HIGH REYNOLDS NUMBER FLOW BETWEEN TWO INFINITE ROTATING DISKS

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1. Introduction

In 1921 von Karman [1] showed that the Navier-Stokes equations for steady viscous axisymmetric flow can be reduced to a set of ordinary differential equations if it is assumed that the axial velocity component is independent of the radial distance from the axis of symmetry. He used these similarity equations to obtain a solution for the flow near an infinite rotating disk. Later Batchelor [2] and Stewartson [3] applied these equations to the problem of steady flow between two infinite disks rotating in parallel planes a finite distance apart.

In this paper we give a systematic treatment of the calculation of analytic approximations to the high Reynolds number flow between two infinite rotating disks when the effects of viscosity only appear in the boundary layers near the disks. Some earlier results obtained by Tam [4] and Mellor, Chapple, and Stokes [5] for particular cases are extended to the general case.

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2. Formulation

The similarity relations first used by von Karman [1] are

$$V_\theta = rH^*(y),\quad V_y = 2F^*(y),\quad V_r = -r\frac{dF^*}{dy}, \tag{2.1}$$

where

$\bar{y} = (r, \theta, y)$ is the position vector in a cylindrical polar coordinate system, and

$\bar{V} = (V_r, V_\theta, V_y)$ is the velocity vector.

Flows for which $(2 \cdot 1)$ holds will be called self-similar. When these relations are

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substituted into the Navier-Stokes equations for steady viscous axisymmetric flow, we see that $F^*(y)$ and $H^*(y)$ are given by

$$2H^* \frac{dH^*}{dy} + 2F^* \frac{d^3F^*}{dy^3} - \nu \frac{d^4F^*}{dy^4} = 0, \tag{2.2a}$$

$$2H^* \frac{dF^*}{dy} - 2F^* \frac{dH^*}{dy} + \nu \frac{d^2H^*}{dy^2} = 0, \tag{2.2b}$$

where ν is the kinematic viscosity.

The boundary conditions corresponding to an infinite disk rotating in the $y = 0$ plane with angular velocity Ω_0 and a second infinite disk in the $y = d$ plane with angular velocity Ω_1 are

$$\begin{aligned} F^*(0) &= \left.\frac{dF^*}{dy}\right|_{y=0} = 0, \quad H^*(0) = \Omega_0, \\ F^*(d) &= \left.\frac{dF^*}{dy}\right|_{y=d} = 0, \quad H^*(d) = \Omega_1, \end{aligned} \tag{2.3}$$

Non-dimensional variables can be defined by

$$\begin{aligned} y &= xd, \\ F^*(y) &= \Omega_0 dF(x), \\ H^*(y) &= \Omega_0 H(x), \end{aligned} \tag{2.4}$$

where it is assumed that $\Omega_0 \neq 0$. Equations (2.2a) and (2.2b) then become

$$2HH' + 2FF''' - \frac{1}{R} F'^{\text{v}} = 0, \tag{2.5a}$$

$$2F'H - 2FH' + \frac{1}{R} H'' = 0, \tag{2.5b}$$

where $R = \Omega_0 d^2/\nu$ is the Reynolds number (or the reciprocal of the Ekman number), and $H'(x) = dH/dx$, etc. The boundary conditions are now given by

$$\begin{aligned} F(0) &= F'(0) = 0, \qquad H(0) = 1, \\ F(1) &= F'(1) = 0, \qquad H(1) = \Omega, \end{aligned} \tag{2.6}$$

where $\Omega = \Omega_1/\Omega_0$.

Since the highest derivatives in the two equations (2.5a) and (2.5b) are multiplied by R^{-1}, it is to be expected that a high Reynolds number expansion will be singular. Hence we shall use the method of matched asymptotic expansions to obtain approximate solutions.

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3. Physical discussion

The general problem of high Reynolds number flow between two infinite rotating disks was first discussed by Batchelor [2] and Stewartson [3]. In the case of flow between a rotating and a stationary disk Batchelor and Stewartson reached different conclusions. Batchelor thought that boundary layers would be formed at both disks and the main body of the fluid would rotate with constant nonzero angular velocity. On the other hand Stewartson thought that only on the rotating disk would a boundary layer develop and the main body of the fluid would not rotate at all. Numerical work by Lance and Rogers [6], Pearson [7], and by Mellor, Chapple and Stokes [5] has shown that both types of flow exist.

The analysis in the next section indicates that the type of flow predicted by Batchelor for the case of a rotating and a stationary disk exists for the more general case of both disks rotating in the same direction. This type of flow which is characterized by boundary layers on both disks and the fluid in the interior region between the boundary layers having constant nonzero angular velocity will in this paper be called type A flow. The type of flow predicted by Stewartson will be called type B flow, and in Section 5 it is shown that this type of flow seems to exist even when the disks rotate in opposite directions. Its main characteristic is that the fluid in the interior region does not rotate.

We use the method of matched asymptotic expansions for obtaining analytic approximations to the flow at high Reynolds number. This method has been used by Tam [4] for the flow between two disks rotating in opposite directions with angular velocities of equal magnitudes, and by Mellor, Chapple, and Stokes [5] who considered the flow between a rotating and a stationary disk.

