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Weak Amenability of a Class of Banach Algebras

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Abstract. We show that, if a Banach algebra $\mathfrak{A}$ is a left ideal in its second dual algebra and has a left bounded approximate identity, then the weak amenability of $\mathfrak{A}$ implies the $(2m+1)$-weak amenability of $\mathfrak{A}$ for all $m \geq 1$.

In a recent paper [2] Dales, Ghahramani and Grøbæk have introduced the concept of n-weak amenability for Banach algebras. They point out the fact that, for $n \geq 1$, $(n+2)$-weak amenability always implies n-weak amenability, and prove further that if a Banach algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}^{**}$, then the weak amenability of $\mathfrak{A}$ also implies the $(2m+1)$-weak amenability of $\mathfrak{A}$ for all $m > 0$. As to the general case, they have raised an open question: Does weak amenability imply 3-weak amenability? This question has been answered in negative by the author in [5]. In this note we consider the Banach algebras which are one sided ideals in their second dual algebras, and discuss sufficient conditions under which weak amenability will imply $(2m+1)$-weak amenability for $m > 0$. We shall also consider an example to show the use of our result.

Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. A linear mapping $D\colon \mathfrak{A} \to X$ is a *derivation* if $D(ab) = a \cdot D(b) + D(a) \cdot b$ for $a, b \in \mathfrak{A}$. For any $x \in X$, the mapping $\delta_x\colon a \mapsto ax - xa$, $a \in \mathfrak{A}$, is a continuous derivation, called an *inner derivation*. Let $\mathcal{B}^1(\mathfrak{A}, X)$ be the space of all continuous derivations from $\mathfrak{A}$ into X and let $\mathcal{Z}^1(\mathfrak{A}, X)$ be the space of all inner derivations from $\mathfrak{A}$ into X. Then the first *cohomology group* of $\mathfrak{A}$ with coefficients in X is the quotient space $\mathcal{H}^1(\mathfrak{A}, X) = \mathcal{B}^1(\mathfrak{A}, X)/\mathcal{Z}^1(\mathfrak{A}, X)$.

For each $n \geq 1$, $\mathfrak{A}^{(n)}$, the n-th conjugate space of $\mathfrak{A}$, is a Banach $\mathfrak{A}$-bimodule, with the module actions defined inductively by

$$\langle u, F \cdot a\rangle = \langle a \cdot u, F\rangle, \quad \langle u, a \cdot F\rangle = \langle u \cdot a, F\rangle, \quad F \in \mathfrak{A}^{(n)}, u \in \mathfrak{A}^{(n-1)}, a \in \mathfrak{A}.$$

A Banach algebra $\mathfrak{A}$ is called *n-weakly amenable* if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\}$. Usually, 1-weakly amenable Banach algebras are called weakly amenable.

Recall that for a Banach algebra $\mathfrak{A}$, its second dual $\mathfrak{A}^{**}$ is a Banach algebra when equipped with the first Arens product which is given by the following formula

$$\langle f, \Phi\Psi\rangle = \langle \Psi f, \Phi\rangle, \quad f \in \mathfrak{A}^*, \quad \Phi, \Psi \in \mathfrak{A}^{**},$$

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where $\Psi f \in \mathfrak{A}^*$ is defined by

$$\langle a, \Psi f\rangle = \langle fa, \Psi\rangle, \quad a \in \mathfrak{A}.$$

We refer to Arens' original paper [1] and the survey paper [3] for properties and references about Arens products. In this note, for $m \geq 1$, we always equip $\mathfrak{A}^{(2m)}$ with the first Arens product.

For a Banach space X we will denote by $\widehat{X}$ (resp. $\hat{x}$) the image of X (resp. $x \in X$) in $X^{(2m)}$ under the canonical mapping. But if no confusion may occur we will keep using X to denote this image. For $m > 0$, the subspace of $X^{(2m+1)}$ annihilating $\widehat{X}$ will be denoted by $X^{\perp}$, *i.e.*, $X^{\perp} = \{F \in X^{(2m+1)} ; F|_{\widehat{X}} = 0\}$. Concerning the Banach algebra $\mathfrak{A}^{(2m)}$ we have:

Lemma 1 *Suppose that $\mathfrak{A}$ is a left, right or two sided ideal in $\mathfrak{A}^{**}$. Then it is also a left, right or two sided ideal in $\mathfrak{A}^{(2m)}$ for all $m \geq 1$.*

