iterating in infinite dimensional space. A standard approach for this is to use successive subspaces $\{V_n\}$ and approximate the solution to the problem in finite dimensional settings. Uniformly contractive systems will be used to show that these finite dimensional approximations do indeed converge to the true solution of (1.1). To formulate the finite dimensional approximating scheme, we shall assume that Banach space X satisfies the following condition.

(V) Suppose that X has a sequence of proper subspaces $\{V_n\}$ and linear projections $P_n : X \to V_n$ for which

$$\lim P_n x = x$$

for each $x \in X$.

We make the following observations.

- $\mu = \sup \|P_n\| < \infty$ since X is complete.
- The subspaces need not be nested, so finite element methods may be applied.
- Any space X with a Schauder basis satisfies condition (V).

The spaces $V_n = P_n(X)$ are usually taken to be finite dimensional, and the equation (1.1) is replaced by

$$x = P_n B(x, x) + P_n y \tag{1.4}$$

which is solved in V_n using the iterative method

$$x_{k+1} = P_n B(x_k, x_k) + P_n y. \tag{1.5}$$

Uniformly contractive systems will be used in Section 2 to show that $z_n \to z_s$, where z_s solves (1.1) and the z_n solve (1.4). These results require the map B only to be bounded and bilinear so the finite rank operators $P_n B$ need only converge pointwise to B. If B is compact, a routine that avoids solving for any of the z_n will be shown to converge to z_s.

Recall that a bilinear operator $B : X \times X \to X$ is *compact* if for any bounded set $S \subset X$, the set $B(S, S)$ is relatively compact. We shall need the following result in Section 2.

Proposition 1.1. *Suppose that X satisfies condition (V). If B is compact, then*

$$\lim \|P_n B - B\| = 0.$$

PROOF: It is clear that P_n converges uniformly to the identity map I on relatively compact sets. Since B is a compact map, for any bounded set $S \subset X$, the set $B(S)$ is relatively compact. Hence

$$\|(P_nB - B)(S)\| = \|(P_n - I)B(S)\| \to 0. \tag{1.6}$$

□

For more details on compact bilinear maps and their applications, see [**11**] and [**4**].

We conclude the paper with a section on approximating solutions using a continuation method similar to that given by Argyros [**2**]. This method is illustrated on Chandrasekhar's equation

$$H(s) = 1 + \lambda H(s) \int_0^1 \frac{s}{s+t} H(t)dt \tag{1.7}$$

and increases the range of positive values of λ for which (1.7) can be solved from 0.424059 given in [**2**] to 0.473571.

2. Solutions to Quadratic Equations

We begin by defining and giving relevant theorems for a uniformly contractive system (UCS). The notion of a UCS was developed and used in [**12**] to provide a general framework for obtaining iterative solutions to a class of multipower equations. We shall use the concept of the UCS in conjunction with the scheme (1.5) discussed in the introduction to construct approximate solutions to equation (1.1). Theorems 2.2, 2.3 and 2.4 stated below are proved in [**12**].

DEFINITION 2.1: Let X be a Banach space, $\{V_n\}$ a sequence of subspaces of X such that

$$\lim_{n\to\infty} dist\,(V_n, x) = 0 \tag{2.1}$$

for each $x \in X$. Let U be a closed set in X and define the sets $U_n = V_n \cap U$ and the operators $Q_n : X \to V_n$. We say that $\{U_n, Q_n\}$ is a *uniformly contractive system* (UCS) if conditions (1) and (2) below hold.

1. There exists a $c \in R$, $0 < c < 1$, and an $N \in \mathbb{N}$ such that if $n \geqslant N$ and $x, y \in U$, then $Q_n(U) \subset U_n$ and $\|Q_n(x) - Q_n(y)\| \leqslant c\,\|x - y\|$.
2. For any $x, y \in U$ and $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $k \geqslant j \geqslant N$ then $\|Q_k(x) - Q_j(y)\| \leqslant c\,\|x - y\| + \varepsilon$.

Note that a space X satisfying condition (V) will satisfy (2.1).

Theorem 2.2. *Let $\{U_n, Q_n\}$ satisfy (1) above. Then condition (2) is equivalent to the existence of a contraction map $Q : U \to U$, defined by $Q(x) = \lim_{n\to\infty} Q_n(x)$, such that*

$$\|Q(x) - Q(y)\| \leqslant c\,\|x - y\|$$

for $x, y \in U$.