It will be assumed that boundary layers are developed on the disks and that the effects of viscosity only appear in these layers. The existence of these layers can be deduced from the approximate solutions for high Reynolds number flow between two disks which rotate with almost equal angular velocities. Their existence is also indicated in the experimental work carried out by Picha and Eckerts [8] for flow between two disks of finite radius.

Approximate solutions are calculated by finding inner expansions, valid in the boundary layers, which are then matched asymptotically to an outer expansion valid in the interior region. This matching procedure produces a sufficient number

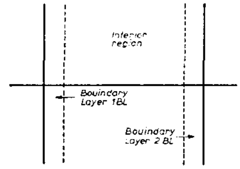

Figure 1. Sketch showing the location of the boundary layers and the interior region.

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of boundary conditions so that in most cases all unknown constants appearing in the solutions can be evaluated.

The location of the different regions of the flow field are shown in Fig. 1; the two boundary layers are designated by $1BL$ and $2BL$.

The boundary value problem for the first term in each of the two expansions valid in $1BL$ and $2BL$ is equivalent to that for the selfsimilar flow near an infinite rotating disk. This problem has been solved numerically by, among others, Rogers and Lance [9], and since their solutions will be used in the following analysis, they are described in Appendix 1.

4. Type *A* flow

In this section we discuss the flow between two disks for which the angular velocity in the interior region is of the same order of magnitude as that in the boundary layers. In this case it is shown that the disks must rotate in the same direction and that the flow is of type A.

In the boundary layers nondimensional variables are defined by

$$y = (\nu/\Omega_0)^{\frac{1}{2}}\tau = d-(\nu/\Omega_0)^{\frac{1}{2}}\sigma,$$
$$F^*(y) = (\nu\Omega_0)^{\frac{1}{2}}f(\tau) = -(\nu\Omega_0)^{\frac{1}{2}}\bar{f}(\sigma),$$
$$H^*(y) = \Omega_0 h(\tau) = \Omega_0\bar{h}(\sigma),$$

where $f(\tau)$ and $h(\tau)$ refer to $1BL$, and $\bar{f}(\sigma)$ and $\bar{h}(\sigma)$ refer to $2BL$, and it is assumed that $\Omega_0 \neq 0$.

The equations for f and h are, from (2.2a) and (2.2b)

$$\begin{aligned} 2hh'+2ff'''-f^{\mathrm{IV}} &= 0, \\ 2hf'-2h'f+h'' &= 0, \end{aligned} \tag{4.1}$$

and similarly for $\bar{f}(\sigma)$ and $\bar{h}(\sigma)$. The boundary conditions on the disks become

$$\begin{aligned} f(0) = f'(0) = 0, &\quad h(0) = 1 \\ \bar{f}(0) = \bar{f}'(0) = 0, &\quad \bar{h}(0) = \Omega_1/\Omega_0. \end{aligned} \tag{4.2}$$

In order that the angular and axial velocity components will be of the same order of magnitude in the three regions, non-dimensional variables in the interior region are defined by

$$y = xd,$$
$$F^*(y) = \varepsilon\Omega_0 dF(x),$$
$$H^*(y) = \Omega_0 H(x),$$

where $\varepsilon = 1/R^{\frac{1}{2}} = (\nu/\Omega_0 d^2)^{\frac{1}{2}}$. Equations (2.2a) and (2.2b) then become

$$2HH'+2\varepsilon^2FF'''-\varepsilon^3F'^V = 0,$$
$$2HF'-2FH'+\varepsilon H'' = 0. \tag{4.3}$$

The inner and outer expansions are obtained by expanding the dependent variables in power series in ε, i.e.

$$f(\tau) = \sum_{n=0}^{\infty} \varepsilon^n f_n(\tau), \qquad h(\tau) = \sum_{n=0}^{\infty} \varepsilon^n h_n(\tau).$$

These expansions are then substituted into the equations for f, h, $\bar{f}$, $\bar{h}$, F, and H, and the coefficients of the powers of ε are equated to zero.

The inner expansion, valid in $1BL$, is matched to the outer expansion, valid in the interior region, near $x = 0$. Similarly the inner expansion in $2BL$ is matched to the outer expansion near $x = 1$. The procedure used in the matching is described in Van Dyke [10] (see Chapter V, pp. 89–90). From the matching we obtain the following conditions on the first two terms of each expansion.

$$\begin{aligned}
&f_0(\tau) \to F_0(0), && \bar{f}_0(\sigma) \to -F_0(0),\\
&f_1(\tau) \sim F_0'(0)\tau + F_1(0), && \bar{f}_1(\sigma) \sim F_0(1)\sigma - F_1(1),\\
&f_0'(\tau) \to 0, && \bar{f}_0'(\sigma) \to 0,\\
&f_1(\tau) \to F_0'(0), && \bar{f}_1'(\sigma) \to F_0'(1),\\
&h_0(\tau) \to H_0(0), && \bar{h}_0(\sigma) \to H_0(1),\\
&h_1(\tau) \sim H_0'(0)\tau + H_1(0), && \bar{h}_1(\sigma) \sim -H_0(1)\sigma + H_1(1),\\
&\text{as } \tau \to \infty; && \text{as } \sigma \to \infty.
\end{aligned} \tag{4.4}$$

From equation (4.3) we see that the equations for the zero order terms of the inner expansion in the interior region are

$$H_0 H_0' = 0,$$
$$H_0 F_0' - H_0' F_0 = 0.$$

Since we are interested in a solution with non-zero swirl in the interior region, we take the solution

$$F_0(x) = -\gamma_0,$$
$$H_0(x) = \pm\beta_0^2, \tag{4.5}$$

where γ_0 and β_0 are non-zero constants. The solution $H_0(x) \equiv 0$ will be discussed in the next section.

