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# IDEAL EXTENSIONS OF Γ-RINGS

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**Abstract**

Given Γ-rings $N_1$ and $N_2$, a construction similar to the Everett sum of rings to find all possible extensions of $N_1$ by $N_2$ is given. Unlike the case of rings, it is not possible to find for any Γ-ring $M$ an ideal extension that has a unity. Furthermore, contrary to the ring case, a Γ-ring with unity can not be characterized as a Γ-ring which is a direct summand in every extension thereof.

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## 1. Introduction

The extension problem is a central one in the study of a specific algebraic structure. It is our purpose here to give a solution to this problem for Γ-rings. Γ-rings were introduced by Nobusawa [5] to provide an algebraic home for the groups Hom $(A, B)$ and Hom $(B, A)$ (where $A$ and $B$ are abelian groups) and the relationship between them. Since its inception, Γ-rings have received much attention, see our references and their references, (for example, Booth and Groenewald [2] and Kyuno [4]).

An extension $E$ of a Γ-ring $M$ by a Γ-ring $N$ is a Γ-ring $E$ satisfying the following conditions: (i) $M$ is isomorphic to an ideal $I$ of $E$ and (ii) $E/I$ is isomorphic to $N$. An extension $E$ of $M$ by $N$ will be denoted by a triple $(f, E, g)$ where $f$ is the Γ-ring isomorphism mapping $M$ onto the ideal $I$ of $E$ and $g$ is the Γ-ring homomorphism from $E$ onto $N$ with $M$ (or its isomorphic image $I$ in $E$) as its kernel. The functions $f$

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and $g$ will be referred to as the functions associated with the extension. In most cases we identify $M$ with $I$ and $N$ with $E/I$. The solutions for the corresponding problem for groups and rings were given by Schreir [8] and Everett [3] respectively (but see also Rédei [7] for an account of both).

As a generalization of rings, it can be expected that the solution of the extension problem for Γ-rings should be along the lines of the ring case. Using the ideas of Petrich [6], who reformulated the ring extension theorem in terms of what he calls the *translational hull* of a ring this is to an extent the case. But there are some striking differences as well. In Section 2, the translational hull of a Γ-ring is described. This is then used in Section 3 to construct extensions of Γ-rings. The solution to the extension problem for Γ-rings as well as criteria for the equivalence of extensions are given. Using the concepts of double homothetisms and holomorphs developed in Sections 4 and 5, a discussion on extensions of Γ-rings to Γ-rings with unity is given in Section 6. The main result here is that, unlike the case for rings, it is not possible to embed any Γ-ring as an ideal in a Γ-ring with unity. Furthermore, contrary to the ring case, where a ring with an identity can be characterized as a ring which is a direct summand in every extension thereof, an example is given to show that this is not the case for Γ-rings.

## 2. Translational hull of a Γ-ring

Γ-rings as a generalization of rings were first defined by Nobusawa [3]. The definition that we will use is the somewhat weaker one due to Barnes [1]. In the sequel $M$ denotes a Γ-ring. We now introduce the translational hull of $M$ along the lines of the ring case (cf. Petrich [6]).

DEFINITION 2.1. Let $\mathscr{E}(M) = \{p : \Gamma \to \operatorname{End}(M^+) | p \text{ is a group homomorphism}\}$.

(1) If $r \in \mathscr{E}(M)$ then $r$ is a *left translation of* $M$ (in which case the argument is written on the right and $r(\gamma)$ is denoted by $r_\gamma$) if

$$r_\gamma(m_1\mu m_2) = (r_\gamma(m_1))\mu m_2 \quad \text{for all } m_1, m_2 \in M, \gamma, \mu \in \Gamma.$$

(2) If $q \in \mathscr{E}(M)$ then $q$ is a *right translation of* $M$, (in which case the argument is written on the left and $(\gamma)q$ is denoted by ${}_\gamma q$) if

$$(m_1\mu m_2)_\gamma q = m_1\mu((m_2)_\gamma q) \quad \text{for all } m_1, m_2 \in M, \gamma, \mu \in \Gamma.$$

(3) A pair $p = (r, q) \in \mathscr{E}(M) \times \mathscr{E}(M)$ is called a *bitranslation of* $M$ if $r$ is a left translation, $q$ is a right translation and for any $m_1, m_2 \in M, \gamma, \mu \in \Gamma$ $m_1\gamma(r_\mu(m_2)) = ((m_1)_\gamma q)\mu m_2$, in which case $(r, q)$ is said to be *linked.*

A bitranslation $p$ will be considered as a double operator with $p(\gamma) = p_\gamma = r_\gamma$ and $(\gamma)p = {}_\gamma p = {}_\gamma q$.

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**Theorem 2.2.** *Let $M$ be a $\Gamma$-ring. Both the sets $\mathscr{E}_\ell(M)$ and $\mathscr{E}_r(M)$ of all the left and right translations of $M$ respectively are $\Gamma$-rings.*

**Proof.** Let $\mathscr{E}_\ell(M) = \{p \in \mathscr{E}(M) | p \text{ is a left translation}\}$. The set $\mathscr{E}_\ell(M)$ is not empty, cf. Example 2.5. Define addition on $\mathscr{E}_\ell(M)$ by $(p^1 + p^2)_\gamma = p^1_\gamma + p^2_\gamma$ for all $p^1, p^2 \in \mathscr{E}_\ell(M)$ and any $\gamma \in \Gamma$. Then $\mathscr{E}_\ell(M)$ is an abelian group with zero element 0 defined by $0_\gamma(m) = 0$ and the additive inverse $-p$ of $p \in \mathscr{E}_\ell(M)$, defined by $(-p)_\gamma(m) = -p_\gamma(m)$. Define the map $(-, -, -): \mathscr{E}_\ell(M) \times \Gamma \times \mathscr{E}_\ell(M) \to \mathscr{E}_\ell(M)$ by $(p^1\gamma p^2)_\mu = p^1_\gamma \circ p^2_\mu$ for all $p^1, p^2 \in \mathscr{E}_\ell(M)$, $\gamma, \mu \in \Gamma$, where $\circ$ is the usual composition of functions. This mapping is well defined and straightforward calculations will confirm that $\mathscr{E}_\ell(M)$ is a $\Gamma$-ring with respect to the operations defined above. By defining, for any $q^1, q^2 \in \mathscr{E}_\Gamma(M)$ and $\gamma \in \Gamma$, $q^1\gamma q^2$ to be the right translation given by ${}_\lambda(q^1\lambda q^2) = {}_\lambda q^1 \circ {}_\gamma q^2$ for all $\lambda \in \Gamma$, we can show similarly that $\mathscr{E}_r(M)$ is a $\Gamma$-ring.

**Definition 2.3.** The set $\mathscr{E}_2(M)$ of all bitranslations of $M$ is called the *translational hull* of $M$.

**Theorem 2.4.** *The translational hull $\mathscr{E}_2(M)$ of a $\Gamma$-ring $M$ is a $\Gamma$-ring with respect to the operations defined by $(p+q)_\gamma = p_\gamma + q_\gamma$, ${}_\gamma(p+q) = {}_\gamma p + {}_\gamma q$, $(p\gamma q)_\lambda = p_\gamma \circ q_\lambda$ and ${}_\lambda(p\gamma q) = {}_\lambda p \circ {}_\gamma q$ for all $p, q \in \mathscr{E}_2(M)$, $\gamma, \lambda, \in \Gamma$.*

**Proof.** The set $\mathscr{E}_2(M)$ is not empty (cf. Example 2.5) and $\mathscr{E}_2(M)$ is an abelian group: We only show $\mathscr{E}_2(M)$ is closed under addition: Let $p, q \in \mathscr{E}_2(M)$. If $p = (r^1, s^1)$ and $q = (r^2, s^2)$, then $p + q = (r^1 + r^2, s^1 + s^2)$. Theorem 2.2 yields that $r^1 + r^2$ and $s^1 + s^2$ are left and right translations respectively. To see that $p + q \in \mathscr{E}_2(M)$, let $m_1, m_2 \in M$ and $\gamma, \mu \in \Gamma$. Then

$$\begin{aligned} m_1\mu((p+q)_\gamma(m_2)) &= m_1\mu(p_\gamma(m_2)) + m_1\mu(q_\gamma(m_2)) \\ &= ((m_1)_\mu p)\gamma m_2 + ((m_1)_\mu q)\gamma m_2 \\ &= ((m_1)_\mu(p+q))\gamma m_2. \end{aligned}$$

Thus $p + q$ is linked and so $p + q \in \mathscr{E}_2(M)$.

Define the map $(-, -, -): \mathscr{E}_2(M) \times \Gamma \times \mathscr{E}_2(M) \to \mathscr{E}_2(M)$ by $(p\gamma q)_\lambda = p_\gamma \circ q_\lambda$ and ${}_\lambda(p\gamma q) = {}_\lambda p \circ {}_\gamma q$. Let $p, q \in \mathscr{E}_2(M)$. If $p = (r^1, s^1)$ and $q = (r^2, s^2)$, then $p\gamma q = (r^1\gamma r^2, s^1\gamma s^2)$ where $r^1\gamma r^2$ and $s^1\gamma s^2$ are defined by $(r^1\gamma r^2)_\lambda = 4^1_\gamma \circ r^2_\lambda$ and ${}_\lambda(s^1\gamma s^2) = {}_\lambda s^1 \circ {}_\gamma s^2$. Then $r^1\gamma r^2 \in \mathscr{E}_l(M)$ and

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$s^1\gamma s^2 \in \mathscr{E}_r(M)$ (from Theorem 2.2). For any $m_1, m_2 \in M$, $\lambda, \mu \in \Gamma$,

$$m_1\mu((p\gamma q)_\lambda(m_2)) = m_1\mu(p_\gamma(q_\lambda(m_2))) = ((m_1)_\mu p)\gamma(q_\lambda(m_2))$$

and

$$((m_1)_\mu(p\gamma q))\gamma m_2 = (((m_1)_\mu p)_\gamma q)\lambda m_2 = ((m_1)_\mu p)\gamma(q_\lambda(m_2)).$$

Hence $p\gamma q$ is linked so that $p\gamma q \in \mathscr{E}_\ell(M)$. The rest of the proof that $\mathscr{E}_2(M)$ is a Γ-ring follows directly from the proofs that $\mathscr{E}_\ell(M)$ and $\mathscr{E}_r(M)$ are Γ-rings.

The Γ-ring $\mathscr{E}_2(M)$ will be used in Section 3 to construct extensions of Γ-rings, while $\mathscr{E}_\ell(M)$ and $\mathscr{E}_r(M)$ are of good use when considering Γ-ring extensions with unity (cf. Section 6). The next example shows that $\mathscr{E}_2(M)$ (and also $\mathscr{E}_\ell(M)$ and $\mathscr{E}_r(M)$) is not empty for any Γ-ring $M$.