We observe that the equations $Q_n(x) = x$ all have unique fixed points $z_n \in U$ by the contraction mapping principle. The next theorem shows that these fixed points converge to z_s, the unique fixed point of the map Q on U.

Theorem 2.3. *Let $\{U_n, Q_n\}$ be a UCS. Then*

$$\lim_{n\to\infty} z_n = z_s, \tag{2.2}$$

where

$$Q(z_s) = z_s. \tag{2.3}$$

Observe that the operators Q_n need not converge uniformly for Theorem 2.3 to hold. If there is uniform convergence, we have the following.

Theorem 2.4. *Let $\{U_n, Q_n\}$ be a UCS such that U is bounded and $\{Q_n\}$ converges to Q uniformly on U. Let $N \in \mathbb{N}$ be given as per condition (1) of Definition 2.1. Beginning with any $k \geqslant N$ and initial guess $x_k \in U_k$, the iterative scheme*

$$x_{n+k+1} = Q_n(x_{n+k}) \tag{2.4}$$

will converge to the fixed point of Q in U:

$$\lim_{n\to\infty} x_{n+k} = z_s = Q(z_s). \tag{2.5}$$

Note that no solution in any individual V_n space need be found for this iterative routine to converge.

In order to apply these UCS results to solve the quadratic equation (1.1) in a space X satisfying condition (V), we define $Q : X \to X$ by

$$Q(x) = B\,(x, x) + y \tag{2.6}$$

and $Q_n : X \to V_n$ by

$$Q_n(x) = P_n B\,(x, x) + P_n y. \tag{2.7}$$

We now give sufficient conditions on B under which the hypotheses for Theorem 2.4 hold.

Proposition 2.5. *Suppose that X satisfies condition (V). If $B : X \times X \to X$ is compact then $\{Q_n\}$ converges uniformly to Q on any bounded set.*

Proof: Apply Proposition 1.1. ∎

Recall that the Fréchet derivative B' of a bilinear operator B is defined by

$$B'(x)(u) = B(x,u) + B(u,x).$$

Note that the maps B' and $B'(x)$ are both linear. We shall make use of the following identities in the sequel.

$$B(u,u) - B(v,v) = B'\left(\frac{u+v}{2}\right)(u-v). \tag{2.8}$$

In the case where $v = 0$, this simpifies to

$$B(u,u) = B'\left(\frac{u}{2}\right)(u). \tag{2.9}$$

In order to prove our uniqueness claim below we require the following theorem, which is a variation on a result due to Rall [**10**].

Theorem 2.6. *Any solution $z \in C = \{x : \|B'(x)\| < 1\}$ to equation (1.1) is unique in C.*

Proof: If $z_1, z_2 \in C$ are solutions to (1.1), then $z_1 - z_2 = B(z_1, z_1) - B(z_2, z_2)$. By the identity (2.8) we have

$$\|z_1 - z_2\| \leqslant \left\|B'\left(\frac{z_1+z_2}{2}\right)\right\| \cdot \|z_1 - z_2\|. \tag{2.10}$$

Note that C is convex by the linearity of B', so $(z_1+z_2)/2 \in C$. By hypothesis, inequality (2.10) can only be true if $z_1 = z_2$. ∎

We can now prove the main result of this section.

Theorem 2.7. *Suppose that X satisfies condition (V). Let $B : X \times X \to X$ be bounded and bilinear, with $y, z \in X$. Define $Q : X \to X$ by*

$$Q(x) = B(x,x) + y.$$

Suppose that

$$a = \frac{1 - \mu\|B'z\|}{\mu\|B'\|} > \sqrt{\frac{2\|Q(z) - z\|}{\|B'\|}}. \tag{2.11}$$

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Then

(1) $Q(x)$ *has a unique fixed point* z_s *in the set* $C = \{x : \mu\|B'x\| < 1\}$.