The equations for $f_0(\tau)$ and $h_0(\tau)$ are equivalent to (4.1), and the boundary conditions at $\tau = 0$ are

$$f_0(0) = f_0'(0) = 0, \qquad h_0(0) = 1.$$

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From (4.4) we see that as $\tau \to \infty$,

$$f_0(\tau) \to F_0(0) = -\gamma_0,$$
$$f_0'(\tau) \to 0,$$
$$h_0(\tau) \to H_0(0) = \pm\beta_0^2.$$

This boundary value problem for $f_0(\tau)$ and $h_0(\tau)$, which is discussed in Appendix 1, corresponds to that for the steady flow near an infinite disk which rotates with angular velocity 1 in a fluid which far from the disk rotates with angular velocity $\pm\beta_0^2$. Now, as explained in Appendix 1, no satisfactory numerical solution has been obtained for this problem when the two angular velocities are of opposite signs. Thus we will here insist that $H_0(x)$ is positive so that $h_0(\tau) \to +\beta_0^2$.

The solution for $f_0(\tau)$ and $h_0(\tau)$ can be written in the form

$$\begin{aligned} f_0(\tau) &= M(\beta_0^2, \tau), \\ h_0(\tau) &= N(\beta_0^2, \tau). \end{aligned} \tag{4.6}$$

The functions $M(\beta_0^2, \tau)$ and $N(\beta_0^2, \tau)$ are described in Appendix 1 and will here be considered as known functions.

The equations for $\bar{f}_0(\sigma)$ and $\bar{h}_0(\sigma)$ are also equivalent to (4.1), and the boundary conditions at $\sigma = 0$ are

$$\bar{f}_0(0) = \bar{f}_0'(0) = 0, \qquad \bar{h}_0(0) = \Omega_1/\Omega_0.$$

As $\sigma \to \infty$ we have from (4.4) that

$$\bar{f}_0(\sigma) \to -F_0(1) = \gamma_0,$$
$$\bar{f}_0'(\sigma) \to 0,$$
$$\bar{h}_0(\sigma) \to H_0(1) = \beta_0^2.$$

As before we insist that $\bar{h}_0(0)$ and $\bar{h}_0(\infty)$ are of the same sign, so we assume that Ω_1/Ω_0 is positive and set it equal to α^2. From Appendix 1 the solutions for $\bar{f}_0(\sigma)$ and $\bar{h}_0(\sigma)$ are

$$\begin{aligned} \bar{f}_0(\sigma) &= \beta_0 P(\alpha^2/\beta_0^2, \beta_0\sigma), \\ \bar{h}_0(\sigma) &= \beta_0^2 Q(\alpha^2/\beta_0^2, \beta_0\sigma), \end{aligned} \tag{4.7}$$

where $\alpha^2 = \Omega_1/\Omega_0$. The functions P and Q are described in Appendix 1.

The value of β_0^2 for a given α is determined by the condition that

$$f_0(\infty) = -\gamma_0 = -\bar{f}_0(\infty).$$

From (4.6) and (6.7) we see that this implies that

$$M(\beta_0^2, \infty) = -\beta_0 P(\alpha^2/\beta_0^2, \infty).$$

This equation has been solved for different values of α^2 using the information given in Appendix 1, and the graphs of β_0^2 vs α^2 and γ_0 vs α^2 are plotted in Fig. 2. A graph of β_0^2 vs α^2 similar to the one given in Fig. 2 was first obtained by Rott and Lewellen [11].

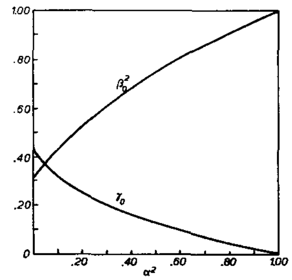

Figure 2. Graphs of β_0^2 and γ_0 against α^2.

In order to obtain the solutions (4.6) and (4.7) for the boundary layers we have implicitly assumed that $1 \geqq \beta_0^2 \geqq \alpha^2$. That this inequality must hold can easily be shown using the functions M, N, P, Q. That the inequality must hold can also be shown by physical arguments. Assume, for example, that

$$1 > \alpha^2 > \beta_0^2 > 0.$$

This means that the fluid in the interior region rotates slower than either of the disks, and it can be expected that the axial flow at the edges of the boundary layers would be towards the disks. This would require the axial flow to change direction somewhere in the interior region which contradicts the fact that $F_0(x)$ is constant.

The higher order terms can now in theory be determined, but since they will only modify the flow slightly, they will not be calculated here.

The main characteristic of the type of flow discussed in this section is that the radial flow takes place in the boundary layers and that the interior region rotates with a constant nonzero velocity. There is a radial inflow near the slower disk and a radial outflow near the faster disk. Thus one of the functions of the interior region which rotates with an angular velocity between those of the disks is to transport fluid from one boundary layer to the other.

