CHARACTER SUMS AND SMALL EIGENVALUES FOR $\Gamma_0(p)$

by HENRYK IWANIEC†

Dedicated to Robert Rankin

1. Introduction. Statement of results. Let Δ denote the Laplace operator acting on the space $L^2(\Gamma/H)$ of automorphic functions with respect to a congruence group Γ, square integrable over the fundamental domain $F=\Gamma/H$. It is known that Δ has a point spectrum

$$\lambda_0=0<\lambda_1\leq\lambda_2\leq\ldots$$

with (Weyl's law)

$$\lambda_n\sim\frac{4\pi}{|F|}n \quad \text{as} \quad n\to\infty$$

and it has a purely continuous spectrum on $[\frac{1}{4},\infty)$ of finite multiplicity equal to the number of inequivalent cusps. The eigenpacket of the continuous spectrum is formed by the Eisenstein series $E_{\mathfrak{a}}(z,s)$ on $s=\frac{1}{2}+it$ where $\mathfrak{a}$ ranges over inequivalent cusps. The eigenfunctions $u_j(z)$ with positive eigenvalues are Maass cusp forms.

A. Selberg's celebrated conjecture [**9**] asserts that all positive eigenvalues lie on the continuous spectrum, i.e.

$$\lambda_1\geq\tfrac{1}{4}. \tag{1.1}$$

Selberg [**9**] succeeded to show that

$$\lambda_1\geq\tfrac{3}{16} \tag{1.2}$$

by using A. Weil's upper bound for Kloosterman sums

$$|\mathcal{S}(m,n;c)|\leq(m,n,c)^{1/2}c^{1/2}\tau(c), \tag{1.3}$$

and S. S. Gelbart and H. Jacquet [**2**] have proved the strict inequality $\lambda_1>3/16$ by a different method (lifting from GL(2) to GL(3)). The conjecture (1.1) is known to be true for subgroups of small index of the modular group, cf. Huxley [**3**].

Let us call exceptional the eigenvalues which do not satisfy the Selberg conjecture, i.e. those with

$$0<\lambda_j<\tfrac{1}{4}.$$

They play a similar role to the real zeros of Dirichlet's L-series in the multiplicative number theory. In fact letting

$$\lambda_j=s_j(1-s_j)$$

it turns out that s_j are zeros of the Selberg zeta-function; thus the exceptional eigenvalues correspond to the real zeros in the segment

$$\tfrac{1}{2}<s_j<1.$$

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The remaining zeros s_j satisfy the Riemann hypothesis, i.e. they lie on the line

$$s_j = \tfrac{1}{2} + it_j, \qquad t_j \text{ real}.$$

Being unable to prove the Selberg eigenvalue conjecture J.-M. Deshouillers and H. Iwaniec [**1**] began to establish statistical results showing a rarity of the s_j in much the same form as the density theorems about the zeros of Dirichlet's L-series. Some of their results proved to be powerful enough to go around the conjecture in a number of important applications. It is not surprising that the matter has something to do with character sums. The first transparent connection was pointed out in [**6**] where the following kind of density theorems were established

$$\sum_{1/2<s_j<1} |F|^{A(s_j-1/2)} \ll |F|^{1+\varepsilon}, \tag{1.4}$$

the constant implied in $\ll$ depending on ε alone. The larger A is the less often exceptional eigenvalues of Γ may occur. J. Szmidt and H. Iwaniec [**6**] considered the Hecke congruence group $\Gamma = \Gamma_0(q)$ of level q (for technical reason we assumed q be prime) showing (1.4) with $A = 24/11$. Here the point is that $A = 24/11 > 2$ because the result with $A = 2$ follows simply by applying Selberg's trace formula with an appropriate test function, see M. N. Huxley [**4**] for example. It is natural to conjecture that (1.4) holds with $A = 4$ (density conjecture). This would contain the Selberg lower bound (1.2) for an individual eigenvalue.

The character sums in question are of the type

$$\sum_a \sum_b \left(\frac{a^2-4}{b}\right). \tag{1.5}$$

In order to estimate them in [**6**] we used A. Weil's (see (3.3)) and D. Burgess' bounds for character sums. The first replaces (1.3) while the second is vital and it yields the desired saving to effect $A > 2$. If the Lindelöf hypothesis for Dirichlet's L-series was used instead of Burgess' bound then we could get the density theorem with $A = 3$.

The problem is also related with the Lindelöf hypothesis for the Rankin zeta-functions. Let us define them. Given a cusp $\mathfrak{a}$ of Γ take $\sigma_\mathfrak{a} \in \mathrm{SL}(2, \mathbb{R})$ (once and for all) such that

$$\sigma_\mathfrak{a}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty \tag{1.6}$$

where $\Gamma_\mathfrak{a}$ is the stabilizer of $\mathfrak{a}$ in Γ. Each cusp form $u_j(z)$, being an eigenfunction of Δ

$$\Delta u_j = \lambda_j u_j,$$

has the Fourier expansion at $\mathfrak{a}$ of type

$$u_j(\sigma_\mathfrak{a} z) = \sqrt{y} \sum_{n \neq 0} \rho_{j\mathfrak{a}}(n) K_{s_j-1/2}(2\pi |n| y) e(nx) \tag{1.7}$$

where the numbers $\rho_{j\mathfrak{a}}(n)$ are called the Fourier coefficients and $K_\nu(y)$ is the McDonald–Bessel function. We assume that the cusp forms u_j form an orthonormal system

$$\langle u_{j_1}, u_{j_2} \rangle = \int_F u_{j_1}(z) \bar{u}_{j_2}(z)\, dz = \delta_{j_1 j_2}.$$

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The Rankin zeta-functions are defined by

$$R_{ja}(s)=\sum_1^\infty |\rho_{ja}(n)|^2 n^{-s}, \qquad \mathrm{Re}\, s>1.$$

They possess meromorphic continuation to the whole complex plane and they satisfy a (vector) functional equation which connects values at s with those at $1-s$. It is reasonable to expect that

$$\rho_{ja}(1) \ll \left(\frac{\mathrm{ch}\, \pi t_j}{q}\right)^{1/2} (\lambda_j q)^\varepsilon$$

and that the analogue of the Lindelöf hypothesis is true

$$R_{ja}(s) \ll \left(\frac{\mathrm{ch}\, \pi t_j}{q}\right)^{1/2} (\lambda_j q\, |s|)^\varepsilon \tag{1.8}$$

on $\mathrm{Re}\, s=\frac{1}{2}$. This would imply the density theorem with $A=3$. What we actually need is a consequence of (1.8), namely that

$$\Lambda_{ja}(N)=\sum_{n\le N} |\rho_{ja}(n)|^2 \gg \frac{N}{|F|} \tag{1.9}$$

for $N \ge q^\varepsilon$. If (1.9) is true for $N=q^\theta$ with $0<\theta<2$ then the density theorem holds with $A=2(2-\theta)$. Therefore the density conjecture is a consequence of another conjecture, that (1.9) is true for all $N\ge q^\varepsilon$. It is disappointing that by present means we are able to show (1.9) only when $N \gg q^{1+\varepsilon}$, compare with Theorem 7.