Proof Assume that $\mathfrak{A}$ is a left ideal of $\mathfrak{A}^{(2m)}$, $m \geq 1$. We prove that it is also a left ideal of $\mathfrak{A}^{(2m+2)}$. First we have the following $\mathfrak{A}$-bimodule direct sum decompositions

$$\mathfrak{A}^{(2m+2)} = (\mathfrak{A}^*)^{\perp} \dotplus (\mathfrak{A}^{**})\hat{} \tag{1}$$

and

$$\mathfrak{A}^{(2m+1)} = (\mathfrak{A})^{\perp} \dotplus (\mathfrak{A}^*)\hat{}. \tag{2}$$

For any $F \in \mathfrak{A}^{(2m+1)}$, let $F = f_1 + \hat{f}_2$, $f_1 \in \mathfrak{A}^{\perp}$, $f_2 \in \mathfrak{A}^*$. Then $af_1 = 0$ for $a \in \mathfrak{A}$, since $\mathfrak{A}$ is a left ideal in $\mathfrak{A}^{(2m)}$. So

$$aF = a\hat{f}_2 = (af_2)\hat{}.$$

For any $\Phi \in \mathfrak{A}^{(2m+2)}$, let $\Phi = \phi + \hat{u}$, $\phi \in (\mathfrak{A}^*)^{\perp}$, $u \in \mathfrak{A}^{**}$. Then

$$\langle F, \Phi a\rangle = \langle (af_2)\hat{}, \phi + \hat{u}\rangle = \langle (af_2)\hat{}, \hat{u}\rangle = \langle F, (ua)\hat{}\rangle.$$

This shows that $\Phi a = (ua)\hat{} \in \widehat{\mathfrak{A}}$ for $a \in \mathfrak{A}$ and $\Phi \in \mathfrak{A}^{(2m+2)}$. Therefore $\mathfrak{A}$ is a left ideal of $\mathfrak{A}^{(2m+2)}$. So the lemma is true when $\mathfrak{A}$ is a left ideal of $\mathfrak{A}^{**}$. For the other two cases the proof is similar. ■

It is known that for a Banach algebra $\mathfrak{A}$ with a bounded approximate identity (b.a.i. in short), if X is a Banach $\mathfrak{A}$-bimodule in which $\mathfrak{A}$ acts trivially on one side, then $\mathcal{H}^1(\mathfrak{A}, X^*) = \{0\}$ (see [4, Proposition 1.5]). The following lemma can be viewed as an extension of this result.

Lemma 2 *Suppose that $\mathfrak{A}$ is a Banach algebra with a left (right) b.a.i.. Suppose that X is a Banach $\mathfrak{A}$-bimodule and Y is a weak* closed submodule of the dual module X^*. If the left (resp. right) $\mathfrak{A}$-module action on Y is trivial, then $\mathcal{H}^1(\mathfrak{A}, Y) = \{0\}$.*

Proof The proof is quite standard. Here we give the proof in the case that $\mathfrak{A}$ has a left b.a.i. and $\mathfrak{A}$ acts trivially on the left in Y. Suppose that $D\colon \mathfrak{A} \to Y$ is a continuous derivation. Let (e_i) be a left b.a.i. of $\mathfrak{A}$, and $f \in Y$ be a weak* cluster point of $\big(D(e_i)\big)$. Since $\mathfrak{A}Y = \{0\}$, we have

$$D(a) = \lim D(e_i a) = fa = fa - af, \quad a \in \mathfrak{A}.$$

Hence D is inner. This shows that $\mathcal{H}^1(\mathfrak{A}, Y) = \{0\}$. ■

With the preceding two lemmas, we can now prove a partial converse to [2, Proposition 1.2] as follows.

Theorem 3 *Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra. If $\mathfrak{A}$ has a left (right) b.a.i. and is a left (resp. right) ideal in $\mathfrak{A}^{**}$, then $\mathfrak{A}$ is $(2m+1)$-weakly amenable for $m \geq 1$.*

Proof We give the prove in the case that $\mathfrak{A}$ has a left b.a.i. and is a left ideal in $\mathfrak{A}^{**}$. The proof for the other case is similar. First, from the $\mathfrak{A}$-bimodule decomposition (2) we have the cohomology group decomposition

$$\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2m+1)}) = \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) \dotplus \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{\perp}).$$

If $\mathfrak{A}$ is weakly amenable, we have $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\}$. $\mathfrak{A}^{\perp}$ is clearly weak* closed submodule of $\mathfrak{A}^{(2m+1)}$. Since $\mathfrak{A}$ is a left ideal in $\mathfrak{A}^{**}$, it is a left ideal in $\mathfrak{A}^{(2m)}$ from Lemma 1. It follows that the left $\mathfrak{A}$-module action on $\mathfrak{A}^{\perp}$ is trivial. Then Lemma 2 leads to $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{\perp}) = \{0\}$. As a consequence we have $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(2m+1)}) = \{0\}$, *i.e.*, $\mathfrak{A}$ is $(2m+1)$-weakly amenable. ■