(2) *This fixed point* z_s *lies in* $\overline{S}(z,b)$, *where*

$$b = a - \sqrt{a^2 - \frac{2\|Q(z) - z\|}{\|B'\|}}. \tag{2.12}$$

(3) *The equations* $Q_n(x) = x$ *have solutions* z_n *for sufficiently large* n, *and these solutions converge to the fixed point* z_s *of* $Q(x)$. *These solutions are unique in* C, *and lie in* $\overline{S}(z,b_n)$, *where*

$$b_n = a - \sqrt{a^2 - \frac{\mu 2\|Q(z) - z\| + \|P_n z - z\|}{\mu\|B'\|}}.$$

(4) *If* B *is compact, then the iterative scheme given in Theorem 2.4 converges to the solution* z_s *of (1.1).*

PROOF: Choose $N \in \mathbb{N}$ so that

$$a^2 > \frac{2(\|P_n z - z\| + \mu\|Q(z) - z\|)}{\mu\|B'\|}$$

for all $n \geqslant N$. Choose $r \in [b,a)$. For $x,w \in \overline{S}(z,r)$ let $\delta = (x+w)/2 - z$. We have

$$\begin{aligned}\|Q_n(x) - Q_n(w)\| &= \left\|B_n'\left(\frac{x+w}{2}\right)(x-w)\right\| \\ &\leqslant \mu\|B'(z+\delta)\| \cdot \|(x-w)\| \\ &\leqslant \mu(\|B'z\| + \|B'\|r)\|x-w\|. \end{aligned} \tag{2.13}$$

Put

$$c = \mu(\|B'z\| + \|B'\|r). \tag{2.14}$$

Now $c < 1$ by the definition of r and a, so Q_n is a contraction on $\overline{S}(z,r)$ for $n \geqslant N$.

To see that $Q_n\left(\overline{S}(z,r)\right) \subset \overline{S}(z,r)$, let $x \in \overline{S}(z,r)$ and set

$$\gamma_n = \mu\|Q(z) - z\| + \|P_n z - z\|.$$

Then

$$\begin{aligned}\|Q_n(x) - z\| &\leqslant \|Q_n(x) - Q_n(z)\| + \|Q_n(z) - P_n z\| + \|P_n z - z\| \\ &\leqslant \left\|B_n'\left(\frac{x+z}{2}\right)(x-z)\right\| + \mu\|Q(z) - z\| + \|P_n z - z\| \\ &\leqslant \mu\left(\left\|B'z + B'\left(\frac{x-z}{2}\right)\right\|\right)\|x-z\| + \gamma_n \\ &\leqslant \mu\|B'z\|r + \mu\frac{\|B'\|}{2}r^2 + \gamma_n. \end{aligned} \tag{2.15}$$

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Thus $\|Q_n(x) - z\| \leqslant r$ if

$$\frac{\mu\|B'\|}{2}r^2 + (\mu\|B'z\| - 1)r + \gamma_n \leqslant 0. \tag{2.16}$$

This quadratic inequality in r is satisfied for $r \in [b_n, a)$, so $Q_n(\overline{S}(z,r)) \subset \overline{S}(z,r)$. Applying the contraction mapping principal, Q_n has a unique fixed point z_n in $\overline{S}(z,a)$, and in fact $z_n \in \overline{S}(z,b_n)$.

Note that all the contractions Q_n have the same contraction factor c defined in (2.14). Since Q_n converges pointwise to Q by condition (V), Q is clearly a contraction and $Q(S(z,r)) \subset \overline{S}(z,r)$. If we put $U_n = V_n \cap \overline{S}(z,r)$, then by Theorem 2.2 $\{U_n, Q_n\}$ is a UCS. Theorem 2.3 yields conclusion (3). Since each $z_n \in \overline{S}(z,b_n)$ and $\lim b_n = b$, conclusion (2) is proved. If B is compact then Q_n converges uniformly to Q, so applying Theorem 2.4 yields conclusion (4).

To prove (1), we note that the contraction mapping theorem guarantees uniqueness in $S(z,a)$. Next consider $x \in S(z,a)$ and write $x = z + \delta$ for some δ with $\|\delta\| < a$. By the linearity of B' we have

$$\|B'x\| \leqslant \|B'z\| + \|B'\delta\| < \|B'z\| + \|B'\|\,a = 1/\mu \leqslant 1. \tag{2.17}$$

This guarantees that

$$S(z,a) \subset C = \{x : \|B'x\| < 1\}. \tag{2.18}$$

The set C contains a solution to (1.1), so by Theorem 2.6 we have uniqueness in C. ☐

REMARK. If we set $V_n = X$, $P_n = I$ for all n so $\mu = 1$, the proof is unchanged for parts (1) and (2). Thus parts (1) and (2) of the theorem hold for any Banach space X with $\mu = 1$.