In this paper we give another treatment of the character sums (1.5) which yields the following improvement over [**6**].

THEOREM 1. *The density theorem* (1.4) *holds for groups* $\Gamma_0(p)$ *with* $A=12/5$.

The present method of estimating the relevant character sums does not depend on the Burgess inequality and is more general.

I benefited a lot from discussions on the subject with H. L. Montgomery to whom I wish to express my thanks as well as to the Mathematics Department of the University of Michigan in Ann Arbor for financial support and a nice atmosphere to work.

2. Estimates for character sums. Let $\mathcal{D}$ be a finite sequence of positive integers (not necessarily distinct) from the interval $[D, 4D]$ with some $D\ge 1$. For any sequence of complex numbers $\beta=(\beta_b)_{1\le b\le B}$ we consider the sum

$$\mathcal{M}(\beta,\mathcal{D})=\sum_{d\in\mathcal{D}}\left|\sum_{1\le b\le B}\beta_b\left(\frac{d}{b}\right)\right|^2$$

with the aim of showing that

$$\mathcal{M}(\beta,\mathcal{D}) \le \mathfrak{d}(B, D, \Delta_1, \Delta_2)\, \|\beta\|^2 \tag{2.1}$$

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where $\mathfrak{d}(B, D, \Delta_1, \Delta_2)$ depends at most on B, D and two other parameters Δ_1, Δ_2 defined by

$$\Delta_1 = \sum_1^\infty v^{1/2} |\mathcal{D}_{v^2}|$$

and

$$\Delta_2 = \max_r r^{1/2} |\mathcal{D}_r|.$$

Here $\mathcal{D}_r$ stands for the subsequence of those elements in $\mathcal{D}$ which are divisible by r and $|\mathcal{D}_r|$ denotes its cardinality. While the first parameter Δ_1 measures how much $\mathcal{D}$ differs from the sequence of squares (on which the characters are trivial) the second one Δ_2 controls the multiplicity $\lambda(d)$ of elements d in $\mathcal{D}$, namely it yields

$$\lambda(d) \le d^{-1/2} \Delta_2 \le D^{-1/2} \Delta_2.$$

Our main result in this section is

THEOREM 2. *We have* (2.1) *with*

$$\mathfrak{d}(B, D, \Delta_1, \Delta_2) = c(\varepsilon)(BD)^\varepsilon \Delta_1^{2/3}\{B + \Delta_2^{1/6} D^{1/3} + \Delta_2^{1/3} D^{-1/6} B\}$$

where ε is any positive number and $c(\varepsilon)$ depends on ε alone.

As a corollary to Theorem 2 we shall deduce

THEOREM 3. *For any A, $B \ge 1$ and $\varepsilon > 0$ we have*

$$\sum_{1 \le a \le A} \left| \sum_{1 \le b \le B} \beta_b \left(\frac{a^2-4}{b} \right) \right|^2 \ll (AB)^\varepsilon (A^{3/2} + A^{2/3} B) \|\beta\|^2,$$

the constant implied in $\ll$ depending on ε alone.

By Cauchy's inequality Theorem 3 yields

COROLLARY. *For any A, $B \ge 1$ and $\varepsilon > 0$ we have*

$$\sum_{1 \le a \le A} \left| \sum_{1 \le b \le B} \beta_b \left(\frac{a^2-4}{b} \right) \right| \ll (AB)^\varepsilon (A^{5/4} + A^{5/6} B^{1/2}) \|\beta\|,$$

the constant implied in $\ll$ depending on ε alone.

In the proof of Theorem 2 we shall appeal to the following simpler result.

THEOREM 4. *Let $\mathcal{D}$ be a sequence of squarefree positive integers $d \le D$ (not necessarily distinct). We then have*

$$\sum_{1 \le b \le B} \left| \sum_{d \in \mathcal{D}} \gamma_d \left(\frac{d}{b} \right) \right|^2 \le c(\varepsilon)(BD)^\varepsilon (|\mathcal{D}| D^{1/2} + |\mathcal{D}|^{1/2} B) \|\gamma\|^2.$$

For clarity we split up the arguments into several lemmas.

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LEMMA 1 (Polya–Vinogradov). *If χ is a nonprincipal character* (mod q) *then*

$$\sum_{1\le n\le N}\chi(n)\ll q^{1/2}\log q.$$

LEMMA 2 (Poisson summation formula). *Let $f(x)$ be a smooth function on $\mathbb{R}$ such that $xf'(x)$ is bounded. We then have*

$$\sum_{n\equiv a\,(\mathrm{mod}\,q)} f(n)=\frac{1}{q}\sum_m e\left(-\frac{am}{q}\right)\hat{f}\left(\frac{m}{q}\right)$$

where $e(z)=e^{2\pi iz}$ and $\hat{f}(y)$ is the Fourier transform of $f(x)$.

LEMMA 3. *For $q>1$, $q\equiv 1$ (mod 8), q squarefree, put*

$$G(q,m)=\sum_{a\,(\mathrm{mod}\,q)}\left(\frac{q}{a}\right)e\left(-\frac{am}{q}\right).$$

We have

$$G(q,0)=0$$

and for $m\neq 0$, we have

$$G(q,m)=\left(\frac{q}{m}\right)\sqrt{q}.$$

Proof. This follows immediately from the quadratic reciprocity law and the well known formula for the Gaussian sum $G(q,1)=\sqrt{q}$.

Combining Lemmas 2 and 3 we infer

LEMMA 4. *Let $f(x)$ be a smooth function on $\mathbb{R}$ such that $xf'(x)$ is bounded, $r\ge 1$, $q>1$, $q\equiv 1$ (mod 8), q squarefree. We then have*

$$\sum_{(n,r)=1} f(n)\left(\frac{q}{n}\right)=\frac{1}{\sqrt{q}}\sum_{k\mid r}\frac{\mu(k)}{k}\sum_{m\neq 0}\left(\frac{q}{km}\right)\hat{f}\left(\frac{m}{kq}\right). \tag{2.2}$$

Proof. By Möbius inversion formula our sum is equal to

$$\sum_{k\mid r}\mu(k)\left(\frac{q}{k}\right)\sum_n f(kn)\left(\frac{q}{n}\right).$$

By Lemma 2 the innermost sum is equal to

$$\sum_{a\,(\mathrm{mod}\,q)}\left(\frac{q}{a}\right)\sum_{n\equiv a\,(\mathrm{mod}\,q)} f(kn)=(kq)^{-1}\sum_m G(q,m)\hat{f}(m/kq).$$

On applying Lemma 3 we complete the proof.