5. Type *B* flow

In the previous section we noticed that there seems to be flows for which the angular velocity in the interior region is identically zero to a first approximation. One of these flows is the type B flow which was discussed briefly in Section 3; it is now considered in more detail in this section.

Only in the interior region will the formulation of the problem be different from the one used in the previous section. In this region non-dimensional variables are defined by

$$y = xd,$$
$$F^*(y) = \varepsilon\Omega_0 dF(x),$$
$$H^*(y) = \varepsilon\Omega_0 H(x),$$

where $\varepsilon = 1/R^{\frac{1}{2}}$. The governing equations (2.2a) and (2.2b) become

$$\begin{aligned} &2HH'+2FF'''-\varepsilon F'^{V} = 0, \\ &2HF'-2FH'+\varepsilon H'' = 0. \end{aligned} \tag{5.1}$$

An outer expansion valid in the interior region is obtained by expanding $F(x)$ and $H(x)$ in power series in ε, i.e.

$$F(x) = \sum_{n=0}^{\infty} \varepsilon^n F_n(x), \qquad H(x) = \sum_{n=0}^{\infty} \varepsilon^n H_n(x).$$

From (5.1) we see that $F_n(x)$ and $H_n(x)$ are given by

$$\begin{aligned} &2\sum_{j=0}^{n} (H_j H'_{n-j} + F_j F'''_{n-j}) - F'^{V}_{n-1} = 0, \\ &2\sum_{j=0}^{n} (H_j F'_{n-j} - H'_j F_{n-j}) + H''_{n-1} = 0, \end{aligned} \tag{5.2}$$

where quantities with negative subscripts are set equal to zero.

Again the inner and outer expansions are matched near $x = 0$ and $x = 1$, and only the conditions on the swirl are different from the ones given in (4.4) and (4.5). The new conditions on the swirl are

$$\begin{aligned} &h_0(\tau) \to 0, \\ &h_1(\tau) \to H_0(0), \end{aligned} \tag{5.3}$$

as $\tau \to \infty$, and

$$\begin{aligned} &\bar{h}_0(\sigma) \to 0, \\ &\bar{h}_1(\sigma) \to H_0(1), \end{aligned} \tag{5.4}$$

as $\sigma \to \infty$.

The equations for the zero order terms in the interior region, obtained from (5.2), are

$$H_0 H'_0 + F_0 F'''_0 = 0,$$
$$H_0 F'_0 - H'_0 F_0 = 0.$$

The second equation can be integrated to give

$$H_0(x) = \beta_0 F_0(x),$$

where β_0 is a constant. Two different types of solution result from the first equation depending on whether $\beta_0 = 0$ or $\beta_0 \neq 0$. The first case which corresponds to $H_0(x) \equiv 0$ will be considered in detail in this section, and in order to simplify the calculations we shall assume that the higher order terms are also identically zero, i.e. $H(x) \equiv 0$. The second case will be discussed briefly in the end of the section.

In the first case, only the fluid in the boundary layer has angular momentum. The two disks act as fans so there is a radial outflow of fluid in the boundary layers. This outflow causes an axial inflow to the boundary layers from the interior region.

Since there is no angular velocity in the interior region, the problem is unaltered in character by a change in the direction of the rotation of one of the disks. Hence we shall only consider $\bar{h}(0) = \Omega_1/\Omega_0 = \alpha^2$, i.e. the disks rotate in the same direction. The boundary conditions on the zero-order terms of the expansion in $2BL$ at $\sigma = 0$ then become

$$\bar{f}_0(0) = \bar{f}_0'(0) = 0, \qquad \bar{h}_0(0) = \alpha^2.$$

From (4.1) we see that the equations for $\bar{f}_0(\sigma)$ and $\bar{h}_0(\sigma)$ are

$$2\bar{h}_0 \bar{h}_0' + 2\bar{f}_0 \bar{f}_0''' - \bar{f}_0^{IV} = 0,$$
$$2\bar{h}_0 \bar{f}_0' - 2\bar{h}_0' \bar{f}_0 + \bar{h}_0'' = 0,$$

and from (4.5) and (5.4) we have the following boundary conditions

$$\bar{f}_0(\infty) = -F_0(1),$$
$$\bar{f}_0'(\infty) = 0,$$
$$\bar{h}_0(\infty) = 0.$$

Hence the solution is

$$\bar{f}_0(\sigma) = \alpha M(0, \alpha\sigma), \qquad \bar{h}_0(\sigma) = \alpha^2 N(0, \alpha\sigma). \tag{5.5}$$

Similarly the zero-order terms in $1BL$ are given by

$$f_0(\tau) = M(0, \tau), \qquad h_0(\tau) = N(0, \tau), \tag{5.6}$$

satisfying the condition

$$f_0(\infty) = F_0(0).$$

From (5.2) we see that, if $H_0(x) \equiv 0$, $F_0(x)$ is given by

$$F_0 F_0''' = 0,$$

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with solution

$$F_0(x) = A+Bx+Cx^2,$$

where A, B, C are constants.

In Appendix 1 $M(0, \infty)$ is given as $-.442$, so since

$$f_0(\infty) = M(0, \infty) = F_0(0),$$

and

$$\bar{f}_0(\infty) = \alpha M(0, \infty) = -F_0(1),$$

we find that

$$F_0(x) = -.442[1-(\alpha+1)x]-C(x-x^2). \tag{5.7}$$

In order to evaluate C we must consider the first-order terms.

