ON A CLASS OF OPERATORS

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Abstract. In this paper we show that the Weyl spectrum of an operator of class W satisfies the spectral mapping theorem for analytic functions and give the equivalent conditions for an operator of the form normal + compact to be polynomially compact.

0. Introduction. Let H be an infinite dimensional Hilbert space and let $B(H)$ be the set of all bounded linear operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T. An operator $T \in B(H)$ is said to be *Fredholm* if its range ran T is closed and both the null space ker T and ker T^* are finite dimensional. The *index* of a Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim \ker T - \dim \ker T^*.$$

It is well known ([**3**]) that $i : \mathcal{F} \to Z$ is a continuous function, where the set $\mathcal{F}$ of Fredholm operators has the norm topology and Z has the discrete topology. The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in C : T - \lambda I \text{ is not Fredholm}\}.$$

A Fredholm operator of index zero is called a *Weyl* operator. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in C : T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([**1**]) that for any operator $T, \sigma_e(T) \subset \omega(T) \subset \sigma(T)$ and $\omega(T)$ is a nonempty compact subset of C.

We say that an operator $T \in B(H)$ is *of class W* if $\sigma_e(T) = \omega(T)$. For example, every normal, compact, quasinilpotent operator is of class W. However, consider the unilateral shift U on l_2 given by

$$U(x_1, x_2, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Then U is hyponormal, $\omega(U) = \sigma(U) = D$(= the closed unit disc) and $\sigma_e(U) = C$(= the unit circle). Hence U is not of class W and so we note that T is not of class W, even if T is hyponormal. By [**2**, Theorem 4.1], every Toeplitz operator is not of class W.

It is also known that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T([**8**]). However if $T_n \to T$ with $T_nT = TT_n$, for all $n \in N$, then

$$\lim \omega(T_n) = \omega(T). \tag{1}$$

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It is known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) \subset f(\omega(T)), \tag{2}$$

by [**1**, Theorem 3.2].

The inclusion (2) , may be proper. (See [**1**, Example 3.3].) If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions.

In this paper we show that the Weyl spectrum of an operator of class W satisfies the spectral mapping theorem for analytic functions and give the equivalent conditions for an operator of the form normal + compact to be polynomially compact.

1. Spectral mapping theorem. The Weyl spectrum of an operator is the disjoint union of the essential spectrum and a particular open set.

LEMMA 1. ([**1**],[**3**]) *For any operator T in $B(H)$,*

$$\omega(T) = \sigma_e(T) \cup \theta(T),$$

where $\theta(T) = \{\lambda : T - \lambda$ is Fredholm and $i(T-\lambda) \neq 0\}$. The union is disjoint.

For example, if U is the unilateral shift, then $\sigma_e(U) = \{\lambda : |\lambda| = 1\}$ and $\theta(U) = \{\lambda : |\lambda| < 1\}$. From Lemma 1, we note that every normal operator is of class W, and that $\sigma_e(T) = \omega(T)$ if and only if the open set $\theta(T)$ is empty. By [**1**, Example 2.12], every compact operator K is of class W. Also it is easy to show that if T is of class W and $\alpha \in C$, then T^* and αT are of class W.

For an example of a nonnormal operator of class W, consider $T = U \oplus U^*$ where U is the unilateral shift. In this case, $\omega(T) = \{\lambda : |\lambda| = 1\} = \sigma_e(T)$. Thus T is of class W. However T is not a normal operator. Thus our class is strictly larger than the class of normal operators.

It is easy to show that the set of operators of class W is closed in $B(H)$, invariant under compact perturbations and closed under similarity.

THEOREM 2. *If T, in $B(H)$, is of the form normal + compact, then T is of class W.*

Proof. Let $T = N + K$, where N is normal and K is compact. If T is not of class W, then by Lemma 1 there exists $\lambda \in C$ such that $T - \lambda$ is Fredholm of nonzero index. But, by [**3**], $T - \lambda - K$ is Fredholm and $i(T-\lambda) = i(T-\lambda-K) = i(N-\lambda) = 0$. This is a contradiction.

We note that if T is a normal operator and f is any continuous complex-valued function on $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$, and so $f(T)$ is of class W ([**1**]). We obtain the following similar result.

THEOREM 3. *If T is of class W and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

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Proof. Suppose that p is any polynomial. Then, by the spectral mapping theorem,

$$p(\omega(T)) = p(\sigma_e(T)) = \sigma_e(p(T)) \subseteq \omega(p(T)).$$

But for any operator $T \in B(H)$, $\omega(p(T)) \subseteq p(\omega(T))$ by (2). Therefore $\omega(p(T)) = p(\omega(T))$, for any polynomial p.

If f is analytic on a neighborhood of $\sigma(T)$ then, by Runge's theorem ([**3**]), there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by [**8**], we have

$$\omega(f(T)) = \lim \omega(p_n(T)) = \lim p_n(\omega(T)) = f(\omega(T)).$$

COROLLARY 4. *If T is of class W and f is analytic on a neighborhood of $\sigma(T)$, then $f(T)$ is of class W.*

Proof. By Theorem 3 and the spectral mapping theorem,

$$\omega(f(T)) = f(\omega(T)) = f(\sigma_e(T)) = \sigma_e(f(T)).$$

Thus $f(T)$ is of class W.

An operator T is said to be *polynomially compact* if there exists a polynomial p such that $p(T)$ is compact.

THEOREM 5. *For an operator T of the form normal + compact, the following are equivalent*:

(1) *T is polynomially compact;*

(2) *there exists an analytic function f on $\sigma(T)$ such that $f(T)$ is compact and f has finitely many zeros on $\omega(T)$;*

(3) *$\omega(T)$ is finite.*

Proof. (1) $\Rightarrow$ (2) trivially.

(2) $\Rightarrow$ (3). By Theorems 2, 3 and [**1**, Example 2.12], we have $f(\omega(T)) = \omega(f(T)) = \{0\}$. Since f has only finite many zeros on $\omega(T)$, it follows that $\omega(T)$ is finite.

(3) $\Rightarrow$ (1). Suppose that $\omega(T)$ is finite. Let p be any nonzero polynomial such that p is 0 on $\omega(T)$. By Theorems 2 and 3, we have $\omega(p(T)) = p(\omega(T)) = \{0\}$. Since T is of the form normal $S+$ compact, $p(T)$ is an operator of the form normal + compact, say $p(T) = p(S) + K$, where K is compact and $p(S)$ is normal. Thus $\omega(p(S)) = \omega(p(T)) = p(\omega(T)) = \{0\}$ by [**1**, Corollary 2.7]. Since $p(S)$ is normal, it follows that $p(S)$ is compact. (See the remarks following [**1**, Corollary 6.3].) Thus $p(T)$ is compact and hence T is polynomially compact.

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