For the purpose of the proof of Theorem 2 it is convenient to take

$$f(x)=\exp\left(-\pi\left(\frac{x}{N}\right)^2\right) \tag{2.3}$$

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with some $N \geq 1$, so

$$\hat{f}(y) = N \exp(-\pi(yN)^2) < y^{-2} \exp(-(yN)^2).$$

Hence, for $|m| > \tau kqN^{-1}$ with some $\tau \geq 1$ to be chosen later, we have

$$\hat{f}\left(\frac{m}{kq}\right) < \left(\frac{\tau kq}{m}\right)^2 \exp(-\tau^2).$$

For the remaining m's we want to separate the variables in $\hat{f}(m/kq)$, so we write $\hat{f}(y)$ as the Mellin transform of the gamma function

$$\hat{f}(y) = \frac{1}{2\pi i} \int_{(\varepsilon)} \pi^{-s} \Gamma(s) N^{1-2s} y^{-2s}\, ds.$$

At this occasion notice that by Stirling's formula

$$|\Gamma(s)| \ll \varepsilon^{-1} \exp\left(-\frac{\pi}{2}|s|\right).$$

Now gathering together the above results we obtain a truncated form of (2.2).

LEMMA 5. *Let* $q > 1$, $q \equiv 1 \pmod 8$ *q square free, $f(x)$ be given by* (2.3) *and* $M_k \geq \tau kqN^{-1}$. *We then have (with some* $|\theta| \leq 1$)

$$\sum_{(n,r)=1} f(n)\left(\frac{q}{n}\right) = 4\theta (rq)^{3/2} \tau^2 \exp(-\tau^2) + \frac{1}{2\pi i} \int_{(2\varepsilon)} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) N^{1-s} \sum_{k \mid r} \mu(k) k^{s-1} q^{s-1/2} \sum_{1 \leq m \leq M_k} m^{-s} \left(\frac{q}{km}\right) ds.$$

By Lemma 5 we immediately obtain

LEMMA 6. *Let $f(x)$ be given by* (2.4), $Q \geq 1$, $M_k = \tau kQ^2N^{-1}$ *and* α_q *be any complex numbers supported in one of the four arithmetic progressions* $q = 1, 3, 5, 7 \pmod 8$. *We then have (with some* $|\theta| \leq 1$)

$$\sum_{1 < q_1, q_2 \leq Q} \mu^2(q_1 q_2) \alpha_{q_1} \alpha_{q_2} \sum_{(n,r)=1} f(n) \left(\frac{q_1 q_2}{n}\right) = 4\theta \tau^2 \exp(-\tau^2) r^{3/2} Q^4 \|\alpha\|^2 + \frac{1}{2\pi i} \int_{(\varepsilon)} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) N^{1-s} \sum_{k \mid r} \mu(k) k^{s-1} \sum_{\substack{1 \leq e \leq Q \\ (e,k)=1}} \mu(e) e^{2s-1} \times \sum_{\substack{1 \leq m \leq M_k \\ (m,e)=1}} m^{-s} \left(\sum_{1 < eh \leq Q} \mu^2(eh) \alpha_{eh} h^{s-1/2} \left(\frac{h}{km}\right)\right)^2 ds.$$

LEMMA 7 (duality principle). *The following two statements are equivalent*

(i) *for all complex numbers* α_m

$$\sum_n \left| \sum_m \alpha_m f(m, n) \right|^2 \leq \nu_f \sum_m |\alpha_m|^2$$

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(ii) *for all complex numbers* β_n

$$\sum_m \left|\sum_n \beta_n f(m,n)\right|^2 \le \nu_f \sum_n |\beta_n|^2.$$

I. Proof of Theorem 4. We have

$$\mathcal{M}(\beta,\mathcal{D}) \le \sum_{1\le b_1,b_2\le B} |\beta_{b_1}\beta_{b_2}| \left|\sum_{d\in\mathcal{D}} \left(\frac{d}{b_1b_2}\right)\right|.$$

By Cauchy's inequality and since $\tau(n) \ll n^\varepsilon$ we get

$$\mathcal{M}^2(\beta,\mathcal{D}) \ll \|\beta\|^4 B^\varepsilon \sum_{1\le n\le B^2} \left|\sum_{d\in\mathcal{D}} \left(\frac{d}{n}\right)\right|^2$$

$$\le \|\beta\|^4 B^\varepsilon \sum_{d_1,d_2\in\mathcal{D}} \left|\sum_{1\le n\le B^2} \left(\frac{d_1d_2}{n}\right)\right|.$$

If $d_1=d_2$ then we use the trivial bound $\Sigma_n \ll B^2$ and if $d_1\ne d_2$ then d_1d_2 is not a square, so by Lemma 1 $\Sigma_n \ll D\log 2D$. Gathering these results together we obtain

$$\mathcal{M}(\beta,\mathcal{D}) \le c(\varepsilon)(BD)^\varepsilon(|\mathcal{D}|\, D^{1/2} + |\mathcal{D}|^{1/2} B)\, \|\beta\|^2.$$

The dual form of the above (see Lemma 7) is just the assertion of Theorem 4.

II. Proof of Theorem 2. Every $d\in\mathcal{D}$ can be factored uniquely as $d=uv^2$ or $2uv^2$ where u is odd and squarefree. Therefore $\mathcal{D}$ can be split up into 8 disjoint subsequences according to the residue class $u \pmod 8$. Clearly it is enough to show (2.1) for each of such subsequences separately. The case of 4 subsequences of numbers $2uv^2$ can be reduced to the case of 4 subsequences of numbers uv^2 simply by changing the coefficients β_b into $\left(\frac{2}{b}\right)\beta_b$. In other words we may assume, without loss of generality, that all elements d in $\mathcal{D}$ have the squarefree parts odd and congruent (mod 8).

Now, we can write

$$\mathcal{M}(\beta,\mathcal{D}) = \sum_v \sum_{u\in\mathcal{D}(v)} |c(u,v)|^2$$

where $\mathcal{D}(v) = \{u;\, uv^2\in\mathcal{D},\ u \text{ squarefree}\}$ and

$$c(u,v) = \sum_{\substack{1\le b\le B\\ (b,v)=1}} \beta_b \left(\frac{u}{b}\right).$$

Hence by Cauchy's inequality we get

$$\mathcal{M}^2(\beta,\mathcal{D}) \le \Delta_1 \sum_v v^{-1/2} \sum_{u\in\mathcal{D}(v)} |c(u,v)|^4$$

$$\le \Delta_1 \sum_v v^{-1/2} \sum_{1\le b_1,b_2,b_3\le B} |\beta_{b_1}\beta_{b_2}\beta_{b_3}| \left|\sum_{u\in\mathcal{D}(v)} c(u,v)\left(\frac{u}{b_1b_2b_3}\right)\right|.$$