The equations for the first-order terms in $1BL$, $f_1(\tau)$ and $h_1(\tau)$, are obtained from equation (4.1). Thus

$$\begin{aligned} &2(h_0 h_1'+h_1 h_0'+f_0 f_1'''+f_1 f_0''')-f_1^{IV} = 0,\\ &2(h_0 f_1'+h_1 f_0'-f_0 h_1'-f_1 h_0')+h_1'' = 0 \end{aligned} \tag{5.8}$$

with the following boundary conditions at $\tau = 0$

$$f_1(0) = f_1'(0) = h_1(0) = 0.$$

Since

$$f_0(\tau) = M(0, \tau),$$
$$h_0(\tau) = N(0, \tau),$$

we can consider (5.8) as a system of linear equations with known coefficients. In Appendix 2 two fundamental solutions have been calculated numerically. If we replace $f_1(\tau)$ by φ and $h_1(\tau)$ by ψ, and let the subscript 1 designate the solution which satisfies

$$\varphi_1(0) = \varphi_1'(0) = 0, \qquad \varphi_1''(0) = 1,$$
$$\psi_1(0) = \psi_1'(0) = 0,$$

and the subscript 2 designate the solution which satisfies

$$\varphi_2(0) = \varphi_2'(0) = \varphi_2''(0) = 0$$
$$\psi_2(0) = 0, \qquad \psi_2'(0) = 1,$$

the solutions for $f_1(\tau)$ and $h_1(\tau)$ can be expressed in the form

$$\begin{aligned} f_1(\tau) &= a_1\varphi_1(\tau)+a_2\varphi_2(\tau),\\ h_1(\tau) &= a_1\psi_1(\tau)+a_2\psi_2(\tau), \end{aligned} \tag{5.9}$$

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where a_1 and a_2 are constants.

The first-order terms in $2BL$, $\bar{f}_1(\sigma)$ and $\bar{h}_1(\sigma)$, are also given by equation (5.8) with f_0 and h_0 replaced by $\bar{f}_0$ and $\bar{h}_0$. Since

$$\bar{f}_0(\sigma) = \alpha M(0, \alpha\sigma),$$
$$\bar{h}_0(\sigma) = \alpha^2 N(0, \alpha\sigma),$$

we see from equation (5.8) that $\bar{f}_1(\sigma)$ and $\bar{h}_1(\sigma)$ are related to $f_1(\tau)$ and $h_1(\tau)$ in the following manner

$$\bar{f}_1(\sigma) = f_1(\alpha\sigma),$$
$$\bar{h}_1(\sigma) = \alpha h_1(\alpha\sigma).$$

Thus since

$$\bar{f}_1(0) = \bar{f}_1'(0) = \bar{h}_1(0) = 0,$$

we have that

$$\begin{aligned} \bar{f}_1(\sigma) &= a_1 \varphi_1(\alpha\sigma) + a_2 \varphi_2(\alpha\sigma), \\ \bar{h}_1(\sigma) &= \alpha a_1 \psi_1(\alpha\sigma) + \alpha a_2 \psi_2(\alpha\sigma). \end{aligned} \tag{5.10}$$

From the matching conditions (4.4), (4.5), (5.3), and (5.4) we see that the first-order terms in the boundary layers must satisfy the following conditions

$$f_1'(\infty) = F_0'(0) = .442(1+\alpha) - C,$$
$$h_1(\infty) = 0,$$
$$\bar{f}_1'(\infty) = F_0'(1) = .442(1+\alpha) + C,$$
$$\bar{h}_1(\infty) = 0.$$

These conditions together with the solutions (5.9) and (5.10) give us three equations, which can be solved for a_1, a_2 and C, using the values of $\varphi_1'(\infty)$, $\varphi_2'(\infty)$, $\psi_1(\infty)$ and $\psi_2(\infty)$ given in Appendix 2. The solutions are

$$a_1 = -.001,$$
$$a_2 = .155,$$
$$C = .442(\alpha - 1).$$

Hence we see that $F_0(x)$ is given by

$$F_0(x) = -.442\{1 - 2x + (1-\alpha)x^2\}. \tag{5.11}$$

Thus we have a complete solution for the zero-order terms. Again in theory the higher order terms could be calculated.

The difference between type B flow, given by (5.5), (5.6), and (5.11), and type A flow which was discussed in the previous section, is that in type B flow the radial velocity is not zero in the interior region but the angular velocity is, while in type A flow the contrary is the case. Since in type B flow the angular velocity is zero in the interior region, it is in this case possible for the disks to rotate in opposite directions.

Discussion of the case $\beta_0 \neq 0$.