EXAMPLE 2.5. Any $m \in M$ determines a bitranslation of $M$ as follows: Define $p^m : \Gamma \to \mathrm{End}(M^+)$ and $q^m : \Gamma \to \mathrm{End}(M^+)$ by $p^m(\gamma) = p^m_\gamma$ and $(\gamma)q^m = {}_\gamma q^m$ where $p^m_\gamma(n) = m\gamma n$ and $(n)_\gamma q^m = n\gamma m$ for all $m, n \in M$ and $\gamma \in \Gamma$. It is clear that $p^m$ is a left translation and $q^m$ is a right translation of $M$. The pair $(p^m, q^m)$ is linked; hence $(p^m, q^m)$ is a bitranslation of $M$ which we will denote by $[m] = (p^m, q^m)$.

DEFINITION 2.6. The bitranslation $[m]$ constructed in Example 2.5 is called the *inner bitranslation of M induced by m*. The set of all inner bitranslations of $M$ will be denoted by $\mathscr{I}(M)$.

The inner bitranslations play an important role in the theory of Γ-rings with unities (cf. Sections 4 and 6).

DEFINITION 2.7. (i) Two bitranslations $p$ and $q$ of $M$ are *amicable* if for any $\gamma, \mu \in \Gamma$ and $m \in M$

$$p_\gamma((m)_\mu q) = (p_\gamma(m))_\mu q \quad \text{and} \quad q_\gamma((m)_\mu p) = (q_\gamma(m))_\mu p.$$

(ii) An *amicable set of bitranslations* of $M$ is a set of bitranslations of $M$ for which all the elements are pairwise amicable.

THEOREM 2.8. *$\mathscr{I}(M)$ is a set of amicable bitranslations of $M$ and $\mathscr{I}(M)$ is an ideal of the Γ-ring $\mathscr{E}_2(M)$ of bitranslations of $M$.*

PROOF. It is straightforward to verify that $\mathscr{I}(M)$ is an ideal of $\mathscr{E}_2(M)$. Any two elements of $\mathscr{I}(M)$ are amicable: Let $[n_1], [n_2] \in \mathscr{I}(M)$. Then for any $m \in M$, $\gamma, \mu \in \Gamma$:

$$[n_1]_\gamma((m)_\mu[n_2]) = n_1\gamma(m\mu n_2) = (n_1\gamma m)\mu n_2 = ([n_1]_\gamma(m))_\mu[n_2] \quad \text{and}$$
$$[n_2]_\gamma((m)_\mu[n_1]) = n_2\gamma(m\mu n_1) = (n_2\gamma m)\mu n_1 = ([n_2]_\gamma(m))_\mu[n_1].$$

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## 3. Extensions of Γ-rings

A Γ-ring can be considered as an Ω-group. As such we immediately have at our disposal the concepts homomorphism, isomorphism, kernel and the isomorphism theorems. The following construction will show, given Γ-rings $M$ and $N$, how an extension of $M$ by $N$ can be constructed. Recall, for $m \in M$, $[m]$ is the inner bitranslation of $M$ induced by $m$.

CONSTRUCTION 3.1. Let $M$ and $N$ be two Γ-rings. Let $(p, F, G)$ be a triple of functions with $p: N \to \mathscr{E}_2(M)$ denoting $p(n)$ by $p^n \in \mathscr{E}_2(M)$, $F: N \times N \to M$ and $g: N \times \Gamma \times N \to M$ satisfying the following conditions for all $n, n_1, n_2, n_3 \in N$ and $\gamma, \mu \in \Gamma$:

(E1) $F(n, 0) = F(0, n) = G(0, \gamma, n) = G(n, \gamma, 0) = 0$; $p^0 = [0]$;
(E2) $p^{n_1}$ is amicable with $p^{n_2}$;
(E3) $p^{n_1} + p^{n_2} - p^{n_1+n_2} = [F(n_1, n_2)]$;
(E4) $p^{n_1}\gamma p^{n_2} - p^{n_1\gamma n_2} = [G(n_1, \gamma, n_2)]$;
(E5) $F(n_1, n_2) = F(n_2, n_1)$;
(E6) $F(n_1, n_2) + F(n_1 + n_2, n_3) = F(n_1, n_2 + n_3) + F(n_2, n_3)$;
(E7) $G(n_1\gamma n_2, \mu, n_3) - G(n_1, \gamma, n_2, \mu n_3) = p^{n_1}_\gamma(G(n_2, \mu, n_3)) -(G(n_1, \gamma, n_2))_\mu p^{n_3}$;
(E8) $G(n_1, \gamma, n_3) + G(n_2, \gamma, n_3) - G(n_1 + n_2, \gamma, n_3) = (F(n_1, n_2))_\gamma p^{n_3} - F(n_1\gamma n_3, n_2\gamma n_3)$;
(E9) $G(n_1, \gamma, n_2) + G(n_1, \gamma, n_3) - G(n_1, \gamma, n_2 + n_3) = p^{n_1}_\gamma(F(n_2, n_3)) - F(n_1\gamma n_2, n_1\gamma n_3)$;
(E10) $G(n_1, \gamma + \mu, n_2) = G(n_1, \gamma, n_2) + G(n_1, \mu, n_2) + F(n_1\gamma n_2, n_1\mu n_2)$.

Let $E = N \times M$ with addition defined on $E$ by

$$(n_1, m_1) + (n_2, m_2) = (n_1 + n_2, F(n_1, n_2) + m_1 + m_2)$$

and a mapping $(-, -, -): E \times \Gamma \times E \to E$ defined by

$$(n_1, m_1)\gamma(n_2, m_2) = (n_1\gamma n_2, G(n_1, \gamma, n_2) + p^{n_1}_\gamma(m_2) + (m_1)_\gamma p^{n_2} + m_1\gamma m_2).$$

Define the functions:

$$f: M \to E \quad \text{by } f(m) = (0, m) \quad \text{for all } m \in M \quad \text{and}$$
$$g: E \to N \quad \text{by } g(n, m) = n \quad \text{for all } (n, m) \in E.$$

DEFINITION 3.2. The triple $(f, E, g)$ of Construction 3.1 is denoted by $E(p, F, G)$ and is called an *E-sum* of the Γ-rings $N$ and $M$.

THEOREM 3.3. *The E-sum $E(p, F, G)$ of the Γ-rings $N$ and $M$ is an extension of $M$ by $N$.*

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**Proof.** Using conditions (E1), (E5) and (E6), it can be verified that $E$ is an abelian group with zero element $(0, 0)$ and $-(n, m) = (-n, -F(n, -n) - m)$. $E$ is a Γ-ring: Let $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in E, \gamma, \mu \in \Gamma$. Then

$$\begin{aligned}
\text{(i) } &((n_1, m_1) + (n_2, m_2))\gamma(n_3, m_3) \\
&= ((n_1 + n_2)\gamma n_3, G(n_1 + n_2, \gamma, n_3) + p^{n_1+n_2}(m_3) \\
&\quad + (F(n_1, n_2) + m_1 + m_2)_\gamma p^{n_3} + (F(n_1, n_2) + m_1 + m_2)\gamma m_3) \\
&= (n_1\gamma n_3 + n_2\gamma n_3, G(n_1 + n_2, \gamma, n_3) + p_\gamma^{n_1}(m_3) + p_\gamma^{n_2}(m_3) \\
&\quad - [F(n_1, n_2)]_\gamma(m_3) \\
&\quad + (F(n_1, n_2))_\gamma p^{n_3} + (m_1)_\gamma p^{n_3} + (m_2)_\gamma p^{n_3} + F(n_1, n_2)\gamma m_3 \\
&\quad + m_1\gamma m_3 + m_2\gamma m_3) \quad \text{(Condition (E3))} \\
&= (n_1\gamma n_3 + n_2\gamma n_3, F(n_1\gamma n_3, n_2\gamma n_3) + G(n_1, \gamma, n_3) \\
&\quad + G(n_2, \gamma, n_3) + p_\gamma^{n_1}(m_3) + p_\gamma^{n_2}(m_3) + (m_1)_\gamma p^{n_3} + (m_2)_\gamma p^{n_3} \\
&\quad + m_1\gamma m_3 + m_2\gamma m_3) \quad \text{(Condition (E8))} \\
&= (n_1, m_1)\gamma(n_3, m_3) + (n_2, m_2)\gamma(n_3, m_3).
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } &(n_1, m_1)(\gamma + \mu)(n_2, m_2) \\
&= (n_1\gamma n_2 + n_1\mu n_2, F(n_1\gamma n_2, n_1\mu n_2) + G(n_1, \gamma, n_2) \\
&\quad + G(n_1, \mu, n_2) + p_\gamma^{n_1}(m_2) \\
&\quad + p_\mu^{n_1}(m_2) + (m_1)_\gamma p^{n_2} + (m_1)_\mu p^{n_2} + m_1\gamma m_2 + m_1\mu m_2) \\
&\qquad\qquad \text{(Condition (E10))} \\
&= (n_1, m_1)\gamma(n_2, m_2) + (n_1, m_1)\mu(n_2, m_2).
\end{aligned}$$

$$\begin{aligned}
\text{(iii) } &(n_1, m_1)\gamma((n_2, m_2) + (n_3, m_3)) \\
&\quad = (n_1, m_1)\gamma(n_2, m_2) + (n_1, m_1)\gamma(n_3, m_3)
\end{aligned}$$

is similar to (i) using conditions (E3) and (E9).

$$\begin{aligned}
\text{(iv) } &(n_1, m_1)\gamma((n_2, m_2)\mu(n_3, m_3)) \\
&= (n_1\gamma(n_2\mu n_3), G(n_1, \gamma, n_2\mu n_3) + p_\gamma^{n_1}(G(n_2, \mu, n_3) + p_\mu^{n_2}(m_3) \\
&\quad + (m_2)_\mu p^{n_3} + m_2\mu m_3) \\
&\quad + (m_1)_\gamma p^{n_2}\mu n_3 + m_1\gamma(G(n_2, \mu, n_3) + p_\mu^{n_2}(m_3) \\
&\qquad\qquad + (m_2)_\mu p^{n_3} + m_2\mu m_3))
\end{aligned}$$