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By Cauchy's inequality again we obtain

$$\mathcal{M}^4(\beta,\mathcal{D}) \le \|\beta\|^6 \Delta_1^2\Big(\sum_{1\le v\le 2D}\frac{1}{v}\Big)\mathcal{S}(\beta,\mathcal{D}) \tag{2.4}$$

where

$$\mathcal{S}(\beta,\mathcal{D}) = \sum_{1\le v\le 2D}\ \sum_{1\le n\le B^3}\tau_3(n)\Big|\sum_{u\in\mathcal{D}(v)} c(u,v)\Big(\frac{u}{n}\Big)\Big|^2.$$

Here we have $\tau_3(n) \ll n^\varepsilon f(n)$ where $f(x)$ is given by (2.3) with $N = B^3$. Accordingly we obtain

$$\mathcal{S}(\beta,\mathcal{D}) \ll B^{3\varepsilon}\mathcal{S}_f(\beta,\mathcal{D}), \tag{2.5}$$

say. Squaring out the innermost sum in $\mathcal{S}_f(\beta,\mathcal{D})$ and changing the order of summation we get

$$\mathcal{S}_f(\beta,\mathcal{D}) = \sum_v \sum_{\substack{u_1,u_2\in\mathcal{D}(v)\\ u_1\ne u_2}} c(u_1,v)c(u_2,v)\sum_n f(n)\Big(\frac{u_1u_2}{n}\Big) + O(B^3\mathcal{M}(\beta,\mathcal{D})). \tag{2.6}$$

Put $r=(u_1,u_2)$, $u_1 = rq_1$, $u_2 = rq_2$, so $1<q_1,\ q_2\le Q = Q(r,v) = 4D/rv^2$. By Lemma 6 we deduce that

$$\begin{aligned}
&\sum_{\substack{u_1,u_2\in\mathcal{D}(v)\\ u_1\ne u_2}} c(u_1,v)c(u_2,v)\sum_n f(n)\Big(\frac{u_1u_2}{n}\Big)\\
&\quad= \sum_r \sum\sum_{\substack{1<q_1,q_2\le Q(r,v)\\ rq_1,rq_2\in\mathcal{D}(v)}} \mu^2(q_1q_2)c(rq_1,v)c(rq_2,v)\sum_{(n,r)=1} f(n)\Big(\frac{q_1q_2}{n}\Big)\\
&\quad\ll \tau^2\exp(-\tau^2)\sum_r r^{3/2}Q^4(r,v)\sum_{rq\in\mathcal{D}(v)}|c(rq,v)|^2\\
&\qquad+ D^\varepsilon B^3\sum_{1\le r\le D}\sum_{k\mid r}\frac{1}{k}\sum_{1\le e\le D}\frac{1}{e}\sum_{1\le m\le M}\Big|\sum_{\substack{1\le h\le H\\ ehr\in\mathcal{D}(v)}}\lambda_h\Big(\frac{h}{m}\Big)\Big|^2,
\end{aligned} \tag{2.7}$$

with

$$M = M(k/r^2v^4) = \tau k r^{-2}v^{-4}D^2B^{-3},$$
$$H = H(erv^2) = 4D/erv^2$$

and some λ_h independent of m such that

$$|\lambda_h| \le h^{-1/2}|c(ehr,v)|.$$

For the innermost sum we apply Theorem 4 giving

$$\sum_m\Big|\sum_h \lambda_h\Big(\frac{h}{m}\Big)\Big|^2 \ll D^\varepsilon\sum_{ehr\in\mathcal{D}(v)}|\lambda_h|^2\{|\mathcal{D}_{erv^2}|\,H^{1/2}(erv^2)+|\mathcal{D}_{erv^2}|^{1/2}M(k/r^2v^4)\}.$$

If $ehr\in\mathcal{D}(v)$ then $ehrv^2\in\mathcal{D}$ so $D<ehrv^2\le 4D$. This yields

$$\sum_{ehr\in\mathcal{D}(v)}|\lambda_h|^2 \ll erv^2D^{-1}\sum_{ehr\in\mathcal{D}(v)}|c(ehr,v)|^2.$$

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Moreover we have $|\mathcal{D}_{erv^2}| \le (erv^2)^{-1/2}\Delta_2$. Hence we conclude that

$$\sum_m \left|\sum_h \lambda_h\left(\frac{h}{m}\right)\right|^2 \ll D^\varepsilon\{\Delta_2 D^{-1/2} + \tau k\, \Delta_2^{1/2} D B^{-3}\} \sum_{ehr\in\mathcal{D}(v)} |c(ehr, v)|^2.$$

Inserting this into (2.7) by (2.6) we infer

$$\begin{aligned}\mathcal{S}_f(\beta, \mathcal{D}) &\ll \tau^2 \exp(-\tau^2) D^4 \sum_v \sum\sum_{rq\in\mathcal{D}(v)} |c(rq, v)|^2 \\ &\quad + D^\varepsilon\{\Delta_2 D^{-1/2} B^3 + \tau\, \Delta_2^{1/2} D\} \sum_v \sum\sum\sum_{ehr\in\mathcal{D}(v)} |c(ehr, v)|^2 \\ &\ll D^\varepsilon\{\Delta_2 D^{-1/2} B^3 + \Delta_2^{1/2} D\}\mathcal{M}(\beta, \mathcal{D})\end{aligned}$$

by taking $\tau = \log 4D$. Finally combining (2.4)–(2.6) we complete the proof of Theorem 2.

III. *Proof of Theorem* 3. We apply Theorem 2 for the sequence

$$\mathcal{D} = \{a^2 - 4;\ A < a \le 2A\}.$$

Therefore $D = A^2$ and it remains to determine the parameters Δ_1 and Δ_2.

If $a^2 \equiv 4 \pmod r$ then there exists a decomposition $r = r_1 r_2$ such that $a \equiv 2 \pmod{r_1}$ and $a \equiv -2 \pmod{r_2}$. Hence $(r_1, r_2) \mid 4$ and if $r = v^2$ then $r_1 = v_1^2$ or $2v_2^2$ and $r_2 = v_2^2$ or $2v_2^2$. This yields

$$\begin{aligned}\sum v^{1/2} |\mathcal{D}_{v^2}| &\le \sum_{r_1, r_2} (r_1 r_2)^{1/4} \sum_{\substack{a\equiv 2 (\mathrm{mod}\ r_1)\\ a\equiv -2 (\mathrm{mod}\ r_2)\\ A<a\le 2A}} 1 \\ &\le \sum_{r_1} r_1^{1/2} \sum_{\substack{a\equiv 2 (\mathrm{mod}\ r_1)\\ A<a\le 2A}} \tau(a+2) + \sum_{r_2} r_2^{1/2} \sum_{\substack{a\equiv -2 (\mathrm{mod}\ r_2)\\ A<a\le 2A}} \tau(a-2) \\ &\ll A^{1+\varepsilon}\left(\sum_{r_1} r_1^{-1/2} + \sum_{r_2} r_2^{-1/2}\right) \ll A^{1+\varepsilon} \log 2A.\end{aligned}$$

We also have

$$|\mathcal{D}_r| \ll \tau(r)\left(\frac{A}{r} + 1\right) \ll r^{-1/2} A^{1+\varepsilon}.$$

Therefore $\Delta_1, \Delta_2 \ll A^{1+\varepsilon}$ and the rest of the proof follows from Theorem 2.