We state Theorem 1 of [2] for comparative purposes.

THEOREM 2.8. *(Argyros) Let B be a bounded bilinear operator on $X \times X$ and suppose y and z belong to X. Define $T : X \to X$ by*

$$T(x) = y + B(x,x).$$

Set

$$a = \frac{1}{2\|B\|} - \|z\|,$$

$$b = a - \sqrt{\left(a^2 - \frac{\|T(z) - z\|}{\|B\|}\right)}$$

and assume b is nonnegative and $a \neq 0$. Then

(i) *T has a unique fixed point in $U(z,a) = \{x \in X : \|x - z\| < a\}$;*
(ii) *this fixed point actually lies in $\overline{U}(z,b)$.*

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REMARKS. We note that since $U(z,a) \subset C = \{x : \|B'x\| < 1\}$, Theorem 2.7 yields greater uniqueness information than Theorem 2.8. Also note that since $\|B'z\| \leqslant 2\|B\| \cdot \|z\|$, Theorem 2.7 gives greater flexibility than Theorem 2.8 in searching for approximate solutions z for which the hypotheses hold. This advantage will be used in Section 3.

It should also be noted in Theorem 2.7 that the operators P_nB need only converge pointwise to B for part (3) to hold—B need not be compact. We state Theorem 7 of [2] for comparison purposes.

THEOREM 2.9. *(Argyros) Consider the quadratic equations*

$$z = y + F_n(x,x) \tag{2.19}$$

where $F_n : X \times X \to X$, $n = 1,2,\ldots$ are bounded symmetric bilinear operators. If

(i) *the sequence $\{F_n\}$ converges to B uniformly as $n \to \infty$,*
(ii) *for each n there exists z_n, satisfying (2.19) and $\sup \|z_n\| < (2\|B\|)^{-1}$, then the sequence $\{z_n\}$ converges to a solution z of (1.1).*

We also observe the following necessary condition on solutions of (1.1).

COROLLARY 2.10. *If the equation*

$$x = B(x,x) + y \tag{2.20}$$

has a solution z_s with

$$\|B'z_s\| < 1,$$

then there is an open ball S about z_s such that for any initial estimate $x_o \in S$, the iterative scheme (1.3) converges to the solution z_s.

PROOF: In the proof of Theorem 2.7, let each $V_n = X$ so $P_n = I$, $\mu = 1$, and choose $z = z_s$. Then (2.11) is satisfied, and by the contraction mapping theorem the iterative routine (1.3) will converge. ∎

The following will be useful in the next section.

PROPOSITION 2.11. *Let $B : X \times X \to X$ be bounded and bilinear, $y \in X$. If*

$$\|B'y\| < \frac{1}{2}$$

then

$$x = y + B(x,x)$$

has a unique solution in $C = \{x \in X : \|B'x\| < 1\}$.

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PROOF: Let $D = \{x : \|B'x\| \leqslant \delta\}$, where $\delta = \|B'y\| + 1/2 < 1$. We shall show that $Q(x) = B(x,x) + y$ has a fixed point in D. Let $u, v \in D$. Then

$$\begin{aligned}\|Q(u) - Q(v)\| = \|B(u,u) - B(v,v)\| &= \left\| B'\left(\frac{u+v}{2}\right)(u-v) \right\| \\ &\leqslant \frac{1}{2}\|B'u + B'v\| \cdot \|u - v\| \leqslant \delta \|u - v\|,\end{aligned}$$

so Q is a contraction on D. Now if $x \in D$, then

$$B'(Q(x)) = B'(B(x,x)) + B'y = B'\left(\frac{B'x}{2}\right) + B'y$$

by identity (2.9). Hence

$$\|B'(Q(x))\| \leqslant \frac{\|B'x\|^2}{2} + \|B'y\| < \delta,$$

so $Q(D) \subset D$. Since D is closed, Q has a fixed point in D by the contraction mapping principle. The uniqueness follows from Theorem 2.6. ∎

REMARK. A similar result (Corollary 2) is proved in [**2**], with the hypothesis "$\|B'y\| < 1/2$" replaced by "$\|B\| \cdot \|y\| < 1/4$". The latter is a stronger assumption since $\|B'\| \leqslant 2\|B\|$.