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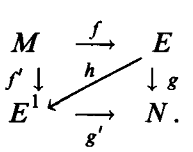
$$
\begin{aligned}
&= ((n_1\gamma n_2)\mu n_3\,,\ G(n_1\,,\gamma\,,n_2\mu n_3) + p_\gamma^{n_1}(G(n_2\,,\mu\,,n_3)) + p_\gamma^{n_1}(p_\mu^{n_2}(m_3)) \\
&\quad + p_\gamma^{n_1}((m_1)_\mu p^{n_3}) + p_\gamma^{n_1}(m_2\mu m_3) + (m_1)_\gamma(p^{n_2}\mu p^{n_3}) \\
&\quad - (m_1)_\gamma[G(n_2\,,\mu\,,n_3)] + m_1\gamma G(n_2\,,\mu\,,n_3) + m_1\gamma p_\mu^{n_2}(m_3) \\
&\quad + m_1\gamma((m_2)_\mu p^{n_3}) + m_1\gamma(m_2\mu m_3)) \quad \text{(Condition (E4))} \\
&= ((n_1\gamma n_2)\mu n_3\,,\ G(n_1\,,\gamma\,,n_2\mu n_3) + p_\gamma^{n_1}(G(n_2\,,\mu\,,n_3)) \\
&\quad + (p^{n_1}\gamma p^{n_2})_\mu(m_3) + p_\gamma^{n_1}((m_2)_\mu p^{n_3})\,,\ p_\gamma^{n_1}(m_2\mu m_3) \\
&\quad + ((m_1)_\gamma p_\mu^{n_2} -_\mu p^{n_3} - m_1\gamma G(n_2\,,\mu\,,n_3) \\
&\quad + m_1\gamma G(n_2\,,\mu\,,n_3) + m_1\gamma p_\mu^{n_2}(m_3) + m_1\gamma((m_2)_\mu p^{n_3}) \\
&\quad + m_1\gamma(m_2\mu m_3)) \\
&= ((n_1\gamma n_2)\mu n_3\,,\ G(n_1\gamma n_2\,,\mu\,,n_3) + (G(n_1\,,\gamma\,,n_2))_\mu p^{n_3} + p_\mu^{n_1\gamma n_2}(m_3) \\
&\quad + [G(n_1\,,\gamma\,,n_2)]_\mu(m_3) + (p_\gamma^{n_1}(m_2))_\mu p^{n_3} + (p_\gamma^{n_1}(m_2))\mu m_3 \\
&\quad + ((m_1)_\gamma p^{n_2})_\mu p^{n_3} + (m_1)_\gamma p^{n_2}\mu m_3 + (m_1\gamma m_2)_\mu p^{n_3} \\
&\quad + (m_1\gamma m_2)\mu m_3) \quad \text{(Conditions (E7), (E4) and (E2))} \\
&= ((n_1\gamma n_2)\mu n_3\,,\ G(n_1\gamma n_2\,,\mu\,,n_3) + p_\mu^{n_1\gamma n_2}(m_3) + (G(n_1\,,\gamma\,,n_2))_\mu p^{n_3} \\
&\quad + (p_\gamma^{n_1}(m_2))_\mu p^{n_3} + ((m_1)_\gamma p^{n_1})_\mu p^{n_3} + (m_1\gamma m_2)_\mu p^{n_3} \\
&\quad + G(n_1\,,\gamma\,,n_2)\mu m_3 \\
&\quad + (p_\gamma^{n_1}(m_2))\mu m_3 + (m_1)_\gamma p^{n_2}\mu m_3 + (m_1\gamma m_2)\mu m_3 \\
&= ((n_1\,,m_1)\gamma(n_2\,,m_2))\mu(n_3\,,m_3)\,.
\end{aligned}
$$

Let $I = \{(0\,,m)|m \in M\}$. Then $I \triangleleft M$ and the function $f\colon M \to E$ defined by $f(m) = (0\,,m)$ for all $m \in M$ is a $\Gamma$-ring isomorphism from $M$ onto the ideal $I$ of $E$. Furthermore, $g\colon E \to N$ defined by $g(n\,,m) = n$ for all $(n\,,m) \in E$ is a surjective $\Gamma$-ring homomorphism with $\ker(g) = I$. Thus $E/M \cong N$ and $E(p\,,F\,,G)$ is an extension of $M$ by $N$.

DEFINITION 3.4. Two extensions $(f\,,E\,,g)$ and $(f'\,,E'\,,g')$ of a $\Gamma$-ring $M$ by a $\Gamma$-ring $N$ are *equivalent* if there exists a $\Gamma$-ring isomorphism $h\colon E \to E'$ such that the following diagram commutes:




The next construction shows how an $E$-sum equivalent to a given extension of $M$ by $N$ can be found.

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CONSTRUCTION 3.5. Let $A$ be an extension of a $\Gamma$-ring $M$ by a $\Gamma$-ring $N$, with $M \triangleleft A$ and $A/M \cong N$. The elements of $N$ will be regarded as cosets determined by $M$ in $A$. Let $k\colon N \to A$ be a function on $N$ with $k(n) \in n$ such that $g \circ k$ is the identity function on $N$ where $g$ is the natural homomorphism of $A$ onto $N = A/M$, subject to $k(0) = 0$. Define the following functions:

(i) $p\colon N \to \mathscr{E}_2(M)$, we write $n \mapsto p^n$, by $p^n_\gamma(m) = k(n)\gamma m$ and $(m)_\gamma p^n = m\gamma k(n)$ for any $m \in M$ and $\gamma \in \Gamma$. In a sense $p^n$ can be regarded as the restriction of $[k(n)]$ to $M$; hence we sometimes write $p^n = [k(n)]|M$.

(ii) $F\colon N\times N \to M$ by $F(n_1, n_2) = k(n_1)+k(n_2)-k(n_1+n_2)$ for all $n_1$, $n_2 \in N$.

(iii) $G\colon N\times\Gamma\times N \to M$ by $G(n_1, \gamma, n_2) = k(n_1)\gamma k(n_2)-k(n_1\gamma n_2)$ for all $n_1, n_2 \in N$, $\gamma \in \Gamma$.

THEOREM 3.6. *The functions $p$, $F$ and $G$ of Construction 3.5 satisfy the conditions of Construction 3.1 to be an $E$-sum $E = E(p, F, G)$ of $N$ and $M$. Furthermore, the extension $A$ is equivalent to the $E$-sum by the equivalence isomorphism $l\colon A \to E$ defined by*

$$l(a) = (g(a), a - k(g(a))) \quad \textit{for all } a \in A.$$

PROOF. To see that $p$, $F$ and $G$ are well defined we observe that:

(i) $n \in N \Rightarrow k(n) \in A \Rightarrow [k(n)] \in \mathscr{E}_2(A)$ and thus $[k(n)]|M \in \mathscr{E}_2(M)$, since $M \triangleleft A$.

$$\text{(ii) } g(k(n_1) + k(n_2) - k(n_1 + n_2)) = g(k(n_1)) + g(k(n_2)) - g(k(n_1 + n_2)) \\ = n_1 + n_2 - (n_1 + n_2) = 0,$$

that is, $k(n_1) + k(n_2) - k(n_1 + n_2) \in \ker(g) = M$ for any $n_1, n_2 \in N$.

$$\text{(iii) } g(k(n_1)\gamma k(n_2) - k(n_1\gamma n_2)) = g(k(n_1))\gamma g(k(n_2)) - g(k(n_1\gamma n_2)) \\ = n_1\gamma n_2 - n_1\gamma n_2 = 0,$$

that is, $k(n_1)\gamma k(n_2) - k(n_1\gamma n_2) \in \ker(g) = M$ for any $n_1, n_2 \in N$, $\gamma \in \Gamma$.

For any $n, n_1, n_2, n_3 \in N$, $\gamma, \mu \in \Gamma$, $m \in M$, the conditions of Construction 3.1 are satisfied (using the definitions of $p$, $F$ and $G$ in Construction 3.5):

(E1) Follows directly from the definitions.

$$\text{(E2)} \qquad p^{n_1}_\gamma((m)_\mu p^{n_2}) = k(n_1)\gamma(m\mu k(n_2)) = (k(n_1)\gamma m)\mu k(n_2) \\ = (p^{n_1}_\gamma(m))_\mu p^{n_2};$$

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(E3) $$p^{n_1} + p^{n_2} - p^{n_1+n_2} = [k(n_1)] + [k(n_2)] - [k(n_1 + n_2)]$$
$$= [k(n_1) + k(n_2) - k(n_1 + n_2)] = [F(n_1, n_2)];$$

(E4) $$p^{n_1}\gamma p^{n_2} - p^{n_1\gamma n_2} = [k(n_1)]\gamma[k(n_2)] - [k(n_1\gamma n_2)]$$
$$= [k(n_1)\gamma k(n_2) - k(n_1\gamma n_2)] = [G(n_1, \gamma, n_2)];$$

(E5) $$F(n_1, n_2) = k(n_1) + k(n_2) - k(n_1 + n_2)$$
$$= k(n_2) + k(n_1) - k(n_2 + n_1) = F(n_2, n_1);$$

(E6)
$$F(n_1, n_2) + F(n_1 + n_2, n_3)$$
$$= k(n_1) + k(n_2) - k(n_1 + n_2) + k(n_1 + n_2) + k(n_3) - k((n_1 + n_2) + n_3$$
$$= k(n_1) + k(n_2 + n_3) - k(n_1 + (n_2 + n_3)) + k(n_2) + k(n_3) - k(n_2 + n_2)$$
$$= F(n_1, n_1 + n_3) + F(n_2, n_3);$$

(E7)
$$G(n_1\gamma n_2, \mu, n_3) - G(n_1, \gamma, n_2\mu n_3$$
$$= k(n_1\gamma n_2)\mu k(n_3) - k(n_1)\gamma(k(n_2\gamma n_3) + k(n_1)\gamma(k(n_2)\mu k(n_3))$$
$$- (k(n_1)\gamma k(n_2))\mu k(n_3)$$
$$= k(n_1)\gamma(k(n_2)\mu k(n_3) - k(n_2\mu n_3)) - (k(n_1)\gamma k(n_2) - k(n_1\gamma n_2))\mu k(n_3)$$
$$= [k(n_1)]_\gamma(G(n_2, \mu, n_3)) - (G(n_1, \gamma, n_2))_\mu[k(n_3)]$$
$$= p_\gamma^{n_1}(G(n_2, \mu, n_3)) - (G(n_1, \gamma, n_2))_\mu p^{n_3};$$

(E8) $$G(n_1, \gamma, n_3) + G(n_2, \gamma, n_3) - G(n_1 + n_2, \gamma, n_3)$$
$$= (k(n_1) + k(n_2) - k(n_1 + n_2))\gamma k(n_3) - (k(n_1\gamma n_3) + k(n_2\gamma n_3)$$
$$- k(n_1\gamma n_3 + n_2\gamma n_3))$$
$$= (F(n_1, n_2))_\gamma[k(n_3)] - F(n_1\gamma n_3, n_2\gamma n_3)$$
$$= (F(n_1, n_2))_\gamma p^{n_3} - F(n_1\gamma n_3, n_2\gamma n_3);$$

(E9) $$G(n_1, \gamma, n_2) + G(n_1, \gamma, n_3) - G(n_1, \gamma, n_2 + n_3)$$
$$= p_\gamma^{n_1}(F(n_2, n_3)) - F(n_1\gamma n_2, n_1\gamma n_3).$$

As in (E8),

(E10) $$(G(n_1, \gamma + \mu, n_2) - G(n_1, \gamma, n_2) - G(n_1, \mu, n_2)$$
$$= k(n_1)\gamma k(n_2) + k(n_1)\mu k(n_2) - k(n_1\gamma n_2 + n_1\mu n_2) - k(n_1)\gamma k(n_2)$$
$$+ k(n_1\gamma n_2) - k(n_1)\mu k(n_2) + k(n_1\mu n_2)$$
$$= k(n_1\gamma n_2) + k(n_1\mu n_2) - k(n_1\gamma n_2 + n_1\mu n_2) = F(n_1\gamma n_2, n_1\mu n_2).$$