3. Quadratic congruences. Let a and $c \ge 1$ be integers and let $\rho(c, a)$ stand for the number of incongruent solutions $x \pmod c$ of

$$x^2 - ax + 1 \equiv 0 \pmod c. \tag{3.1}$$

Our aim here is to evaluate $\rho(c, a)$ on average with respect to a and c. We have

$$\sum_{a (\mathrm{mod}\ c)} \rho(c, a) = \phi(c), \tag{3.2}$$

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so trivially

$$\sum_{1\le a\le A}\rho(c,a)=\frac{\phi(c)}{c}A+O(c).$$

The error term $O(c)$ proves to be too big for our applications in mind. On applying A. Weil's bounds for character sums (see Lemma 8) we can reduce the error term to $O(c^{1/2}\tau(c))$ which is still not satisfactory. In two papers [**5**], [**6**] sharper results were established on average with respect to c by means of D. Burgess' inequality.

In this section we improve the result (25) of [**6**] by an appeal to the corollary to Theorem 3.

THEOREM 5. *Let* A, $C\ge 1$ *and* $q\ge 1$, q *squarefree. For any* $\varepsilon>0$ *we have*

$$\mathcal{B}(A,C;q)=\sum_{\substack{1\le c\le C\\ c\equiv 0 (\mathrm{mod}\ q)}}\left|\sum_{2<a\le A}\rho(c,a)-\frac{\phi(c)}{c}A\right|$$

$$\ll (AC)^{\varepsilon}\{A^{5/6}+q^{1/4}A^{5/8}+A^{1/3}C^{1/6}\}\frac{C}{q},$$

the constant implied in $\ll$ *depending on* ε *alone.*

Proof. Every c can be uniquely factored as $c=kl$ where k is squarefree, $4l$ is squareful and $(k,4l)=1$. For notational simplicity in the sequel we do not repeat these properties of k and l, so the reader should keep them in mind to the end of the proof. Since $\rho(c,a)$ is multiplicative in c and k is squarefree and odd we have

$$\rho(c,a)=\rho(l,a)\rho(k,a)=\rho(l,a)\sum_{b\mid k}\left(\frac{a^2-4}{b}\right).$$

For a parameter X to be chosen later we partition $\rho(c,a)=\rho_1(c,a)+\rho_2(c,a)$ where

$$\rho_1(c,a)=\rho(l,a)\sum_{b\mid k,\, bl\le X}\left(\frac{a^2-4}{b}\right)$$

and

$$\rho_2(c,a)=\rho(l,a)\sum_{b\mid k,\, bl>X}\left(\frac{a^2-4}{b}\right).$$

The first term $\rho_1(c,a)$ contributes to the main term. We have

$$\sum_{2<a\le A}\rho_1(c,a)=\sum_{\substack{b\mid k\\ bl\le X}}\sum_{2<a\le A}\rho(l,a)\left(\frac{a^2-4}{b}\right)$$

$$=\sum_{\substack{b\mid k\\ bl\le X}}\sum_{\lambda (\mathrm{mod}\ l)}\rho(l,\lambda)\sum_{\substack{2<a\le A\\ a\equiv\lambda (\mathrm{mod}\ l)}}\left(\frac{a^2-4}{b}\right).$$

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We evaluate the innermost sum by

LEMMA 8. *If* b, $l \geq 1$, $(b, l) = 1$, b *squarefree then*

$$\sum_{\substack{2<a\leq A\\ a\equiv\lambda \pmod l}} \left(\frac{a^2-4}{b}\right) = \frac{\mu(b)}{bl} A + O(\tau(b) b^{1/2} \log 2A).$$

Proof. It follows in a standard way from A. Weil's bounds for character sums, precisely from (see [**8**])

$$\left|\sum_{a \pmod b} \left(\frac{a^2-4}{b}\right) e\left(\frac{ah}{b}\right)\right| \leq b^{1/2}\tau(b) \tag{3.3}$$

and that for $h=0$ the sum is equal to $\mu(b)$.

By Lemma 8 and (3.2) we further infer

$$\sum_{2<a\leq A} \rho_1(c, a) = A \frac{\phi(l)}{l} \sum_{\substack{b\mid k\\ bl\leq X}} \mu(b) b^{-1} + O((lX)^{1/2}(AC)^{\varepsilon})$$

$$= \frac{\phi(c)}{c} A + O\left\{\left(\frac{lA}{X} + (lX)^{1/2}\right)(AC)^{\varepsilon}\right\}.$$

Hence

$$\mathcal{B}_1 = \sum_{\substack{1\leq c\leq C\\ c\equiv 0 \pmod q}} \left|\sum_{2<a\leq A} \rho_1(c, a) - \frac{\phi(c)}{c} A\right|$$

$$\ll (AC)^{\varepsilon} \sum_{\substack{kl\leq C\\ kl\equiv 0 \pmod q}} \{lAX^{-1} + (lX)^{1/2}\}$$

$$\ll (AC)^{\varepsilon} C \sum_{l\leq C} \frac{(l, q)}{lq} \{lAX^{-1} + (lX)^{1/2}\}$$

$$\ll (AC)^{\varepsilon} \{AC^{1/2}X^{-1} + X^{1/2}\} \frac{C}{q}. \tag{3.4}$$

Now it remains to estimate

$$\mathcal{B}_2 = \sum_{\substack{1\leq c\leq C\\ c\equiv 0 \pmod q}} \left|\sum_{2<a\leq A} \rho_2(c, a)\right|.$$

Let $L \geq 1$ be a parameter to be chosen later. We split up the summation over c into two sums

$$\mathcal{B}_3 = \sum_{\substack{kl\leq C, l\leq L\\ kl\equiv 0 \pmod q}} \left|\sum_{2<a\leq A} \rho_2(kl, a)\right|$$

and

$$\mathcal{B}_4 = \sum_{\substack{kl\leq C, l> L\\ kl\equiv 0 \pmod q}} \left|\sum_{2<a\leq A} \rho_2(kl, a)\right|.$$

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First we estimate $\mathcal{B}_4$ by essentially elementary means. We have $|\rho_2(kl, 0)| \le \rho(l, a)\tau(k)$ and

$$\sum_{\substack{k \le C/l \\ k \equiv 0 (\mathrm{mod}\; q/(l,q))}} \tau(k) \ll \frac{(l, q)}{lq} C^{1+\varepsilon}.$$

Hence

$$\mathcal{B}_4 \ll \frac{C^{1+\varepsilon}}{q} \sum_{L < l \le C} \frac{(l, q)}{l} \sum_{2 < a \le A} \rho(l, a).$$

Since l is squareful the smallest n such that $l \mid n^2$ satisfies $l^{1/2} \le n \le l^{3/4}$. Moreover $\rho(l, a) \le \rho(n^2, a) \ll (a^2-4, n^2)^{1/2} n^{\varepsilon}$, so

$$\begin{aligned}\mathcal{B}_4 &\ll q^{-1} C^{1+\varepsilon} \sum_{L^{1/2} < n \le C^{3/4}} (n, q) n^{-4/3} \sum_{2 < a \le A} (a^2-4, n^2)^{1/2} \\ &\ll q^{-1} C^{1+\varepsilon} L^{-1/6} A. \end{aligned} \tag{3.5}$$

This bound is admissible for $L = A$.