3. A Continuation Algorithm

It is often the case that we seek a solution to

$$x = y + \lambda B(x,x), \quad \lambda \geqslant 0 \tag{3.1}$$

for large λ, but finding an approximate solution $z \in X$ for which Theorem 2.7 applies may be difficult or impractical. One way to handle this problem is the *continuation* technique, whereby (3.1) is solved for small enough λ so that an initial guess z can be easily found for which Theorem 2.7 applies. An approximate solution z_n is found in some V_n space, and λ is increased with z_n used as an initial guess for the new equation (3.1) with larger λ. This process is repeated until the desired large λ is reached and a satisfactory approximation obtained. In this section we present an algorithm and a theorem that make this precise for our problem of solving quadratic equations in a space that satisfies condition (V). In particular, we give conditions under which the desired large λ can be reached in a finite number of repetitions of the continuation process. This scheme is then illustrated on Chandrasekhar's equation, extending the range of λ values from 0.424059 given in [**2**] to 0.473571.

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CONTINUATION ALGORITHM.

1. Choose λ_0 small enough so that (3.1) is guaranteed to have a solution z_0 for which $\mu\lambda_0 \|B'z_0\| < 1$. We note that this is always possible (for example, $\lambda_0 = 0$).

2. Choose n sufficiently large so that

$$x = P_n y + \lambda_0 P_n B(x, x)$$

has a solution z_n in V_n that satisfies

$$1 - \mu\lambda_0 \|B'z_n\| > \mu\sqrt{2\lambda_0 \|B'\|}\sqrt{\|E_{0,n}\|} \tag{3.2}$$

where the "error" for z_n is

$$E_{0,n} = \lambda_0 B(z_n, z_n) + y - z_n.$$

That such an n exists follows from Theorem 2.7 (3), for

$$\lim_{n\to\infty} z_n = z_0 \text{ and } \lim_{n\to\infty} E_{0,n} = 0.$$

3. Solve

$$1 - \mu\lambda_1 \|B'z_n\| = \mu\sqrt{2\lambda_1 \|B'\|}\sqrt{(\lambda_1 - \lambda_0)\|B(z_n, z_n)\| + \|E_{0,n}\|} \tag{3.3}$$

for λ_1.

CLAIM. For each λ satisfying $\lambda_0 \leqslant \lambda < \lambda_1$, equation (3.1) has a solution z_λ for which $\mu\lambda \|B'z_\lambda\| < 1$.

PROOF: It is clear that replacing λ_1 by λ in (3.3) will yield

$$1 - \mu\lambda \|B'z_n\| > \mu\sqrt{2\lambda \|B'\|}\sqrt{(\lambda - \lambda_0)\|B(z_n, z_n)\| + \|E_{0,n}\|}.$$

Define Q_λ by $Q_\lambda(x) = \lambda B(x, x) + y$. Then

$$\|Q_\lambda(z_n) - z_n\| \leqslant (\lambda - \lambda_0)\|B(z_n, z_n)\| + \|E_{0,n}\|,$$

so

$$\frac{1 - \mu\lambda \|B'z_n\|}{\mu\lambda \|B'\|} > \sqrt{\frac{2\|Q_\lambda(z_n) - z_n\|}{\lambda \|B'\|}} \tag{3.4}$$

The Claim then follows from Theorem 2.7. ∎

4. If λ_1 is not large enough, return to step (1) with λ_0 replaced by $\lambda_1 - \varepsilon$ for small $\varepsilon > 0$.

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REMARKS. Observe that inequality (3.2) is satisfied if it holds when upper bounds for $\|B'z_n\|$ and $\|B'\|$ are used in place of $\|B'z_n\|$ and $\|B'\|$, respectively.

Information on the location and uniqueness of each "intermediate" solution z_n in this algorithm can be obtained from Theorem 2.7.

There is some question about whether this algorithm will eventually reach the desired large λ value. The next result gives conditions that guarantee this convergence—in a finite number of steps.