In this case the solutions of $F_0(x)$ and $H_0(x)$ are

$$(5.12)\qquad \begin{aligned} F_0(x) &= A_0+A_1 \sin \beta_0 x+A_2 \cos \beta_0 x, \\ H_0(x) &= \beta_0 F_0(x), \end{aligned}$$

where A_0, A_1, A_2 are constants. As before the zero-order terms in the boundary layers are given by

$$\begin{aligned} \bar{f}_0(\sigma) &= (|\Omega_1/\Omega_0|)^{\frac{1}{2}}M(0, (|\Omega_1/\Omega_0|)^{\frac{1}{2}}\sigma), \\ \bar{h}_0(\sigma) &= (\Omega_1/\Omega_0)N(0, (|\Omega_1/\Omega_0|)^{\frac{1}{2}}\sigma), \end{aligned}$$

and

$$\begin{aligned} f_0(\tau) &= M(0, \tau), \\ h_0(\tau) &= \dot{N}(0, \tau). \end{aligned}$$

The boundary conditions on $F_0(x)$ and $H_0(x)$ are obtained from the matching conditions (4.4), (4.5), (5.3), and (5.4):

$$\begin{aligned} F_0(0) &= f_0(\infty) = -.442, \\ F_0'(0) &= f_1'(\infty), \\ H_0(0) &= h_1(\infty), \\ F_0(1) &= -\bar{f}_0(\infty) = .442, \\ F_0'(1) &= \bar{f}_1'(\infty), \\ H_0(1) &= \bar{h}_1(\infty). \end{aligned}$$

Since the first-order terms in the boundary layers are not known, we see that the matching process only gives two boundary conditions on $F_0(x)$. Now the solution (5.12) for $F_0(x)$ contains four unknown constants which cannot therefore all be evaluated. Thus it seems that the flow in the interior region is not uniquely determined; at present it is not known if this non-uniqueness is caused by the perturbation scheme.

If the solutions for $F_0(x)$ and $H_0(x)$ are substituted into the equations for $F_1(x)$ and $H_1(x)$, a solution for $F_1(x)$ can be obtained. The third and fourth derivatives of this solution become infinite at certain values of x; the physical interpretation of this phenomenon is not yet clear.

6. Conclusions

The analysis in the previous sections has indicated that there are two main types of high Reynolds number flow between two infinite rotating disks when the effects of viscosity are assumed to be important only near the disks. In a type A flow, which is only possible when the disks rotate in the same direction, the radial velocity is zero in the interior region and the angular velocity equals a nonzero constant in this region. Since the angular velocity is zero for a type B flow, it is possible to obtain this type of flow when the disks do not rotate in the same direction. The physical interpretation of the sinusoidal solution for the interior region, obtained at the end of Section 5, is uncertain, and it is not clear if this solution points to the existence of flows with viscous behaviour outside the boundary layers on the disks.

From their numerical solutions for the flow between a rotating and a stationary disk Mellor, Chapple, and Stokes [5] came to the conclusion that there exist two types of one-cell flow of which one tends to the type A flow as the Reynolds number tends to infinity while the other tends to the type B flow. The results obtained in this paper seem to indicate that the first type of one-cell solutions can only occur when the disks rotate in the same direction or when one disk is stationary. The other type seems to be possible even when the ratio of the angular velocities of the disks is negative.

In order to consider the nonuniqueness of the flow for high Reynolds number it is necessary to discuss the boundary conditions that may be imposed on the flow far from the axis of rotation. We shall not go into the details of this problem here but will only describe two physical flow situations where type A and type B flows are obtained in part of the flow field.

The high Reynolds number flow between two disks of finite radius which rotate with almost equal angular velocities and which are contained in a rotating cylinder of equal radius is discussed in Rasmussen [12], Chapter 6. If the angular velocity of the cylinder is equal to that of the faster disk, the following flow is obtained. On the two disks there are Ekman layers of thickness $R^{-\frac{1}{2}}$ and on the cylinder a Stewartson layer of thickness $R^{-\frac{1}{4}}$. The flow in the Ekman layers and the interior region outside the boundary layers is selfsimilar, i.e. satisfies the von Karman similarity relations. In the interior region the angular and axial velocity components are constant and nonzero while the radial velocity component is zero. The fluid flows radially outwards in the Ekman layer on the faster disk into the Stewartson layer on the cylinder through which it flows to the Ekman layer on the slower disk. From this layer it moves into the interior region where it then flows back to the Ekman layer on the faster disk. Thus we see that the flow in the Ekman layers and the interior region forms a type A flow.

It is not certain if a type A flow will be obtained if the difference between the angular velocities of the disks and the cylinder is not small. Since in this case the governing equations cannot be linearized, the analysis of the bounaary layer on the cylinder becomes rather difficult.

Picha and Eckerts [8] studied experimentally the high Reynolds number flow between two finite disks. When the disks were rotating freely, i.e. not contained in a cylinder, they found that no rotation of the fluid core could be observed for the case of counterrotating disks. This would seem to indicate that a type B flow is obtained in this case.

Throughout the treatment in this paper of high Reynolds number flow between two disks it has been assumed that the effects of viscosity only appear in the boundary layers on the disks. There seems to be no physical reason why a flow should not exist in which a viscous shear layer appears somewhere in the flow field. An analysis of this problem has been started but is not yet completed.

Appendix 1

In this appendix we describe the numerical solutions obtained by Rogers and Lance [9] for the one-disk problem with no radial flow at infinity.