Now it remains to estimate $\mathcal{B}_3$. Letting $\nu(c)$ be the sign of the innermost sum we obtain

$$\mathcal{B}_3 = \sum_{\substack{kl \le C, l < L \\ kl \equiv 0 (\mathrm{mod}\; q)}}\sum \nu(kl) \sum_{2 < a \le A} \rho(l, a) \sum_{\substack{b \mid k \\ b > X/l}} \left(\frac{a^2-4}{b}\right).$$

Then writing $k = br$ we get by Cauchy's inequality

$$\begin{aligned}\mathcal{B}_3 &= \sum_{\substack{rl \le C \\ l < L, r < C/X}}\sum \sum_{2 < a \le A} \rho(l, a) \sum_{\substack{X/l < b < C/lr \\ brl \equiv 0 (\mathrm{mod}\; q)}} \nu(brl) \left(\frac{a^2-4}{b}\right) \\ &\ll \sum_{\substack{rl \le C \\ l < L, r < C/X}}\sum \left\{ A(l) \sum_{2 < a \le A} \left| \sum_{B_1 < b \le B_2} \nu(bqrl/(rl, q)) \left(\frac{a^2-4}{b}\right) \right|^2 \right\}^{1/2} \end{aligned} \tag{3.6}$$

where $B_1 = X(rl, q)/lq$, $B_2 = C(rl, q)/lq$ and

$$A(l) = \sum_{2 < a \le A} \rho^2(l, a) \le \frac{2A}{l} \sum_{a (\mathrm{mod}\; l)} \rho^2(l, a).$$

For any $c \ge 1$ we have

$$\sum_{a (\mathrm{mod}\; c)} \rho^2(c, a) = \#\{x, y (\mathrm{mod}\; c); (xy, c) = 1, (x-y)(xy+1) \equiv 0 (\mathrm{mod}\; c)\} \ll c^{1+\varepsilon}.$$

Therefore

$$A(l) \ll A^{1+\varepsilon}. \tag{3.7}$$

By (3.6), (3.7) and Theorem 3 we obtain

$$\begin{aligned}\mathcal{B}_3 &\ll (AC)^{\varepsilon} \sum_{l < L} \sum_{r < C/X} \left\{ A^{5/4} C^{1/2} \left(\frac{(rl, q)}{rlq}\right)^{1/2} + A^{5/6} C \frac{(rl, q)}{rlq} \right\} \\ &\ll (AC)^{\varepsilon} \sum_{l < L} \left\{ A^{5/4} C X^{-1/2} \left(\frac{(l, q)}{lq}\right)^{1/2} + A^{5/6} C \frac{(l, q)}{lq} \right\} \\ &\ll (AC)^{\varepsilon} \{ A^{5/4} C X^{-1/2} q^{-1/2} + A^{5/6} C q^{-1} \}. \end{aligned} \tag{3.8}$$

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Combining (3.4)–(3.8) we obtain

$$\mathcal{B}(A, C; q) \ll (AC)^\varepsilon \{A^{5/6} + q^{1/2}A^{5/4}X^{-1/2} + X^{1/2} + AC^{1/2}X^{-1}\}\frac{C}{q}$$

for any $X>0$. On putting $X = q^{1/2}A^{5/4} + A^{2/3}C^{1/3}$ we complete the proof.

By partial summation we infer from Theorem 5 the following

COROLLARY. *Let $0<\alpha\le 1$, $\xi\ge 0$, $q\ge 1$, q squarefree and $C\ge 1$. We then have*

$$\sum_{\substack{1\le c\le C\\ c\equiv 0 (\mathrm{mod}\, q)}} \left|\sum_{|a|\le \alpha c} e\left(\frac{a}{c}\xi\right)\left(\rho(c, a) - \frac{\phi(c)}{c}\right)\right|$$
$$\ll (1+\alpha\xi)\left\{\left(\frac{C}{q}\right)^{1/2} + (\alpha C)^{5/6} + q^{1/4}(\alpha C)^{5/8} + (\alpha C)^{1/3}C^{1/6}\right\}\frac{C^{1+\varepsilon}}{q}. \quad (3.9)$$

REMARK. We included the terms with $|a|\le 2$ using trivial bounds $\rho(qc_0, a) \ll c_0^{1/2}(c_0, q)^{1/2}(qc_0)^\varepsilon$ for q squarefree.

4. Estimates for sums of Kloosterman sums. In this section we apply the corollary to Theorem 5 to estimate sums of Kloosterman sums $\mathcal{S}(n, n; c)$ over moduli $c\equiv 0 \pmod q$ as well as over the coefficients n. It is the latter parameter which yields an extra saving compared to the Weil upper bound (1.3). By contrast the conjecture of Y. V. Linnik **[7]** and A. Selberg **[9]** predicts a cancellation of terms $\mathcal{S}(m, n; c)$ in sums over the moduli c. In fact the analogue of the Linnik–Selberg conjecture, namely the following statement

$$\sum_{c\equiv 0 (\mathrm{mod}\, q)} g\left(\frac{X}{c}\right)\mathcal{S}(m, n; c) \ll X^{1+\varepsilon}$$

for a smooth function $g(\xi)$ compactly supported in $\mathbb{R}^+$, and any $X\ge 1$ is equivalent to the eigenvalue conjecture (1.1).

We first prove the following general result.

THEOREM 6. *Let $f_0(\xi)$ be a smooth function supported in* $[1, 2]$, *$N\ge 1$, $C\ge 4N$, $q\ge 1$, q squarefree. Put $f(n) = f_0(n/N)$. We then have*

$$\sum_{\substack{C<c\le 2C\\ c\equiv 0 (\mathrm{mod}\, q)}} \left|\sum_n f(n)\mathcal{S}(n, n; c)\right| \ll \left\{\left(\frac{C}{q}\right)^{1/2} + \left(\frac{C}{N}\right)^{5/6} + q^{1/4}\left(\frac{C}{N}\right)^{5/8} + \left(\frac{C}{N}\right)^{1/3}C^{1/6} + \frac{C}{N^2}\right\}\frac{(CN)^{1+\varepsilon}}{q}.$$

Proof. We have

$$\mathcal{S}(n, n; c) = \sum_{a (\mathrm{mod}\, c)} e\left(\frac{a}{c}n\right)\rho(c, a).$$

By Poisson's formula

$$\sum_n f(n)e\left(\frac{a}{c}n\right) = \sum_h \hat{f}\left(h+\frac{a}{c}\right).$$