THEOREM 3.1. *Suppose that $\lambda_E > 0$ and equation (3.1) has a solution z_λ with $\mu\|B'z_\lambda\| < 1$ for all λ, $0 \leqslant \lambda \leqslant \lambda_E$. Then after a finite number of iterations, the algorithm given above will obtain a λ_1 for which $\lambda_E \leqslant \lambda_1$.*

PROOF: For each λ, $0 \leqslant \lambda \leqslant \lambda_E$, the inequality (3.4) and a continuity argument on λ guarantee some $\delta_\lambda^n > 0$ for which $t \in (\lambda - \delta_\lambda^n, \lambda + \delta_\lambda^n) \Rightarrow x = y + tB(x,x)$ has a solution z_t with $\mu\|B'z_t\| < 1$. The open sets $(\lambda - \delta_\lambda^n, \lambda + \delta_\lambda^n)$ form an open cover of the compact set $[0, \lambda_E]$, so there exists some δ such that $0 < \delta < \delta_\lambda^n$ for all $\lambda \in [0, \lambda_E]$. Therefore each iteration of the algorithm increases λ by at least δ. $\square$

The final result of the algorithm given is an estimate $z_n \in V_n$. This is less than satisfactory, since the true solution to (3.1) must be of the form $y + f$, where $f \in$ *Range*(B). To get an approximation of this form, one approach is to find

$$\widehat{z} = y + B(z_n, z_n).$$

While calculating $y + B(z_n, z_n)$ is more expensive than an iteration in V_n, the following result shows that $\widehat{z}$ must be an improvement on z_n. Numerical experiments suggest that one such calculation is worth the price in many situations.

PROPOSITION 3.2. *Let $B, F : X \times X \to X$ be bounded and bilinear. Suppose that the equations*

$$x = \widetilde{y} + F(x,x) \tag{3.5}$$

and

$$x = y + B(x,x)$$

have solutions z_n and z_s, respectively, with

$$\|B'z_n\|, \ \|B'z_s\| < 1.$$

Then

$$\|z_s - \widehat{z}\| < \|z_s - z_n\|.$$

PROOF: We calculate:

$$\|z_s - \hat{z}\| = \|B(z_s, z_s) - B(z_n, z_n)\| = \left\| B'\left(\frac{z_s + z_n}{2}\right)(z_s - z_n) \right\|$$
$$\leqslant \frac{\|B'(z_n)\| + \|B'(z_s)\|}{2} \|z_s - z_n\| < \|z_s - z_n\|.$$

□

We now illustrate the continuation algorithm to approximate solutions z_λ to the Chandrasekhar equation (1.7).

EXAMPLE. Equation (1.7) is usually solved in $C[0,1]$ for physical reasons [**5**, **2**]. We shall first seek solutions in $L^2[0,1]$, taking advantage of certain properties of this space, and then show that such solutions lie in $C[0,1]$. For these reasons, let $X = L^2[0,1]$ and let V_n be the span of the first n Legendre polynomials $P_0, \dots P_{n-1}$. Observe that $\mu = 1$ in a Hilbert space. Define $B : X \times X \to X$ by

$$B(f,g)(s) = f(s) \int_0^1 \frac{s}{s+t} g(t) dt.$$

We seek the maximal λ value for which our algorithm applies. A first estimate of $y = 1$ is natural. From Proposition 2.11, any $\lambda_0 < 1/(2\|B'y\|)$ will satisfy step 1 of the algorithm. We bound $\|B'y\|$ as follows. For $f \in X$, $\|f\| \leqslant 1$, we have

$$\|B'y(f)\| \leqslant \left\| \int_0^1 \frac{s}{s+t} f(t) dt \right\| + \left\| f(s) \int_0^1 \frac{s}{s+t} dt \right\|.$$

By Cauchy-Schwartz, we obtain

$$\left\| \int_0^1 \frac{s}{s+t} f(t) dt \right\|^2 \leqslant \|f\|^2 \cdot \int_0^1 \int_0^1 \left(\frac{s}{s+t}\right)^2 dt ds \leqslant 1 - \ln 2$$

and

$$\left\| f(s) \int_0^1 \frac{s}{s+t} dt \right\| \leqslant \|f\| \cdot \sup_s \int_0^1 \frac{s}{s+t} dt \leqslant \ln 2.$$

Therefore

$$\|B'y\| \leqslant \sqrt{1 - \ln 2} + \ln 2,$$

so we set $\lambda_0 = 0.40 < 1/(2\|B'y\|)$. Similar arguments yield

$$\|B'\| \leqslant 2\|B\| \leqslant 2 \sup_s \sqrt{\int_0^1 \left(\frac{s}{s+t}\right)^2 dt} = \frac{1}{\sqrt{2}}.$$