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Thus $E = E(p, F, G)$ is an $E$-sum of $N$ and $M$ with associated functions $f' : M \to N \times M$ and $g' : N \times M \to N$ with $f'(m) = (0, m)$ for all $m \in M$ and $g'(n, m) = n$ for all $(n, m) \in N \times M$. From Theorem 3.3, $E(p, F, G)$ is an extension of $M$ by $N$. The mapping $l$ is a $\Gamma$-ring isomorphism: $l$ is well defined since $a \in A \Rightarrow g(a) \in N \Rightarrow k(g(a)) \in A$ and $g(a-k(g(a)) = g(a)-g(k(g(a))) = g(a)-g(a) = 0$. Thus $a-k(g(a+M)) \in \ker(g) = M$ for any $a \in A$. Let $a_1, a_2 \in E$, $\gamma \in \Gamma$, then

$$\begin{aligned}
&l(a_1) + l(a_2) \\
&\quad = (g(a_1) + g(a_2), F(g(a_1), g(a_2)) + a_1 - k(g(a_1)) + a_2 - k(g(a_2))) \\
&\quad = (g(a_1 + a_2), k(g(a_1)) + k(g(a_2)) - k(g(a_1) + g(a_2)) \\
&\qquad + a_1 - k(g(a_1)) + a_2 - k(g(a_2)) \\
&\quad = (g(a_1 + a_2)), a_1 + a_2 - k(g(a_1 + a_2)) = l(a + a_2) \quad \text{and} \\
&l(a_1)\gamma l(a_2) = (g(a_1)\gamma g(a_2), G(g(a_1), \gamma, g(a_2)) \\
&\qquad\qquad + p_\gamma^{g(a_1)}(a_2 - k(g(a_2))) + (a_1 - k(g(a_1)))_\gamma p^{g(p_2)} \\
&\qquad\qquad + (a_1 - kg(a_1))\gamma(a_2 - k(g(a_2))) \\
&\quad = (g(a_1\gamma a_2), k(g(a_1))\gamma k(g(a_2)) - k(g(a_1)\gamma g(a_2) + k(g(a_1))\gamma a_2 \\
&\qquad - k(g(a_1))\gamma k(g(a_2)) + a_1\gamma k(g(a_2)) - k(g(a_1))\gamma k(g(a_2)) \\
&\qquad + a_1\gamma a_2 - a_1\gamma k(g(a_2)) + k(g(a_1))\gamma k(g(a_2)) \\
&\quad = (g(a_1\gamma a_2), a_1\gamma a_2 - k(g(a_1\gamma a_2))) \\
&\quad = l(a_1\gamma a_2).
\end{aligned}$$

It is straightforward to verify that $l$ is a bijection. Lastly, consider the diagram

$$\begin{array}{ccc}
M & \xrightarrow{i} & A \\
f' \downarrow & \swarrow_{l} & \downarrow g \\
E & \xrightarrow[g']{} & N,
\end{array}$$

where $i : M \to A$ is the inclusion. If $m \in M$, then $(l \circ i)(m) = l(m) = (g(m + M), m - k(g(m + M))) = (g(M), m - k(g(M))) = (0, m - k(0)) = (0, m) = f'(m)$. Thus $l \circ i = f'$. If $a \in A$, then $(g' \circ l)(a) = g'(l(a)) = g'(g(a+M), a-k(g(a+M))) = g(a+M) = g(a)$. Thus $g' \circ l = g$. Hence the diagram commutes, which shows that the extensions $A$ and $E = E(p, F, G)$ are equivalent.

To conclude this section, we give necessary and sufficient conditions for the equivalence of any two $E$-sums, thus also for any two extensions of a

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Γ-ring $M$ by a Γ-ring $N$. In fact, we determine all equivalences between two extensions of $M$ by $N$.

**Theorem 3.7.** *Let $E(p, F, G)$ and $E'(p', F', G')$ be any two $E$-sums of the $\Gamma$-rings $N$ and $M$. Let $k\colon N \to M$ be any function with $k(0) = 0$ satisfying the following conditions for any $n, n_1, n_2 \in N$, $\gamma \in \Gamma$:*

(I1) $F'(n_1, n_2) - F(n_1, n_2) = k(n_1) + k(n_2) - k(n_1 + n_2)$;

(I2) $G'(n_1, \gamma, n_2) - G(n_1, \gamma, n_2) = k(n_1)\gamma k(n_2) - k(n_1\gamma n_2) + p_\gamma^{n_1}(k(n_2)) + (k(n_1))_\gamma p^{n_2}$;

(I3) $(p')^n - p^n = [k(n)]$.

*Then the function $l\colon E(p, F, G) \to E'(p', F', G')$ defined by $l(n, m) = (n, m - k(n))$ is an equivalence isomorphism. Conversely, every equivalence isomorphism between two extensions of $M$ by $N$ is of this form for some function $k$ satisfying the conditions* (I1) *to* (I3) *above.*

**Proof.** It is clear that $l$ is a bijection. We show $l$ is a homomorphism: If $(n_1, m_1), (n_2, m_2) \in E$, $\gamma \in \Gamma$, then

$$\begin{aligned} l(n_1, m_1) + l(n_2, m_2) &= (n_1 + n_2, F'(n_1, n_2) + m_1 - k(n_1) + m_2 - k(n_2)) \\ &= (n_1 + n_2, F(n_1, n_2) - k(n_1 + n_2) + m_1 + m_2) \\ &\qquad \text{(Condition (I1))} \\ &= l((n_1, m_1) + (n_2, m_2)) \end{aligned}$$

and

$$\begin{aligned} l(n_1, m_1)\gamma l(n_2, m_2) &= (n_1\gamma n_2, G'(n_1, \gamma, n_2) + (p')_\gamma^{n_1}(m_2 - k(n_2)) \\ &\quad + (m_1 - k(n_1))_\gamma (p')^{n_2} \\ &\quad + (m_1 - k(n_1))\gamma(m_2 - k(n_2))) \\ &= (n_1\gamma n_2, G'(n_1, \gamma, n_2) + (p')_\gamma^{n}(m_2) - (p')_\gamma^{n_1}(k(n_2)) \\ &\quad + (m_1)_\gamma (p')^{n_2} - (k(n_1)_\gamma (p')^{n_2} \\ &\quad + m_1\gamma m_2 - m_1\gamma k(n_2) - k(n_1)\gamma m_2 + k(n_1)\gamma k(n_2)) \\ &= (n_1\gamma n_2, G(n_1, \gamma, n_2) + k(n_1)\gamma k(n_2) - k(n_1\gamma n_2) \\ &\quad + p_\gamma^{n_1}(k(n_2)) + (k(n_1))_\gamma p^{n_2} + p_\gamma^{n_1}(m_2) \\ &\quad + [k(n_1)]_\gamma(m_2) - p_\gamma^{n_1}(k(n_2)) - [k(n_1)]_\gamma(k(n_2)) + (m_1)_\gamma p^{n_2} \\ &\quad + (m_1)_\gamma[k(n_2)] - (k(n_1))_\gamma p^{n_2} \\ &\quad - (k(n_1))_\gamma[k(n_2)] + m_1\gamma m_2 - m_1\gamma k(n_2) - k(n_1)\gamma m_2 \\ &\quad + f(n_1)\gamma f(n_2) \quad \text{(Conditions (I2) and (I3))} \end{aligned}$$

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$$\begin{aligned}&= (n_1\gamma n_2\,,\, G(n_1\,,\,\gamma\,,\,n_2) - k(n_1\gamma n_2) + p_\gamma^{n_1}(m_2)\\ &\quad + (m_1)_\gamma p^{n_2} + m_1\gamma m_2)\\ &= l(n_1\gamma n_2\,,\, G(n_1\,,\,\gamma\,,\,n_2) + p_\gamma^{n_1}(m_2) + (m_1)_\gamma p^{n_2} + m_1\gamma m_2\\ &= l(n_1\,,\,m_1)\gamma(n_2\,,\,m_2)).\end{aligned}$$

Consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & E \\ f'\downarrow & \overset{l}{\swarrow} & \downarrow g \\ E' & \xrightarrow[g']{} & N \end{array}$$

where $f\,,\,g\,,\,f'$ and $g'$ are the functions associated with the extensions $E$ and $E'$ respectively. If $m \in M$, then $(l \circ f)(m) = l(f(m)) = l(0\,,\,m) = (0\,,\,m - k(0)) = (0\,,\,m) = f'(m)$ and, if $(n\,,\,m) \in E$, then $(g' \circ l)(n\,,\,m) = g'(l(n\,,\,m)) = g'(n\,,\,m - k(n)) = n = g(n\,,\,m)$. Thus the diagram commutes and $l$ is an equivalence isomorphism. Conversely, let $l\colon E \to E'$ be any equivalence isomorphism between two extensions $E$ and $E'$ of $M$ and $N$. If $f\,,\,f'\,,\,g$ and $g'$ are as before, the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & E \\ f'\downarrow & \overset{l}{\swarrow} & \downarrow g \\ E' & \xrightarrow[g']{} & N \end{array}$$

will commute. Thus $g'(l(n\,,\,m)) = g(n\,,\,m) = n$ for any $(n\,,\,m) \in E$. To satisfy this, $l(n\,,\,m)$ must be of the form $(n\,,\,h(n\,,\,m))$ where $h$ is some function from $N \times M$ into $M$. But $l(f(m)) = f'(m)$ for any $m \in M$, that is, $l(0\,,\,m) = (0\,,\,m)$. Thus $l(0\,,\,m) = (0\,,\,h(0\,,\,m))$ so that $h(0\,,\,m) = m$. Consequently, we have the existence of a function $h\colon N \times M \to M$ satisfying $h(0\,,\,m) = m$ for all $m \in M$. Define $k\colon N \to M$ by $k(n) = -h(n\,,\,0)$ for all $n \in N$. Then $l(n\,,\,m) = l(0+n\,,\,F(0\,,\,n)+m+0) = l((0\,,\,m)+(n\,,\,0)) = l(0\,,\,m) + l(n\,,\,0) = (0\,,\,h(0\,,\,m)) + (n\,,\,h(n\,,\,0)) = (0\,,\,m) + (n\,,\,-k(n)) = (n\,,\,m - k(n))$. Hence $l$ is of the required form and it remains to be shown that $k$ satisfies conditions (I1) to (I3).