Suppose that $|a/c|\le\frac{1}{2}$, then

$$\hat{f}\left(h+\frac{a}{c}\right) \ll (hN)^{-2} \quad \text{if} \quad h\ne 0$$

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and

$$\hat{f}\left(\frac{a}{c}\right) \ll N^{-2} \quad \text{if} \quad |a/c| \geq \alpha = N^{\varepsilon-1}.$$

Hence

$$\sum_n f(n)\mathcal{S}(n, n; c) = \int f(\xi) \sum_{|a|<\alpha c} e\left(\frac{a}{c}\xi\right)\rho(c, a)\, d\xi + O(cN^{-2}).$$

Now, by (3.9) our sum is equal to

$$\sum_{\substack{C<c\leq 2C \\ c\equiv 0 (\text{mod } q)}} \frac{\phi(c)}{c} \left|\int f(\xi) \sum_{|a|\leq \alpha c} e\left(\frac{a}{c}\xi\right) d\xi\right|$$
$$+ O\left(\left\{\left(\frac{C}{q}\right)^{1/2} + \left(\frac{C}{N}\right)^{5/6} + q^{1/4}\left(\frac{C}{N}\right)^{5/8} + \left(\frac{C}{N}\right)^{1/3} C^{1/6} + \frac{C}{N^2}\right\} \frac{(CN)^{1+\varepsilon}}{q}\right).$$

Here, the innermost sum is equal to

$$\frac{e(a\xi) - e(-a\xi)}{e(\xi/c) - 1} + O(1).$$

Notice that $\xi/c \leq 2N/C \leq \frac{1}{2}$ and $\alpha\xi \geq N^\varepsilon$. Therefore integrating over ξ yields

$$\left|\int f(\xi) \sum_{|a|\leq \alpha c} e\left(\frac{a}{c}\xi\right) d\xi\right| \ll N.$$

Gathering the above results together we complete the proof.

From Theorem 6 it is easy to deduce the following

COROLLARY. *Let $g(\xi)$ be a smooth function supported in $[1, \sqrt{2}]$. For $q \geq 1$, q squarefree and $X \geq 2q$ we have*

$$\sum_{c\equiv 0(\text{mod } q)} \frac{1}{c} \left|\sum_n \frac{1}{n} \exp\left(-\frac{n}{q}\right) g\left(\frac{kn}{c} X\right) \mathcal{S}(n, n; c)\right| \ll (X^{5/6} + q^{1/4}X^{5/8}) \frac{X^\varepsilon}{q},$$

the constant implied in $\ll$ depending on $g(\xi)$ and ε at most.

Proof. The partial sum with $c \leq C_1 = X^{4/3}$ by Weil's upper bound (1.3) is

$$\sum_{c\leq C_1} \ll q^{-1}X^{2/3+\varepsilon}$$

and the partial sum with $c > C_2 = qX \log X$ by the trivial estimate $|\mathcal{S}(n, n; c)| \leq c$ is

$$\sum_{c>C_2} \leq \frac{1}{q} \sum_{c>X\log X} \sum_{nX>c} \exp(-n) \ll \frac{1}{q}.$$

We split the remaining range of summation over c into $\ll \log X$ subintervals of the type $(C, \sqrt{2}C)$ with $C_1 < C \leq C_2$. For each of the resulting sums separately Theorem 6 is

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applicable with $N = C/4\pi X$ and

$$f(x) = \frac{1}{x}\exp\left(-\frac{xC}{qX}\right)g\left(\frac{4\pi xC}{c}\right)$$

giving

$$\sum_{\substack{C<c\leq 2C\\ c\equiv 0(\mathrm{mod}\, q)}} \frac{1}{c}\left|\sum_n \frac{1}{n}\exp\left(-\frac{n}{q}\right)g\left(\frac{4\pi n}{c}X\right)\mathcal{S}(n, n; c)\right|$$

$$\ll \left\{\left(\frac{C}{q}\right)^{1/2} + X^{5/6} + q^{1/4}X^{5/8} + X^{1/3}C^{1/6} + \frac{X^2}{C}\right\}\frac{C^\varepsilon}{q}.$$

Gathering the above results together we complete the proof.

5. Lower bounds for Fourier coefficients of cusp forms. We shall show a prototype of (1.9). Our method is so special that it requires q be prime. Thus $\Gamma_0(q)$ has two inequivalent cusps ∞ and 0. Let $u_j(z)$ be a Maass cusp form whose Fourier expansions at $\mathfrak{a} = 0$ and $\mathfrak{a} = \infty$ are given by (1.7). Put

$$c_{j0} = \sum_1^\infty \frac{q}{n}\exp\left(-\frac{n}{q}\right)|\rho_{j0}(n)|^2$$

and

$$c_{j\infty} = \sum_1^\infty \frac{1}{n}\exp(-n)\,|\rho_{j\infty}(n)|^2.$$

THEOREM 7. *If λ_j is an exceptional eigenvalue then*

$$c_j = c_{j0} + c_{j\infty} \geq \sqrt{3}.$$

Proof. This result is Lemma 3 of [**6**]. Let $P(Y)$ stand for the euclidean strip

$$P(Y) = \{z = x + iy;\, |x| \leq \tfrac{1}{2},\, y \geq Y\}.$$

One can find positive numbers $Y_\mathfrak{a}$ such that

$$F \subseteq \bigcup_\mathfrak{a} \sigma_\mathfrak{a} P(Y_\mathfrak{a}). \tag{5.1}$$

Hence and by the Fourier expansions (1.7) we obtain

$$1 = \int_F |u_j(z)|^2\, dz \leq \sum_\mathfrak{a} \int_{\sigma_\mathfrak{a} P(Y_\mathfrak{a})} |u_j(z)|^2\, dz$$

$$= \sum_\mathfrak{a} \int_{P(Y_\mathfrak{a})} |u_j(\sigma_\mathfrak{a} z)|^2\, dz = 2\sum_\mathfrak{a}\sum_1^\infty |\rho_{j\mathfrak{a}}(n)|^2 \int_{2\pi n Y_\mathfrak{a}}^\infty K_{it_j}^2(y)\frac{dy}{y}$$

because $\rho_{j\mathfrak{a}}(n) = \rho_{j\mathfrak{a}}(-n)$. We have $0 < it_j < \frac{1}{2}$, so

$$\int_A^\infty K_{it_j}^2(y)\frac{dy}{y} \leq \int_A^\infty K_{1/2}^2(y)\frac{dy}{y} = \frac{\pi}{2}\int_A^\infty e^{-2y}\frac{dy}{y^2} \leq \frac{\pi}{2A}e^{-2A}.$$

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This yields

$$\sum_{\mathfrak{a}} \frac{1}{Y_{\mathfrak{a}}} \sum_{1}^{\infty} \frac{1}{n} \exp(-4\pi n Y_{\mathfrak{a}}) |\rho_{j\mathfrak{a}}(n)|^2 \geq 2. \tag{5.2}$$

Now notice that $\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_0 = \begin{pmatrix} 0 & -1/\sqrt{q} \\ \sqrt{q} & 0 \end{pmatrix}$ satisfy (1.6) and that $Y_\infty = \sqrt{3}/2$ and $Y_0 = \sqrt{3}/2q$ satisfy (5.1) completing the proof of Theorem 7.