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A choice of $n = 1$ is not large enough to satisfy step 2 of the continuation algorithm, so we begin with $n = 2$ for our initial V_n space. The following table gives the results of the algorithm. At each step, the iterative routine (1.5) was carried out in V_n space until consecutive iterates differed by less than 10^{-12} in the L^2 norm. Each column represents one iteration of the algorithm. For each n value, the λ values increase until reaching a limiting value. At this point, we increase n as directed by step 2 of the algorithm and continue.

n	2	2	$\cdots$	2	6	$\cdots$	6	9	$\cdots$	9
λ_0	.4000	.4102	$\cdots$	.4371	.4371	$\cdots$	.4708	.4708	$\cdots$	.473571
λ_1	.4102	.4198	$\cdots$	.4371	.4428	$\cdots$	.4708	.4712	$\cdots$	.473571
$\lambda_1 \lVert B' z_n \rVert$	.7123	.7531	$\cdots$	.8342	.8521	$\cdots$	.9645	.9651	$\cdots$	.979623

We now show that the solutions guaranteed are not only in $X = L^2[0,1]$, but are in fact continuous.

Lemma 3.4. *If f is a $L^2[0,1]$ solution to the Chandrasekhar equation (1.7), then $f \in C[0,1]$.*

Proof: Suppose that f solves (1.7), and define F_f by

$$F_f(s) = \int_0^1 \frac{s}{s+t} f(t)\, dt.$$

Obviously $F_f(0) = 0$, and $\lim_{s \to 0} F_f(s) = 0$ since

$$|F_f(s)| \leqslant \|f\| \sqrt{\int_0^1 \left(\frac{s}{s+t}\right)^2 dt}.$$

Hence F_f is continuous at 0. Now F_f is clearly continuous for $s > 0$, so $F_f(s)$ is continuous on $[0,1]$. By hypothesis $f(s) = 1 + f(s) F_f(s)$, so $F_f(s) \neq 1$ for $s \in [0,1]$, and thus

$$f(s) = \frac{1}{1 - F_f(s)}.$$

We conclude that $f \in C[0,1]$. $\square$

Remarks. The Legendre polynomials were used solely for simplicity; wavelet bases have been shown to be superior in many aspects [**12**]. Certainly, a Banach space with a multiresolution analysis satisfies condition (V).

All computations for the example were done using *Maple V* on a 486 PC. No *Pentium* chips were involved in this work.

With $n = 9$, we obtain $\lambda \approx 0.473571$. By increasing n this λ value can be increased. It has been shown elsewhere [5] that the maximum value for λ is 0.5—no solution exists for larger λ values. The next result shows that this maximum value cannot be achieved by our algorithm.

THEOREM 3.5. *Let $BL(X \times X, X)$ denote the bounded bilinear operators on $X \times X$ into X, and suppose that $y \in X$. Then the set $\mathcal{O}$ of all $B \in BL(X \times X, X)$ such that*

$$B(x, x) + y = x \tag{3.6}$$

has a solution z with $\|B'z\| < 1$ is open (in the operator topology) in $BL(X \times X, X)$.

PROOF: Suppose that $F \in \mathcal{O}$ has solution z with $\|F'z\| < 1$. It is easy to verify that $\|B - F\| < \varepsilon \Rightarrow \|B' - F'\| < 2\varepsilon$ when $B \in BL(X \times X, X)$. Thus we can choose $\varepsilon > 0$ sufficiently small so that

$$\frac{1 - \|B'z\|}{\|B'\|} > \sqrt{\frac{2\|B(z,z) + y - z\|}{\|B'\|}} = \sqrt{\frac{2\|B(z,z) - F(z,z)\|}{\|B'\|}}$$

holds. Then by Theorem 2.7, equation (3.6) has a solution s with $\|B's\| < 1$. We conclude that there is an open ball of radius ε about F that is contained in set $\mathcal{O}$. ∎

Now if Chandrasekhar's equation (1.7) with $\lambda = 0.5$ had a solution z with $\|0.5B'z\| < 1$, then by Theorem 3.5 equation (1.7) would have a solution for some $\lambda > 0.5$, which is false as noted in the remarks before Theorem 3.5.

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