The governing equations are

$$2H\frac{dH}{dy}+2F\frac{d^3F}{dy^3}-\nu\frac{d^4F}{dy^4}=0, \tag{A1.1}$$

$$2H\frac{dF}{dy}-2F\frac{dH}{dy}+\nu\frac{d^2H}{dy^2}=0, \tag{A1.2}$$

with boundary conditions

$$F(0)=\left.\frac{dF}{dy}\right|_{y=0}=0, \qquad H(0)=\Omega_0\geqq 0,$$

$$\frac{dF}{dy}\to 0, \qquad H\to\Omega_\infty\geqq 0 \quad \text{as } y\to\infty.$$

Here Ω_0 and Ω_∞ are the angular velocities of the disk and the fluid at infinity, respectively.

When the disk rotates faster than the fluid at infinity, i.e. when $\Omega_0\geqq\Omega_\infty$, and $\Omega_0\neq 0$ we define dimensionless variables by

$$y=(\nu/\Omega_0)^{\frac{1}{2}}x,$$

$$F(y)=(\nu\Omega_0)^{\frac{1}{2}}M(s^2,x),$$

$$H(y)=\Omega_0 N(s^2,x),$$

where $s^2=\Omega_\infty/\Omega_0$ and $0\leqq s^2\leqq 1$. Equations (A1.1) and (A1.2) then become

$$2NN'+2MM'''-M'^{V}=0,$$

$$2M'N-2MN'+N''=0,$$

with boundary conditions

$$M(s^2, 0) = \left.\frac{dM}{dx}\right|_{x=0} = 0, \qquad \left.\frac{dM}{dx}\right|_{x\to\infty} = 0,$$

$$N(s^2, 0) = 1 \qquad N(s^2, \infty) = s^2.$$

When the disk rotates slower than the fluid at infinity, i.e. when $\Omega_0 < \Omega_\infty$ and $\Omega_\infty > 0$, nondimensional variables are defined by

$$y = (\nu/\Omega_\infty)^{\frac{1}{2}}x,$$
$$F(y) = (\nu\Omega_\infty)^{\frac{1}{2}}P(t^2, x),$$
$$H(y) = \Omega_\infty Q(t^2, x),$$

where $t^2 = \Omega_0/\Omega_\infty$ and $0 \leqq t^2 < 1$. Equations (A1.1) and (A1.2) then become

$$2QQ' + 2PP''' - P'^{V} = 0,$$
$$2QP' - 2PQ' + Q'' = 0,$$

with boundary conditions

$$P(t^2, 0) = \left.\frac{dP}{dx}\right|_{x=0} = 0, \qquad \left.\frac{dP}{dx}\right|_{x\to\infty} = 0,$$

$$Q(t^2, 0) = t^2, \qquad Q(t^2, \infty) = 1.$$

Rogers and Lance integrated these two boundary value problems numerically for several values of s^2 and t^2. They also attempted to obtain numerical solutions for Ω_∞/Ω_0 negative. In this case which corresponds to the disk and the fluid rotating in opposite directions, the numerical procedure did not converge, and they were unable to obtain a solution. It has not yet been proved that a solution does not exist but in this paper we have assumed that this is the case.

Since the treatment in Sections 4 and 5 only require the relationships between s^2 and $M(s^2, \infty)$ and between t^2 and $P(t^2, \infty)$, these are the only parts of the solutions which are reproduced here in Tables 1 and 2. More details can be found in [9] where a different notation is used.

TABLE 1

s^2	$M(s^2, \infty)$	s^2	$M(s^2, \infty)$
0.0	—.44223	0.6	—.21539
0.1	—.45885	0.8	—.10401
0.2	—.43088	0.9	—.05101
0.4	—.32998	1.0	0.00000

TABLE 2

t^2	$P(t^2, \infty)$	t^2	$P(t^2, \infty)$
0.0	.68481	0.6	.22689
0.1	.59758	0.8	.10636
0.2	.51540	0.9	.05155
0.4	.36304	1.0	0.00000

Appendix 2

If we let

$$\varphi(\tau) = f_1(\tau),$$
$$\psi(\tau) = h_1(\tau),$$

we can write equations (5.8) for the first order terms in the boundary layers in the form

(A2.1) $$\varphi''' - 2f_0\varphi'' + 2f_0'\varphi' - 2f_0''\varphi + 2h_0\psi = 0,$$

(A2.2) $$\psi'' - 2f_0\psi' + 2f_0'\psi - 2h_0'\varphi + 2h_0\varphi' = 0.$$

The zero order terms $f_0(\tau)$ and $h_0(\tau)$ are given by

$$f_0(\tau) = M(0, \tau),$$
$$h_0(\tau) = N(0, \tau).$$

These terms can be approximated by the first five terms of an approximate solution calculated by Fettis [13]. Thus

$$f_0(\tau) = \tfrac{1}{2}C + \frac{1}{4C^3}(e^{2C\tau} - 2e^{C\tau}) + \frac{1}{576C^7}(e^{4C\tau} - 24e^{3C\tau} + 6e^{2C\tau} - 52e^{C\tau}),$$

$$h_0(\tau) = e^{C\tau} - \frac{1}{12C^4}(e^{3C\tau} - e^{C\tau}) - \frac{1}{1152C^8}(3e^{5C\tau} - 32e^{4C\tau} + 48e^{3C\tau} - 19e^{C\tau}),$$

where $C = -.8759$.