(I1) $$\begin{aligned}&l(n_1\,,\,m_1) + l(n_2\,,\,m_2) = l((n_1\,,\,m_1) + (n_2\,,\,m_2))\\ &\Rightarrow (n_1 + n_2\,,\,F'(n_1\,,\,n_2) + m_1 - k(n_1) + m_2 - k(n_2))\\ &\quad = (n_1 + n_2\,,\,F(n_1\,,\,n_2) + m_1 + m_2 - k(n_1 + n_2))\\ &\Rightarrow F'(n_1\,,\,n_2) - k(n_1) - k(n_2) = F(n_1\,,\,n_2) - k(n_1 + n_2)\\ &\Rightarrow F'(n_1\,,\,n_2) - F(n_1\,,\,n_2) = k(n_1) + k(n_2) - k(n_1 + n_2)\end{aligned}$$

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(I3)

$$\begin{aligned}(0, p_\gamma^n(m)) &= (0, p_\gamma^n(m) - k(0)) = l(0, p_\gamma^n(m))\\ &= l(n\gamma 0, G(n, \gamma, 0) + p_\gamma^n(m) + (0)_\gamma p^0 + 0\gamma m) = l((n, 0)\gamma(0, m))\\ &= l(n, 0)\gamma l(0, m) = (n, 0 - k(n))\gamma(0, m - k(0)) = (n, -k(n))\gamma(0, m)\\ &= (n\gamma 0, G'(n, \gamma, 0) + p_\gamma'^n(m) + (-k(n))_\gamma p'^0 - k(n)\gamma m)\\ &= (0, p_\gamma'^n(m) - [k(n)]_\gamma(m)).\end{aligned}$$

Thus $p_\gamma^n(m) = p_\gamma'^n(m) - [k(n)]_\gamma(m)$, that is, $(p'^n - p^n)_\gamma = [k(n)]_\gamma$. Similarly, ${}_\gamma(p'^n - p^n) = {}_\gamma[k(n)]$. Hence $p'^n - p^n = [k(n)]$.

(I2) $l(n_1, m_1)\gamma l(n_2, m_2) = l((n_1, m_1)\gamma(n_2, m_2))$

$$\begin{aligned}&\Rightarrow (n_1\gamma n_2, G(n_1, \gamma, n_2) + p_\gamma'^{n_1}(m_2 - k(n_2)) + (m_1 - k(n_1))_\gamma p'^{n_2}\\ &\quad + (m_1 - k(n_1))\gamma(m_2 - k(n_2))\\ &\qquad = (n_1\gamma n_2, G(n_1, \gamma, n_2) + p_\gamma^{n_1}(m_2) + (m_1)_\gamma p^{n_1}\\ &\qquad\quad + m_1\gamma m_2 - k(n_1\gamma n_2))\\ &\Rightarrow (n_1\gamma n_2, G'(n_1, \gamma, n_2) + p_\gamma^{n_1}(m_2) + k(n_1)\gamma m_2 - p_\gamma^{n_1}(k(n_2))\\ &\quad - k(n_1)\gamma k(n_2) + (m_1)_\gamma p^{n_2}\\ &\quad + m_1\gamma k(n_2) - (k(n_1))_\gamma p^{n_2} - k(n_1)\gamma k(n_2) + m_1\gamma m_2 - m_1\gamma k(n_2)\\ &\quad - k(n_1)\gamma m_2 + k(n_1)\gamma k(n_2))\\ &\qquad = (n_1\gamma n_2, G(n_1, \gamma, n_2) + p_\gamma^{n_1}(m_2) + (m_1)_\gamma p^{n_2}\\ &\qquad\quad + m_1\gamma m_2 - k(n_1\gamma n_2)) \quad \text{(using Condition (I3))}\\ &\Rightarrow G'(n_1, \gamma, n_2) - p_\gamma^{n_1}(k(n_2)) - (k(n_1))_\gamma p^{n_2} - k(n_1)\gamma k(n_2)\\ &\qquad = G(n_1, \gamma, n_2) - k(n_1\gamma n_2)\\ &\Rightarrow G'(n_1, \gamma, n_2) - G(n_1, \gamma, n_2) = k(n_1)\gamma k(n_2) - k(n_1\gamma n_2)\\ &\quad + p_\gamma^{n_1}(k(n_2)) + (k(n_1))_\gamma p^{n_2}.\end{aligned}$$

EXAMPLE 3.8. Let $M$ and $N$ be any $\Gamma$-rings. Define the functions $F$, $G$ and $p$ for an extension of $M$ by $N$ by $F(n_1, n_2) = G(n_1, \gamma, n_2) = 0$ and $p^n = [0]$ for any $n, n_1, n_2 \in N$, $\gamma \in \Gamma$. Then the $E$-sum of $N$ and $M$ defined by these functions is the same as the direct sum $N \oplus M$ of $N$ and $M$. Therefore, there always exists at least one extension of $M$ by $N$. From Theorem 3.7 it follows that any extension $E$ of $M$ by $N$ will be equivalent to $N \oplus M$ iff there exists a function $k: N \to M$ with $k(0) = 0$ satisfying for all $n, n_1, n_2 \in N$, $\gamma \in \Gamma$:

(i) $F'(n_1, n_2) = k(n_1) + k(n_2) - k(n_1 + n_2)$;

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(ii) $G'(n_1, \gamma, n_2) = k(n_1)\gamma k(n_2) - k(n_1\gamma n_2)$;
(iii) $p'^n = [k(n)]$.

DEFINITION 3.9. Let $M$ and $N$ be any Γ-rings. An extension of $M$ by $N$ for which $F(n_1, n_2) = G(n_1, \gamma, n_2) = 0$ for all $n_1, n_2 \in N$, $\gamma \in \Gamma$, is called a *factor free extension* of $M$ by $N$ and will be denoted by $N\#M$.

## 4. Double homothetisms of a Γ-ring

The double homothetism of a ring (cf. Rédei [7]) is an important tool in the study of rings with identity. It is also of much use in the Γ-ring case.

DEFINITION 4.1. A *double homothetism* $\rho$ of a Γ-ring $M$ is a bitranslation of $M$ that is amicable with itself.

It is hard not to find double homothetisms of a Γ-ring $M$: from Theorem 2.8 we have that the set $\mathcal{J}(M)$ of all inner bitranslations of a Γ-ring $M$ is a Γ-ring of amicable double homothetisms of $M$. For this reason the elements of $\mathcal{J}(M)$ will be called the *inner double homothetisms* of $M$.

The proof of the next result is a straightforward application of Zorn's Lemma:

THEOREM 4.2. *Every set of amicable double homothetisms of a* Γ*-ring* $M$ *is contained in a maximal set of amicable double homothetisms of a* $P$*-ring* $M$.

THEOREM 4.3. *The sub* Γ*-ring of* $\mathscr{E}_2(M)$ *generated by any set of amicable double homothetisms of* $M$ *is always a* Γ*-ring of amicable double homothetisms of* $M$.

PROOF. Let $A$ be a set of pair wise amicable bitranslations of $M$, and let $B$ be the sub Γ-ring of $\mathscr{E}_2(M)$ generated by $A$. Then $B$ consists of all sums of the form $\sum a_i - \sum b_i + \sum c_i\gamma_i d_i$, $a_i, b_i, c_i, d_i \in A$. It can be proved by straightforward calculation that the elements of $B$ are pairwise amicable, and hence are in particular double homothetism.

In view of the above result, any maximal set of amicable double homothetisms of a Γ-ring $M$ must be a Γ-ring and it will be called a *maximal* Γ*-ring of amicable double homothetisms of* $M$.

From Theorem 4.2 it follows that any set of amicable double homothetisms of a $P$-ring $M$ is contained in at least one maximal Γ-ring of amicable double homothetisms of $M$.

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THEOREM 4.4. *For any $\Gamma$-ring $M$, $\mathcal{I}(M)$ is contained in every maximal $\Gamma$-ring of amicable double homothetisms of $M$.*

PROOF. Let $p$ be any double homothetism of $M$, $[n] \in \mathcal{I}(M)$, $m \in M$ and $\gamma \in \Gamma$. Then

$$p_\gamma((m)_\mu[n]) = p_\gamma(m\mu n) = (p_\gamma(m))\mu n = (p_\gamma(m))_\mu[n] \quad \text{and}$$
$$[n]_\gamma((m)_\mu p) = n\gamma((m)_\mu p) = (n\gamma m)_\mu p = ([n]_\gamma(m))_\mu p.$$

Thus $[n]$ and $p$ are amicable. Hence if $A$ is a maximal $\Gamma$-ring of amicable double homothetism of $M$, then $A \cup \mathcal{I}(M)$ is a set of amicable double homothetisms of $M$. Let $B$ be the sub $\Gamma$-ring of $\mathcal{E}_2(M)$ generated by $A \cup \mathcal{I}(M)$. Then $B$ is a $\Gamma$-ring of amicable double homothetisms of $M$ by Theorem 4.3 and $A \subseteq B$. Hence $A = B$, by the maximality of $A$, whence $\mathcal{I}(M) \subseteq A$, as required.

THEOREM 4.5. *If $\mathcal{P}$ is any $\Gamma$-ring of amicable double homothetisms of $M$, define the functions:*

(i) $F: \mathcal{P} \times \mathcal{P} \to M$ by $F(p_1, p_2) = 0$ for all $p_1, p_2 \in \mathcal{P}$,
(ii) $G: \mathcal{P} \times \Gamma \times \mathcal{P} \to M$ *by* $G(p_1, \gamma, p_2) = 0$ for all $p_1, p_2 \in \mathcal{P}$, $\gamma \in \Gamma$ and
(iii) $p: \mathcal{P} \to \mathcal{E}_2(M)$ by $p(p_1) = p^{p_1} = p_1$ for all $p_1 \in \mathcal{P}$.

*Then the triple $(p, F, G)$ defines a factor free extension $\mathcal{P}\#M$ of $M$ by $\mathcal{P}$.*

PROOF. Because every amicable double homothetism of $M$ is a bitranslation of $M$, $p$ is well defined. The conditions of Construction 3.1 are clearly satisfied.

The operations in the $\Gamma$-ring $\mathcal{P}\#M$ of Theorem 4.5 are given, for all $(p_1, m_1), (p_2, m_2) \in \mathcal{P} \times M$ and $\gamma \in \Gamma$ by:

$$(p_1, m_1) + (p_2, m_2) = (p_1 + p_2, m_1 + m_2) \quad \text{and}$$
$$(p_1, m_1)\gamma(p_2, m_2) = (p_1\gamma p_2, p_{1\gamma}(m_2) + (m_1)_\gamma p_2 + m_1\gamma m_2).$$

THEOREM 4.6. *The inner double homothetisms of all the extensions of a $\Gamma$-ring $M$ induce all the double homothetisms of $M$.*

PROOF. Let $M$ be a $\Gamma$-ring and let $E$ be any extension of $M$. Let $[a] \in \mathcal{I}(E)$, then $[a] = (p^a, q^a)$ where $p^a$ and $q^a$ is a left and a right translation of $E$ respectively, with $p^a_\gamma(b) = a\gamma b$ and $(b)_\gamma p^a = b\gamma a$ for all $b \in E$. Let $m \in M$. Then $p^a_\gamma(m) = a\gamma m \in M$ and $(m)_\gamma q^a = m\gamma a \in M$ because $M \triangleleft E$. Hence the restrictions of both $p^a$ and $q^a$ to $M$, say $p^a|M$ and $q^a|M$, are

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left and right translations respectively of $M$ and $p = (p^a|M, q^a|M)$ is a double homothetism of $M$.