Now let us consider an example. Suppose that S is an infinite set and s_0 a fixed element in S. Define an algebra product in $\ell^1(S)$ in the following way.

$$a \cdot b = a(s_0)b, \quad a, b \in \ell^1(S). \tag{3}$$

It is easily verified that with this product $\ell^1(S)$ is a Banach algebra. We shall denote it by $\big(\ell^1(S), \cdot\big)$, or $\ell^1(S)$ in short. It has a left identity e_0 defined by

$$e_0(s) = \begin{cases} 1 & \text{if } s = s_0 \\ 0 & \text{if } s \neq s_0. \end{cases}$$

But it has no right approximate identity. $\ell^1(S)$ is also a left ideal in $\ell^1(S)^{**}$. In fact, for $u \in \ell^1(S)^{**}$, $u = \text{wk}^*\text{-}\lim a_\alpha$, with (a_α) a bounded net in $\ell^1(S)$, we have

$$u \cdot a = \text{wk}^*\text{-}\lim a_\alpha \cdot a = \lim a_\alpha(s_0) a \in \ell^1(S), \quad a \in \ell^1(S).$$

Here we have used the fact that $\lim a_\alpha(s_0)$ exists. It is also easy to see that $\ell^1(S)$ is not a right ideal of $\ell^1(S)^{**}$. The $\ell^1(S)$-bimodule actions on the dual module $\ell^1(S)^* = \ell^\infty(S)$ are in fact formulated as follows.

$$a \cdot f = \langle a, f \rangle e_0^*, \quad f \cdot a = a(s_0) f, \quad a \in \ell^1(S), f \in \ell^\infty(S), \tag{4}$$

where e_0^* is the element of $\ell^\infty(S)$ satisfying $e_0^*(s_0) = 1$, and $e_0^*(s) = 0$ for $s \neq s_0$.

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Assertion 1 *The Banach algebra* $\left(\ell^1(S), \cdot\right)$ *is weakly amenable.*

Proof Suppose that $D\colon \ell^1(S) \to \ell^\infty(S)$ is a derivation. Then for $a, b \in \ell^1(S)$, from equations (3) and (4),

$$\begin{aligned} a(s_0)D(b) = D(a \cdot b) &= a \cdot D(b) + D(a) \cdot b \\ &= \langle a, D(b)\rangle e_0^* + b(s_0)D(a). \end{aligned}$$

By taking $b = a$, we see $\langle a, D(a)\rangle = 0$ for all $a \in \ell^1(S)$. This in turn implies that

$$\langle a, D(b)\rangle = -\langle b, D(a)\rangle, \quad a, b \in \ell^1(S).$$

So

$$\begin{aligned} D(a) = D(e_0 \cdot a) &= \langle e_0, D(a)\rangle e_0^* + a(s_0)D(e_0) \\ &= -\langle a, D(e_0)\rangle e_0^* + a(s_0)D(e_0) \\ &= D(e_0) \cdot a - a \cdot D(e_0), \quad a \in \ell^1(S). \end{aligned}$$

Therefore D is inner. This shows that $\left(\ell^1(S), \cdot\right)$ is weakly amenable and the proof is complete. ■

By using Theorem 3, Assertion 1 induces immediately the following:

Assertion 2 *For* $m \geq 0$, $\left(\ell^1(S), \cdot\right)$ *is* $(2m+1)$*-weakly amenable.*

Note The algebra $\left(\ell^1(S), \cdot\right)$ is not $2m$-weakly amenable for any $m \geq 1$.

Proof From [2, Proposition 1.2] it suffices to show that $\left(\ell^1(S), \cdot\right)$ is not 2-weakly amenable. Let $E = \{e_0^*\}^\perp \subset \ell^1(S)^{**}$. Then for every $u \in E$ and every $a \in \mathfrak{A}$, from equation (4), $u \cdot a = 0$. This implies that any linear mapping from $\ell^1(S)$ into E is a derivation. Especially $D\colon a \mapsto a(s_1)u$ for some nonzero $u \in E$ and s_1 $(\neq s_0) \in S$ is a continuous non-inner derivation from $\ell^1(S)$ into $\ell^1(S)^{**}$. Therefore $\left(\ell^1(S), \cdot\right)$ is not 2-weakly amenable. ■

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