6. Proof of the density theorem. We begin by applying Kuznetsov's formula for the Hecke group $\Gamma = \Gamma_0(q)$, see [**1**]. Let $\{u_j(z)\}$ be the orthonormal basis of Maass cusp forms whose Fourier expansions at a cusp $\mathfrak{a}$ are given by (1.7). Let $E_{\mathfrak{c}}(z, s)$ be the Eisenstein series associated with the cusp $\mathfrak{c}$ whose Fourier expansion at $\mathfrak{a}$ is given by

$$E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z, s) = \text{constant term} + \sqrt{y} \sum_{n \neq 0} \phi_{\mathfrak{c}\mathfrak{a}n}(s) K_{s-1/2}(2\pi |n| y) e(nx).$$

Let $\{\Psi_{jk}(z)\}_{1 \leq j \leq \theta_k}$ be an orthonormal basis of the space $\mathfrak{M}_k^0(\Gamma)$ of holomorphic cusp forms of weight k whose Fourier expansion at $\mathfrak{a}$ is given by

$$\psi_{jk}(\sigma_{\mathfrak{a}} z) = j(\sigma_{\mathfrak{a}}, z)^k \sum_{1}^{\infty} \psi_{jk}(\mathfrak{a}, n) e(nz).$$

Let $f(x)$ be a smooth function supported in $(0, \infty)$. Define

$$\hat{f}(t) = \frac{\pi}{2i \operatorname{sh} \pi t} \int_0^\infty [J_{2it}(x) - J_{-2it}(x)] f(x) \frac{dx}{x},$$

$$\tilde{f}(k) = \int_0^\infty J_k(x) f(x) \frac{dx}{x},$$

$$V_0(\mathfrak{a}, n) = \sum_{0 < \lambda_j < 1/4} \frac{\hat{f}(t_j)}{\operatorname{ch} \pi t_j} |\rho_{j\mathfrak{a}}(n)|^2,$$

$$V_1(\mathfrak{a}, n) = \sum_{\lambda_j \geq 1/4} \frac{\hat{f}(t_j)}{\operatorname{ch} \pi t_j} |\rho_{j\mathfrak{a}}(n)|^2,$$

$$V_2(\mathfrak{a}, n) = \frac{1}{2\pi} \sum_{k \text{ even}} i^k \tilde{f}(k-1) \frac{(k-1)!}{(4\pi n)^{k-1}} \sum_{1 \leq j \leq \theta_k} |\psi_{jk}(\mathfrak{a}, n)|^2,$$

$$V_3(\mathfrak{a}, n) = \frac{1}{\pi} \sum_{\mathfrak{c}} \int_{-\infty}^{\infty} \hat{f}(t) |\phi_{\mathfrak{c}\mathfrak{a}n}(\tfrac{1}{2} + it)|^2 \, dt,$$

$$\mathcal{S}(\mathfrak{a}, n) = \sum_{c > 0} \frac{1}{c} \mathcal{S}_{\mathfrak{a}\mathfrak{a}}(n, n; c) f\left(\frac{4\pi n}{c}\right).$$

Here $\mathcal{S}_{\mathfrak{a}\mathfrak{a}}(n, n; c)$ is the generalized Kloosterman sum. In case of $\Gamma = \Gamma_0(q)$, $\mathfrak{a} = 0$ or ∞ we have $c \equiv 0 \pmod{q}$ and the Kloosterman sums $\mathcal{S}_{\mathfrak{a}\mathfrak{a}}(n, n; c)$ coincide with the classical ones $\mathcal{S}(n, n; c)$.

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The sum formula of Kuznetsov says that

$$\sum_{i=0}^{3} V_i(\mathfrak{a}, n) = \mathcal{S}(\mathfrak{a}, n). \tag{6.1}$$

We take $f(x) = g(xX)$ where $g(\xi)$ is a smooth function whose graph is

and $X \geq 2q$. We have $\tilde{f}(k-1) \ll k^{-2}$ and for real t, $\hat{f}(t) \ll (t^2+1)^{-1} \log X$. This together with Theorem 2 of Deshouillers and Iwaniec [**1**] shows that the series $V_i(\mathfrak{a}, n)$ with $i = 1, 2, 3$ converges rapidly and that

$$V_i(\mathfrak{a}, n) \ll \left(1 + \frac{n}{q}\right)(nX)^{\varepsilon}, \qquad i = 1, 2, 3. \tag{6.2}$$

Hence by (6.1)

$$V_0(\mathfrak{a}, n) = \mathcal{S}(\mathfrak{a}, n) + O\left(\left(1 + \frac{n}{q}\right)(nX)^{\varepsilon}\right). \tag{6.3}$$

Multiply both sides of (6.3) by

$$\frac{q}{n} \exp\left(-\frac{n}{q}\right) \quad \text{if} \quad \mathfrak{a} = 0$$

and by

$$\frac{1}{n} \exp(-n) \quad \text{if} \quad \mathfrak{a} = \infty$$

and sum over $n = 1, 2, \ldots,$ and $\mathfrak{a} = 0, \infty$ getting (see Theorem 7)

$$\begin{aligned} \sum_{0<\lambda_j<1/4} c_j \frac{\hat{f}(t_j)}{\operatorname{ch} \pi t_j} &= \sum_{1}^{\infty} \frac{q}{n} \exp\left(-\frac{n}{q}\right) \mathcal{S}(0, n) \\ &\quad + \sum_{1}^{\infty} \frac{1}{n} \exp(-n) \mathcal{S}(\infty, n) \\ &\quad + O(qX^{\varepsilon}) \\ &\ll (X^{5/6} + q^{1/4} X^{5/8} + qX^{\varepsilon}), \end{aligned}$$

the last inequality following from the Corollary to Theorem 6.

On the left hand side the arguments t_j of $\hat{f}(t_j)$ are purely imaginary. Using the power series expansion of the Bessel functions $J_{2it_j}(x)$ we deduce that

$$\hat{f}(t_j) \gg X^{2|t_j|} = X^{2(s_j - 1/2)}.$$

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Combining this with (6.4) and Theorem 6 we conclude that

$$\sum_{1/2<s_j<1} X^{2(s_j-1/2)} \ll (X^{5/6}+q^{1/4}X^{5/8}+q)X^{\varepsilon}$$

Putting $X=q^{6/5}$ we complete the proof.

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POLISH ACADEMY OF SCIENCES,
MATHEMATICS INSTITUTE,
UL. SNIADECKICH 8,
(00-950) WARSAW
POLAND

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