Conversely, let $p$ be any double homothetism of $M$. We show that $p$ is induced by an inner double homothetism of some extension of $M$. Because any double homothetism is amicable with itself, $\{p\}$ is a set of amicable double homothetisms. From Theorem 4.2, $p$ is an element of some maximal Γ-ring $\mathscr{P}$ of amicable double homothetisms. Form the factor free extension Γ-ring $\mathscr{P}\#M$. Then $(p, 0) \in \mathscr{P}\#M$; thus $[(p, 0)] \in \mathscr{I}(\mathscr{P}\#M)$ where $[(p, 0)]_\gamma(q, m) = (p, 0)\gamma(q, m)$ and $(q, m)_\gamma[(p, 0)] = (q, m)\gamma(p, 0)$. Consider $[(p, 0)]_\gamma|M$ and ${}_\gamma[(p, 0)]|M$. We identify $M$ with the subset $\{(0, m)|m \in M\}$ of $\mathscr{P}\#M$. Then $[(p, 0)]_\gamma(0, m) = (p, 0)\gamma(0, m) = (0, p_\gamma(m))$ and $(0, m)_\gamma[(p, 0)] = (0, m)\gamma(p, 0) = (0, (m)_\gamma p)$. Hence $[(p, 0)]_\gamma|M = p_\gamma$ and ${}_\gamma[(p, 0)]|M = {}_\gamma p$. Thus $p$ is induced by the inner double homothetism $[(p, 0)]$ of $\mathscr{P}\#M$.

## 5. The holomorph of a Γ-ring

Definition 5.1. A sub Γ-ring $I$ of a Γ-ring $M$ is called *characteristic* if it is invariant under any double homothetism of $M$, that is, if $p$ is any double homothetism of $M$, then $p_\gamma(I) \subseteq I$ and $(I)_\gamma p \subseteq I$ for all $\gamma \in \Gamma$.

Theorem 5.2. *A sub Γ-ring $I$ of a Γ-ring $M$ is characteristic iff it is an ideal in every extension of $M$.*

Proof. Use definition 5.1 and Theorem 4.6.

If $I$ is a characteristic sub Γ-ring of $M$, $i \in I$, $m \in M$, $\gamma \in \Gamma$, then $[m] \in \mathscr{I}(M)$, that is, $i\gamma m = (i)_\gamma[m] \in I$ and $m\gamma i = [m]_\gamma(i) \in I$. Thus $I \lhd M$. Also, if $I$ is a characteristic sub Γ-ring of $M$ and $p$ is any double homothetism of $M$, $p$ will induce a double homothetism of $I$, since $p_\gamma(I) \subseteq I$ and $(I)_\gamma p \subseteq I$.

Definition 5.3. A *holomorph* of a Γ-ring $M$ is a factor free extension $\mathscr{P}\#M$ of $M$ by any maximal Γ-ring $\mathscr{P}$ of amicable double homothetisms of $M$.

Theorem 5.4. *The inner double homothetisms of all the holomorphs of a Γ-ring $M$ induce all the double homothetisms of $M$.*

Proof. As in the proof of Theorem 4.6.

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**Theorem 5.5.** *A sub $\Gamma$-ring of a $\Gamma$-ring $M$ is characteristic iff it is an ideal in all the homomorphs of $M$.*

**Proof.** Follows directly from Definition 5.1 and Theorem 5.4.

## 6. Unities of Γ-rings

Unities in $\Gamma$-rings differ from unities in rings in the very important way that they are not necessarily unique. Contrary to the ring case, we will show that not every $\Gamma$-ring can be embedded as an ideal in a $\Gamma$-ring with unity. A $\Gamma$-ring $M$ has a *left (right) unity* if there exists elements $e_1, e_2, \dots, e_s \in M$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ such that $\sum_{i=1}^{s} e_i\gamma_i m = m (\sum_{i=1}^{s} m\gamma_i e_i = m)$ for any $m \in M$. For examples see Kyuno [4]. It is possible for a $\Gamma$-ring to have more than one unity.

**Definition 6.1.** A $\Gamma$-ring $M$ has a *left (right) double homothetism unity* if there exist double homothetisms $p^1, p^2, \dots, p^s$ of $M$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ such that

$$\sum_{i=1}^{s} p^i_{\gamma_i}(m) = m \qquad \left(\sum_{i=1}^{s} (m)_{\gamma_i} p^i = m\right) \quad \text{for all } m \in M.$$

A ring has a unity iff it has only inner double homothetisms (cf. Réidei [7, p. 197]). The next theorem and the following examples show that this is not the case for $\Gamma$-rings.

**Theorem 6.2.** *A $\Gamma$-ring has a left and a right unity iff it has a left and right double homothetism unity and only inner double homothetisms.*

**Proof.** Assume $M$ has a left and a right unity, that is, there exist $e_1, e_2, \dots, e_s \in M$, $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ and $a_1, a_2, \dots, a_t \in M$, $\lambda_1, \lambda_2, \dots, \lambda_t \in \Gamma$ such that

$$\sum_{i=1}^{s} e_i\gamma_i m = m \quad \text{and} \quad \sum_{j=1}^{t} m\lambda_j a_j = m \quad \text{for all } m \in M.$$

For each $i = 1, 2, \dots, s$, $[e_i] \in \mathscr{I}(M)$ and $\sum_{i=1}^{s} [e_i]_{\gamma_i}(m) = \sum_{i=1}^{s} e_i\gamma_i m = m$ for all $m \in M$. Hence the double homothetisms $[e_1], [e_2], \dots, [e_s]$ of $M$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ form a left double homothetism unity for $M$. Likewise the double homothetism $[a_1], [a_2], \dots, [a_t]$ of $M$ and $\lambda_1, \lambda_2, \dots, \lambda_t \in \Gamma$ form a right double homothetism unity for $M$. Let $p$ be any double

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homothetism of $M$, $m \in M$, $\mu \in \Gamma$. Then

$$p_\mu(m) = \sum_{i=1}^{s} e_i\gamma_i(p_\mu(m)) = \sum_{i=1}^{s}((e_i)_{\gamma_i}p)\mu m = \left(\sum_{i=1}^{s}(e_i)_{\gamma_i}p\right)\mu m$$
$$= \left[\sum_{i=1}^{s}(e_i)_{\gamma_i}p\right]_\mu (m).$$

Also,

$$(m)_\mu p = \sum_{j=1}^{t}((m)_\mu p)\lambda_j a_j = \sum_{j=1}^{t} m\mu(p_{\lambda_j}(a_j)) = m\mu\left(\sum_{j=1}^{t} p_{\lambda_j}(a_j)\right)$$
$$= (m)_\mu\left[\sum_{j=1}^{t} p_{\lambda_j}(a_j)\right].$$

We now show that $\sum_{i=1}^{s}(e_i)_{\lambda_i}p = \sum_{j=1}^{t} p_{\lambda_j}(a_i)$:

$$\sum_{i=1}^{s}(e_i)_{\gamma_i}p = \sum_{j=1}^{t}\left(\sum_{i=1}^{s}(e_i)_{\gamma_i}p\right)\lambda_j a_j = \sum_{i=1}^{s}\sum_{j=1}^{t} e_i\gamma_i(p_{\gamma_j}(a_j))$$
$$= \sum_{i=1}^{s} e_i\gamma i\left(\sum_{j=1}^{t} p_{\lambda_j}(a_j)\right) = \sum_{j=1}^{t} p_{\lambda_j}(a_j).$$

Thus

$$p_\mu = \left[\sum_{i=1}^{s}(e_i)_{\gamma_i}p\right]_\mu = \left[\sum_{j=1}^{t} p_{\lambda_j}(a_j)\right]_\mu \text{ and } {}_\mu p = {}_\mu\left[\sum_{j=1}^{t} p_{\gamma_j}(a_i)\right] = {}_\mu\left[\sum_{i=1}^{s}(e_{i_{\gamma_i}}p\right].$$

Hence $p$ is the inner double homothetism of $M$ induced by $\sum_{i=1}^{s}(e_i)_{\gamma_i}p = \sum_{j=1}^{t} p_{\lambda_j}(a_j)$. Therefore, $M$ has only inner double homothetisms. Conversely, suppose $M$ has only inner double homothetisms and $M$ has a left and a right double homothetism unity. Let $p^1, p^2, \ldots, p^s$ be the double homothetisms and $\gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma$ for which $\sum_{i=1}^{s} p^i_{\gamma_i}(m) = m$. Because $M$ has only inner double homothetisms, there exists $e_1, e_2, \ldots, e_s \in M$ such that $p^i = [e_i]$. Similarly, if $q^1, q^2, \ldots, q^t$ and $\lambda_1, \lambda_2, \ldots, \lambda_t \in \Gamma$ is a right double homothetism unity of $M$ there exists $a_1, a_2, \ldots, a_s \in M$ such that $q^j = [a_j]$. If $m \in M$, then

$$\sum_{i=1}^{s} e_i\gamma_i m = \sum_{i=1}^{s}[e_i]_{\gamma_i}(m) = \sum_{i=1}^{s} p^i_{\gamma_i}(m) = m \quad \text{and}$$
$$\sum_{j=1}^{t} m\lambda_j a_j = \sum_{j=1}^{t}(m)_{\lambda_j}[a_j] = \sum_{j=1}^{t}(m)_{\lambda_j}p^j = m.$$

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Thus $e_1, e_2, \dots, e_s \in M$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ form a left unity of $M$, while $a_1, a_2, \dots, a_t \in M$ and $\lambda_1, \lambda_2, \dots, \lambda_t \in \Gamma$ form a right unity of $M$.

The next two examples will show that the two conditions required in Theorem 6.2 are independent.

EXAMPLE 6.3. Let $M = \mathbb{Z}_2 = \{0, 1\}$ and $\Gamma = \{\gamma\} \cong \mathbb{Z}_1 = \{0\}$. Define the mapping $(-, -, -): M \times \Gamma \times M \to M$ by $m_1 \gamma m_2 = 0$ for all $m_1, m_2 \in M$, $\gamma \in \Gamma$. Then $M$ is a $\Gamma$-ring. $\mathrm{End}(M^+) = \{f_0, f_1\}$, where $f_0(m) = 0$ and $f_1(m) = m$ for all $m \in M$. Then $\mathscr{E}(M) = \{p_0\}$ where $p_0(\gamma) = f_0$ for all $\gamma \in \Gamma$. It follows that $M$ has only one double homothetism $p$ with $p_\gamma(m) = (m)_\gamma p = 0$ for any $m \in M$, $\gamma \in \Gamma$. Moreover, $p$ is the inner double homothetism induced by both 0 and 1 in $M$. Thus $M$ has only inner double homothetisms. Since $p_\gamma(m) = 0$ for any $m \in M$, $\gamma \in \Gamma$, we have $\sum_{i=1}^{s} p^i_{\gamma_i}(1) = 0 \neq 1$ for any $\gamma_i \in \Gamma$ and any double homothetisms $p^i$ of $M$. Hence $M$ does not have a left double homothetism unity. Similarly, it does not have a right double homothetism unity. It is also clear from the definition that $M$ does not have a unity although all its double homothetisms are inner double homothetisms.

The next example shows that the existence of a left and a right double homothetism unity is not sufficient to ensure the existence of a left and a right unity.