Let $\varphi_1(\tau)$ and $\psi_1(\tau)$ be the solutions of equations (A2.1) and (A2.2) satisfying the initial conditions

(A2.3) $$\varphi_1(0) = \varphi_1'(0) = 0, \qquad \varphi_1''(0) = 1,$$
$$\psi_1(0) = \psi_1'(0) = 0,$$

and $\varphi_2(\tau)$ and $\psi_2(\tau)$ the solutions satisfying

(A2.4) $$\varphi_2(0) = \varphi_2'(0) = \varphi_2''(0) = 0,$$
$$\psi_2(0) = 0, \qquad \psi_2'(0) = 1.$$

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These solutions were calculated by integrating (A2.1) and (A2.2) using a fourth-order Runge Kutta procedure. The computations were carried out at the University of Queensland Computer Centre using a subprogram produced by the centre. The solutions are tabulated below.

TABLE 3

The solutions of equations (A2.1) and (A2.2) which satisfy the initial conditions (A2.3)

	$\varphi_1(\tau)$	$\varphi_1'(\tau)$	$\varphi_1''(\tau)$	$\psi_1(\tau)$	$\psi_1'(\tau)$
0	0.00000	0.00000	1.00000	0.00000	0.00000
1	0.49230	0.97280	0.92385	−0.27003	−0.73857
2	1.88919	1.77445	0.64885	−1.56356	−1.75863
3	3.93182	2.25745	0.33186	−3.50294	−1.97760
4	6.31810	2.48409	0.14492	−5.31467	−1.58912
5	8.85853	2.58406	0.06763	−6.63958	−1.06577
6	11.47048	2.63515	0.03895	−7.48156	−0.64210
7	14.12258	2.66689	0.02581	−7.97206	−0.36115
8	16.80085	2.68824	0.01734	−8.24178	−0.19378
9	19.49665	2.70233	0.01116	−8.38420	−0.10049
10	22.20376	2.71117	0.00681	−8.45717	−0.05079
11	24.91780	2.71644	0.00395	−8.49371	−0.02515
12	27.63589	2.71944	0.00220	−8.51167	−0.01226
13	30.35623	2.72108	0.00118	−8.52037	−0.00589
14	33.07780	2.72195	0.00062	−8.52453	−0.00280
15	35.80000	2.72241	0.00032	−8.52649	−0.00132
16	38.52254	2.72263	0.00016	−8.52742	−0.00062
17	41.24523	2.72275	0.00008	−8.52785	−0.00028
18	43.96801	2.72280	0.00004	−8.52804	−0.00013
19	46.69083	2.72283	0.00002	−8.52814	−0.00006
20	49.41367	2.72284	0.00001	−8.52818	−0.00003
21	52.13652	2.72285	0.00000	−8.52820	−0.00001
22	54.85937	2.72285	0.00000	−8.52820	−0.00001
23	57.58223	2.72285	0.00000	−8.52821	−0.00000
24	60.30508	2.72286	0.00000	−8.52821	−0.00000
25	63.02794	2.72286	0.00000	−8.52821	−0.00000
26	65.75079	2.72286	0.00000	−8.52821	−0.00000

TABLE 4

The solutions of equations (A2.1) and (A2.2) which satisfy the initial conditions (A2.4)

	$\varphi_2(\tau)$	$\varphi_2'(\tau)$	$\varphi_2''(\tau)$	$\psi_2(\tau)$	$\psi_2^{\cdot}(\tau)$
0	0.00000	0.00000	0.00000	0.00000	1.00070
1	0.06232	0.22959	0.59392	0.98123	0.93732
2	0.71400	1.17916	1.21786	1.73081	0.48777
3	2.53251	2.46583	1.27135	1.90585	−0.11345
4	5.59543	3.61365	0.99808	1.60356	−0.43369
5	9.65312	4.44771	0.67574	1.13519	−0.46798
6	14.39241	4.98808	0.41911	0.71105	−0.36975
7	19.55782	5.31380	0.24516	0.40216	−0.24980
8	24.97390	5.50047	0.13735	0.20321	−0.15337
9	30.53106	5.60340	0.07439	0.08485	−0.08826
10	36.16489	5.65843	0.03919	0.01823	−0.04848
11	41.83924	5.68712	0.02017	−0.01775	−0.02570
12	47.53450	4.70175	0.01019	−0.03659	−0.01326
13	53.24034	4.70909	0.00506	−0.04621	−0.00669
14	58.95146	5.71271	0.00248	−0.05102	−0.00332
15	64.66515	5.71447	0.00120	−0.05339	−0.00162
16	70.38009	5.71532	0.00057	−0.05454	−0.00078
17	76.09564	5.71572	0.00027	−0.05509	−0.00037
18	81.81146	5.71591	0.00013	−0.05535	−0.00018
19	87.52742	5.71600	0.00006	−0.05548	−0.00008
20	93.24344	5.71604	0.00003	−0.05553	−0.00004
21	98.95950	5.71606	0.00001	−0.05556	−0.00002
22	104.67556	5.71607	0.00001	−0.05557	−0.00001
23	110.39163	5.71607	0.00000	−0.05558	−0.00000
24	116.10770	5.71607	0.00000	−0.05558	−0.00000
25	121.82378	5.71608	0.00000	−0.05558	−0.00000
26	127.53985	5.71608	0.00000	−0.00558	−0.00000

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