EXAMPLE 6.4. Let $M = \mathbb{Z}_2 = \{0, 1\}$ and $\Gamma = \mathbb{Z}_2 = \{0, 1\}$ and define the mapping $(-, -, -): M \times \Gamma \times M \to M$ by $m_1 \gamma m_2 = 0$ for all $m_1, m_2 \in M$, $\gamma \in \Gamma$. Then $M$ is a $\Gamma$-ring. $\mathrm{End}(M^+) = \{f_0, f_1\}$ with $f_0(m) = 0$ and $f_1(m) = m$ for all $m \in M$. Define a double homothetism $p$ as follows:

$$p_0(m) = (m)_0 p = f_0(m) = 0 \quad \text{and}$$
$$p_1(m) = (m)_1 p = f_1(m) = m \quad \text{for all } m \in M.$$

Simple calculations will verify that $p$ is a double homothetism of $M$. Since $p_1(m) = m$ for any $m$ from $M$, the double homothetism $p$ with $1 \in \Gamma$ is a left double homothetism unity of $M$. Similarly, $p$ and $1 \in \Gamma$ is also a right double homothetism unity of $M$. For any $n \in M$, $[n]_1(1) = 0 \neq 1$, while $p_1(1) = 1$, hence $p$ is not an inner double homothetism of $M$. As in Example 6.3, this $\Gamma$-ring does not have a left nor a right unity.

THEOREM 6.5. *If $M$ is a $\Gamma$-ring that has a left or a right unity, then $M$ is isomorphic to the $\Gamma$-ring $\mathscr{I}(M)$ of all inner double homothetisms of $M$.*

PROOF. Define the mapping $f: M \to \mathscr{I}(M)$ by $f(m) = [m]$.

Straightforward calculations will show that $f$ is a surjective $\Gamma$-ring homomorphism. $f$ is injective: let $e_1, e_2, \dots, e_s \in M$, $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$

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be a right unity of $M$. If $m_1, m_2 \in M$ such that $f(m_1) = f(m_2)$, then $[m_1] = [m_2]$, that is, $[m_1]_{\gamma_i}(e_i) = [m_2]_{\gamma_i}(e_i)$ for $i = 1, 2, \ldots, s$. Thus

$$\begin{aligned} &\sum_{i=1}^{s}[m_1]_{\gamma_i}(e_i) = \sum_{i=1}^{s}[m_2]_{\gamma_i}(e_i) \\ &\Rightarrow \sum_{i=1}^{s} m_1\gamma_i(e_i) = \sum_{i=1}^{s} m_2\gamma_i(e_i) \\ &\Rightarrow m_1 = m_2. \end{aligned}$$

Hence $f$ is an isomorphism from $M$ onto $\mathscr{I}(M)$. The result follows similarly if $M$ has a left unity.

THEOREM 6.6. *A $\Gamma$-ring has a left and right unity iff it has a left and a right double homothetism unity and it is a direct summand in all its extensions.*

PROOF. If $M$ has a left and right unity, then $M$ has a left and a right double homothetism unity (Theorem 6.2). Let $E$ be any extension of $M$ and let $a \in E$. Then $[a] \in \mathscr{I}(E)$. In view of Theorem 4.6 there exists a double homothetism $p$ of $M$ with $p_\gamma(m) = [a]_\gamma(m)$ and $(m)_\gamma p = (m)_\gamma[a]$ for all $m \in M$ and $\gamma \in \Gamma$. Since $M$ has a left and a right unity, it has only inner double homothetisms (Theorem 6.2). Hence there exists an $m \in M$ such that

$$[a]_\lambda(n) = p_\lambda(n) = [m]_\lambda(n) \quad \text{and} \quad (n)_\lambda[a] = (n)_\gamma p = (n)_\lambda[m] \quad \text{for all } n \in M,\ \lambda \in \Gamma.$$

This $m$ is uniquely determined by $a$. Indeed, if $m' \in M$ with $[a]_\lambda(n) = [m']_\lambda(n)$ and $(n)_\lambda[a] = (n)_\lambda[m']$ for all $n \in M$ and $\lambda \in \Gamma$, then $[m]_\gamma(n) = [m']_\gamma(n)$ for all $n \in M$, $\lambda \in \Gamma$. Thus, if $e_1, e_2, \ldots, e_s \in M$ and $\gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma$ is a right unity of $M$, then

$$m = \sum_{i=1}^{s} m\gamma_i e_i = \sum_{i=1}^{s}[m]_{\gamma_i}(e_i) = \sum_{i=1}^{s}[m']_{\gamma_i}(e_i) = \sum_{i=1}^{s} m'\gamma_i e_i = m'.$$

Since $[a]_\lambda(n) = [m]_\lambda(n)$ and $(n)_\lambda[a] = (n)_\lambda[m]$, $a\lambda n = m\lambda n$ and $n\lambda a = n\lambda m$ or $(a-m)\lambda n = n\lambda(a-m) = 0$ for all $\lambda \in \Gamma$, $n \in M$. Because $m$ is uniquely determined by $a$, the element $b = a - m$ of $E$ is uniquely determined by $a$. Thus $b$ is an element of $M$ for which $b\lambda n = n\lambda b = 0$ for all $n \in M$ and $\lambda \in \Gamma$, that is, $b\lambda M = M\lambda b = 0$ for all $\lambda \in \Gamma$. Hence $b \in B = \{c \in E \mid c\lambda M = M\lambda c = 0 \text{ for all } \lambda \in \Gamma\}$. From the definition of $b$, we have that $a = b + m$ where both $b \in B$ and $m \in M$ are uniquely determined by $a$. Since $a$ was an arbitrary element of $E$, it follows that

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every element of $E$ can be written as a unique expression as a sum of an element of $B$ and an element of $M$. To complete the proof, we show that $B$ is an ideal of $E$: Let $b, b_1, b_2 \in B$, $a \in E$, $\mu \in \Gamma$, then

$$(b_1 - b_2)\lambda M = b_1\lambda M - b_2\lambda M = 0 - 0 = 0 \quad \text{and}$$
$$M\lambda(b_1 - b_2) = M\lambda b_1 - M\lambda b_2 = 0 - 0 = 0 \quad \text{for all } \lambda \in \Gamma.$$

Hence $b_1 - b_2 \in B$. Also, $(b\mu a)\lambda M = b\mu(a\lambda M) \subseteq b\mu M = 0$ and

$$M\lambda(b\mu a) = (M\lambda b)\mu a = 0\mu a = 0 \quad \text{for all } \lambda \in \Gamma.$$

Hence $b\mu a \in B$. Likewise $a\mu b \in B$ and hence $E = B \oplus M$. Conversely, suppose that $M$ is a direct summand in all its extensions. In particular, it is also a direct summand in any holomorph $\mathscr{P}\#M$ of $M$. Thus $\mathscr{P}\#M$ is a factor-free $E$-sum of $\mathscr{P}$ (any maximal set of amicable double homothetisms of $M$) and $M$. Let the functions of $\mathscr{P}\#M$ be given by

$F': \mathscr{P} \times \mathscr{P} \to M$ by $F'(p_1, p_2) = 0$ for all $p_1, p_2 \in \mathscr{P}$,

$G': \mathscr{P} \times \Gamma \times \mathscr{P} \to M$ by $G'(p_1, \gamma, p_2) = 0$ for all $p_1, p_2 \in \mathscr{P}$, $\gamma \in \Gamma$ and

$p': \mathscr{P} \to \mathscr{E}_2(M)$ by $p'(p_1) = p^{p_1} = p_1$ for all $p_1 \in \mathscr{P}$.

Because $M$ is a direct summand of $\mathscr{P}\#M$, $\mathscr{P}\#M$ is equivalent to the direct sum $\mathscr{P} \oplus M$ of $\mathscr{P}$ and $M$. The latter is an extension of $M$ by $\mathscr{P}$ with respect to the functions defined as follows:

$F: \mathscr{P} \times \mathscr{P} \to M$ by $F(p_1, p_2) = 0$ for all $p_1, p_2 \in \mathscr{P}$,

$G: \mathscr{P} \times \Gamma \times \mathscr{P} \to M$ by $G(p_1, \gamma, p_2) = 0$ for all $p_1, p_2 \in \mathscr{P}$, $\gamma \in \Gamma$ and

$p: \mathscr{P} \to \mathscr{E}_2(M)$ by $p(p_1) = p^{p_1} = 0$ for all $p_1 \in \mathscr{P}$.

Thus Theorem 3.7 shows the existence of a function $f: \mathscr{P} \to M$ with $f(0) = 0$, satisfying the following conditions for all $p_1, p_2 \in \mathscr{P}$, $\gamma \in \Gamma$:

(i) $F'(p_1, p_2) - F(p_1, p_2) = f(p_1) + f(p_2) - f(p_1 + p_2)$, that is, $f(p_1) + f(p_2) = f(p_1 + p_2)$;

(ii) $G'(p_1, \gamma, p_2) - G(p_1, \gamma, p_2) = f(p_1)\gamma f(p_2) - f(p_1\gamma p_2) + p_\gamma^{p_1}(f(p_2)) + (f(p_1))_\gamma p^{p_2}$, that is, $f(p_1)\gamma f(p_2) = f(p_1\gamma p_2)$;

(iii) $p'^{p_1} - p^{p_1} = [f(p_1)]$, that is, $p_1 = [f(p_1)]$. From Condition (iii) above it follows that for any $p \in \mathscr{P}$, $\gamma \in \Gamma$ and $m \in M$

$$p_\gamma(m) = [f(p)]_\gamma(m) \quad \text{and} \quad (m)_\gamma p = (m)_\gamma[f(p)].$$

Hence $p$ is the inner double homothetism of $M$ induced by $f(p)$. Therefore any $\mathscr{P}$ consists of only inner double homothetisms of $M$. Let $p$ be any double homothetism of $M$. Then $p$ is amicable with itself. Hence $\{p\}$ is a set of amicable double homothetisms of $M$. Thus it is contained in some maximal set $\mathscr{P}_0$ of amicable double homothetisms of $M$ (Theorem 4.2).

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From the previous part of the proof $p$ must be an inner double homothetism of $M$. Hence $M$ have only inner double homothetisms. Since $M$ has both a left and a right double homothetism unity, it must have both a left and a right unity (Theorem 6.2).

COROLLARY 6.7. *Every* $\Gamma$*-ring that has both a left and a right unity has only one holomorph.*

PROOF. Let $\mathscr{P}_1\#M$ and $\mathscr{P}_2\#M$ be two holomorphs of $M$, where $\mathscr{P}_1$ and $\mathscr{P}_2$ are maximal sets of amicable double homothetisms of $M$. Because $M$ has both a left and a right unity, it has only inner double homothetisms (Theorem 6.2). Hence $\mathscr{P}_1, \mathscr{P}_2 \subseteq \mathscr{I}(M)$. But $\mathscr{I}(M) \subseteq \mathscr{P}_1$ and $\mathscr{I}(M) \subseteq \mathscr{P}_2$ (Theorem 4.4), which yields $\mathscr{P}_1 = \mathscr{I}(M) = \mathscr{P}_2$. Thus $\mathscr{P}_1\#M = \mathscr{P}_2\#M$ so that $M$ has only one holomorph.

COROLLARY 6.8. *In a* $\Gamma$*-ring that has both a left and a right unity, all ideals are characteristic.*

PROOF. Let $M$ be a $\Gamma$-ring that has both a left and a right unity. Let $I \triangleleft M$. Theorem 6.6 yields that $M$ is a direct summand in all its holomorphs $\mathscr{P}\#M$. Thus $\mathscr{P}\#M = \mathscr{P} \oplus M$. Since $I \triangleleft M$, $\{0\} \oplus I \triangleleft \mathscr{P} \oplus M = \mathscr{P}\#M$ with $I \cong \{0\} \oplus I$. Thus $I$ is an ideal in every holomorph of $M$. Hence $I$ is a characteristic sub $\Gamma$-ring of $M$.

Corollaries 6.7 and 6.8 coincide with the corresponding results for rings (cf. Rédei [7]). The next result gives a necessary condition for a $\Gamma$-ring to be embedded in an ideal in a $\Gamma$-ring with a left or a right unity.

THEOREM 6.9. *Let* $M$ *be any* $\Gamma$*-ring. If* $M$ *can be embedded as an ideal in a* $\Gamma$*-ring with left (right) unity, then* $M$ *has a left (right) double homothetism unity.*

PROOF. Suppose $M$ can be embedded as an ideal in a $\Gamma$-ring $E$ that has a left unity. Then there exists $e_1, e_1, \dots, e_s \in E$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ such that $\sum_{i=1}^{s} e_i\gamma_i a = a$ for all $a \in E$. Since $e_i \in E$, each $[e_i] \in \mathscr{I}(E)$. Thus there exists for each $i = 1, 2, \dots, s$ a double homothetism $p^i$ of $M$ such that $p^i_\gamma(m) = [e_i]_\gamma(m)$ for all $m \in M$, $\gamma \in \Gamma$ (Theorem 4.6). Hence for any $m \in M$ we have that

$$\sum_{i=1}^{s} p^i_{\gamma_i}(m) = \sum_{i=1}^{s} [e_i]_{\gamma_i}(m) = \sum_{i=1}^{s} e_i\gamma_i m = m.$$

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This shows that the double homothetisms $p_1, p_2, \dots, p^s$ of $M$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ form a left double homothetism unity of $M$. Similar arguments show that if $E$ has a right unity, then $M$ must have a right double homothetism unity.

For the special case where $M$ is a $\Gamma$-ring with $m\gamma n = 0$ for all $m, n \in M$ and $\gamma \in \Gamma$, we have the converse:

**Theorem** 6.10. *If $M$ is a $\Gamma$-ring that has a left (ring) double homothetism unity and $m\gamma n = 0$ for all $m, n \in M$, $\gamma \in \Gamma$, then $M$ can be embedded as an ideal in a $\Gamma$-ring with a left (right) unity.*

**Proof.** Suppose $M$ is a $\Gamma$-ring that has a left double homothetism unity. Let $E_\ell$ be the direct sum of the groups $\mathscr{E}_\ell(M)$ (all left translations of $M$) and $M$, that is, $E_\ell = \mathscr{E}_\ell(M) \oplus M$. Define the mapping $(-, -, ): E_\ell \times \Gamma \times E_\ell$ by

$$(p^1, m_1)\gamma(p^2, m_2) = (p^1\gamma p^2, p^1_\gamma(m_2)) \text{ for all } (p^1, m_1), (p^2, m_2) \in \mathscr{E}_\ell(M) \times M.$$

$E_\ell$ is a $\Gamma$-ring: we only show one of the requirements, the others being easy to verify. Let $(p^1, m_1), (p^2, m_2), (p^3, m_3) \in E_\ell$ $\gamma, \mu \in \Gamma$. Then

$$\begin{aligned}(p^1, m_1)\gamma[(p^2, m_2)\mu(p^3, m_3)] &= (p^1, m_1)\gamma(p^2\mu p^3, p^2_\mu(m_3)) \\ &= (p^1\gamma(p^2\mu p^3), p^1_\gamma(p^2_\mu(m_3))) \\ &= ((p^1\gamma p^2)\mu p^3, (p^1\gamma p^2)_\mu(m_3)) \\ &= (p^1\gamma p^2, p^1_\gamma(m_2))\mu(p^3, m_3) \\ &= [(p^1, m_1)\gamma(p^2, m_2)]\mu(p^3, m_3).\end{aligned}$$

The subset $M' = \{(0, m) | m \in M\}$ of $E_\ell$ is an ideal of $E_\ell$ and is isomorphic (as a $\Gamma$-ring) to $M$. Let the double homothetisms $p^1, p^2, \dots, p^s$ of $M$ and $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ form a left double homothetism of $M$. Define for each $i = 1, 2, \dots, s$ an element $q^i$ of $\mathscr{E}_\ell(M)$ by $q^i_\gamma = p^i_\gamma$ for all $\gamma \in \Gamma$. Then for any $p \in E_\ell$, $m \in M$:

$$\left[\sum_{i=1}^s q^i\gamma_i p\right]_\lambda (m) = \sum_{i=1}^s (q^i\gamma_i p)_\lambda(m) = \sum_{i=1}^s q^i_{\gamma_i}(p_\lambda(m)) = p_\lambda(m).$$

Thus, $\sum_{i=1}^s q^i\gamma_i p = p$ for all $p \in \mathscr{E}_\ell(M)$. Also, for each $i = 1, 2 \dots, s$,

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$(q^i, 0) \in E_\ell$. If $(p, m)$ is any element of $E_\ell$, then

$$\sum_{i=1}^{s}(q^i, 0)\gamma_i(p, m) = \sum_{i=1}^{s}(q^i\gamma_i p, q^i_{\gamma_i}(m)) = \left(\sum_{i=1}^{s} q^i\gamma_i p, \sum_{i=1}^{s} q^i_{\gamma_i}(m)\right)$$
$$= (p, \sum_{i=1}^{s} p^i_{\gamma_i}(m)) = (p, m).$$

Thus, $(q^1, 0), (q^2, 0), \ldots, (q^s, 0) \in E_\ell$ and $\gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma$ form a left unity of $E_\ell$. Hence $M$ can be embedded as an ideal in a Γ-ring with left unity. Similarly, if $M$ has a right double homothetism unity, then $M$ is isomorphic to an ideal of $E_r = M \oplus \mathscr{E}_r(M)$, which is a Γ-ring with right unity where $\mathscr{E}_r$ is the set of right translations of $M$.

Corollary 6.11. *A Γ-ring that does not have a left (right) double homothetism unity, cannot be embedded as an ideal in a Γ-ring with left or right unity.*

Corollary 6.12. *If $M$ is any Γ-ring that has only inner double homothetisms and there exists a Γ-ring $E$ with a left (right) unity such that $M \lhd E$, then $M$ has a left (right) unity.*

The Γ-ring of Example 6.3 is an example of a Γ-ring that does not have a left nor a right double homothetism unity. Consequently it is also an example of a Γ-ring that cannot be embedded as an ideal in a Γ-ring with left or right unity. In the same way it can be shown that any Γ-ring $M$ with $M \neq \{0\}$ and $\Gamma = \{0\}$ has only one double homothetism, namely the inner double homothetism induced by 0. Thus any such Γ-ring has neither a left nor a right double homothetism unity and cannot, therefore, be embedded as an ideal in a Γ-ring with left or right unity.

Example 6.13. Let $M = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $\Gamma = \{\gamma_0, \gamma_1\} \cong \mathbb{Z}_2$. Define a mapping $(-,-,-)\colon M \times \Gamma \times M \to M$ with $m_1\gamma m_2$ for any $m_1, m_2 \in M$, $\gamma \in \Gamma$ given by

$$m\gamma_0 n = 0 \quad \text{for all } m, n \in M \quad \text{and}$$
$$m\gamma_1 n = \left\{\begin{matrix} 2 & \text{if } m, n \in \{1, 3\} \\ 0 & \text{otherwise} \end{matrix}\right\}.$$

Then $M$ is a Γ-ring. $\mathrm{End}(M^+) = \{f_0, f_1, f_2, f_3\}$, where

$$f_0(m) = 0 \quad \text{and} \quad f_1(m) = m \quad \text{for all } m \in M,$$
$$f_2(0) = 0, \quad f_2(1) = 3, \quad f_2(2) = 2 \text{ and } f_2(3) = 1,$$
$$f_3(0) = f_3(2) = 0 \quad \text{and} \quad f_3(1) = f_3(3) = 2.$$

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$\mathscr{E}(M) = \{p_0, p_1\}$, where

$$p_\circ(\gamma) = f_\circ \quad \text{for all } \gamma \in \Gamma, \qquad p_1(\gamma_0) = f_0 \quad \text{and} \quad p_1(\gamma_1) = f_3.$$

Thus, for any double homothetism $p$ of $M$, $p_\gamma(m)$ is either equal to $f_0(m)$ or $f_3(m)$ for any $m \in M$. Thus $p_\gamma(m) = 0$ or $p_\gamma(m) = 2$ for any $m \in M$, $\gamma \in \Gamma$ and any double homothetism $p$ of $M$. Hence $p_\gamma(1) = 0$ or $p_\gamma(1) = 2$ for any $\gamma$ and $p$. Also, any finite sum of elements from the subset $\{0, 2\}$ of $M$ is always equal to 0 or 2. Thus for any double homothetisms $p^1, p^2, \dots, p^s$ of $M$ and any $\gamma_1, \gamma_2, \dots, \gamma_\Gamma \in \Gamma$

$$\sum_{i=1}^{s} p^i_{\gamma_i}(1) = 0 \neq 1 \quad \text{or} \quad \sum_{i=1}^{s} p^i_{\gamma_i}(1) = 2 \neq 1.$$

Hence $M$ does not have a left double homothetism unity so that it cannot be embedded as an ideal in a $\Gamma$-ring with a left unity. Similarly, $M$ does not have a right double homothetism unity so that $M$ also cannot be embedded as an ideal in a $\Gamma$-ring with a right unity.

## References

[1] W. E. Barnes, 'On the $\Gamma$-rings of Nobusawa', *Pacific J. Math.* **18** (1966), 411–422.
[2] G. L. Booth and N. J. Groenewald, 'On uniformly strongly prime gamma rings', *Bull. Austral. Math. Soc.* **37** (1988), 437–445.
[3] C. J. Everett, 'An extension theory for rings', *Amer. J. Math.* **64** (1942), 363–370.
[4] S. Kyuno, 'A gamma ring with the right and left unities', *Math. Japon.* **24** (1979), 191–193.
[5] N. Nobusawa, 'On a generalization of the ring theory', *Osaka J. Math.* **1** (1964), 81–89.
[6] M. Petrich, 'Ideal extensions of rings', *Acta Math. Hungar.* **45** (1985), 263–283.
[7] L. Rédei, *Algebra*, Volume 1 (Pergamon Press, Oxford, 1967).
[8] O. Schreier, 'Über die Erweiterung von Gruppen', *Monatsh. Math.* **34** (1926), 165–180.

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