Canad. J. Math. Vol. **66** (3), 2014 pp. 625–640
http://dx.doi.org/10.4153/CJM-2013-016-5

Classifying the Minimal Varieties of Polynomial Growth

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Abstract. Let $\mathcal{V}$ be a variety of associative algebras generated by an algebra with 1 over a field of characteristic zero. This paper is devoted to the classification of the varieties $\mathcal{V}$ that are minimal of polynomial growth (*i.e.*, their sequence of codimensions grows like n^k, but any proper subvariety grows like n^t with $t < k$). These varieties are the building blocks of general varieties of polynomial growth.

It turns out that for $k \le 4$ there are only a finite number of varieties of polynomial growth n^k, but for each $k > 4$, the number of minimal varieties is at least $|F|$, the cardinality of the base field, and we give a recipe for their construction.

1 Introduction

One of the more challenging problems in PI-theory is that of classifying the algebras, up to PI-equivalence, by means of some numerical parameters (invariants) that can be explicitly computed. In characteristic zero, unlike the Lie or Jordan case [1, 5], the sequence of codimensions of an associative algebra is exponentially bounded ([22]), and few such parameters related to the asymptotic behavior of such a sequence have been successfully applied.

For instance, the exponent (measuring the exponential growth of the sequence of codimensions) has been used in order to classify the minimal varieties of algebras of exponential growth. It turns out that for any given exponent $d \ge 2$, there are only a finite number of PI-algebras of exponent d, and they can be explicitly described as upper block triangular matrix algebras with entries in the Grassmann algebra [12, 13].

In this paper we address the problem of classifying the minimal varieties of polynomial growth. We recall that by [15, 16], given a PI-algebra A, its sequence of codimensions either grows exponentially or is polynomially bounded. The exponential growth of such a sequence was determined in [10, 11], and it turns out to be an integer called the PI-exponent of A. In the language of varieties of algebras, the exponent of a variety is the PI-exponent of a generating algebra.

Among varieties, prominent role is played by the minimal varieties. Recall that $\mathcal{V}$ is a minimal variety of exponential growth $d \ge 2$ if $\exp(\mathcal{V}) = d$ and $\exp(\mathcal{U}) < d$, for every proper subvariery $\mathcal{U}$. As we mentioned above, minimal varieties have been classified.

Received by the editors September 18, 2012; revised April 10, 2013.
Published electronically May 15, 2013.
The third author was partially supported by RFBR grant 09-01-00303 and SSC-1983.2008.1
AMS subject classification: **16R10**, 16P90.
Keywords: polynomial identity, codimension, polynomial growth, T-ideal.

Now let $\mathcal{V}$ be a variety of algebras and let $c_n(\mathcal{V})$, $n = 1, 2, \dots$, be its sequence of codimensions. It is well known that if $c_n(\mathcal{V})$ is polynomially bounded, then $c_n(\mathcal{V}) \approx qn^k$ for some integer k and rational number q [7].

We say that $\mathcal{V}$ is a minimal variety of polynomial growth n^k if $c_n(\mathcal{V}) \approx an^k$ asymptotically for some $a \neq 0$ and $c_n(\mathcal{U}) \approx bn^t$ with $t < k$ for any proper subvariety $\mathcal{U}$ of $\mathcal{V}$.

By a well-known theorem of Kemer, there are only two varieties of algebras of almost polynomial growth (*i.e.,* their growth is exponential but any proper subvariety grows polynomially): the variety generated by the Grassmann algebra G and the variety generated by the algebra of 2×2 upper triangular matrices UT_2.

In [19, 20] the second author classified all minimal subvarieties of $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$, and it turns out that there are only a finite number of them. She also gave, for each such variety, a finite dimensional generating algebra.

The relevance of such classification relies in the fact that these were the building blocks that allowed the author to give a complete classification of the subvarieties of $var(G)$ and $var(UT_2)$.

Inspired by these positive results we shall try to classify the minimal varieties of polynomial growth in general. We shall restrict ourselves to varieties generated by algebras with 1.

We shall classify explicitly all minimal varieties of polynomial growth n^k for $k \leq 5$ and give a recipe for classifying all minimal varieties of polynomial growth n^k, $k > 5$. It will tuns out that for $k \leq 4$, there are only a finite number of minimal varieties of polynomial growth n^k, but for $k \geq 5$, the number of minimal varieties is at least $|F|$, the cardinality of the base field.

2 Preliminaries

Throughout this paper F is a field of characteristic zero and A is an associative F-algebra with 1. Let $F\langle X\rangle$ denote the free associative algebra over F on a countable set $X = \{x_1, x_2, \dots\}$. For any $n \geq 0$, we denote by P_n the space of multilinear polynomials in $x_1, \dots, x_n$ and we set $P_0 = \mathrm{span}\{1\}$. For a PI-algebra A, we denote by $\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$ the T-ideal of $F\langle X\rangle$ of polynomial identities of A. Recall that

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}, \quad n = 0, 1, 2, \dots,$$

is called the *sequence of codimensions of A.*

A distinguished subspace of P_n is given by Γ_n, the space of proper polynomials in $x_1, \dots, x_n$. Recall that $f(x_1, \dots, x_n) \in \Gamma_n$ is a proper polynomial if it is a linear combination of products of (long) left normed Lie commutators $[x_{i_1}, \dots, x_{i_k}]$; we also set $\Gamma_0 = \mathrm{span}\{1\}$.

Recall that the sequence

$$c_n^p(A) = \dim_F \frac{\Gamma_n}{\Gamma_n \cap \mathrm{Id}(A)}, \quad n = 0, 1, 2, \dots,$$

https://doi.org/10.4153/CJM-2013-016-5 Published online by Cambridge University Press

is called the *sequence of proper codimensions of A*.

The relation between ordinary and proper codimensions of a unitary algebra A is well known and was described by Drensky in [3]: we have $c_n(A) = \sum_{i=0}^{n} \binom{n}{i} c_i^p(A)$.

In [7] it was also proved that if A is a unitary algebra, then the sequence of (ordinary) codimensions is polynomially bounded if and only if $c_m^p(A) = 0$ for some even integer $m \geq 2$. Moreover, in such a case, $c_n^p(A) = 0$ for all $n \geq m$. Therefore, if A is a unitary algebra whose codimensions are polynomially bounded, the codimensions of A can be written as a finite sum

$$c_n(A) = \sum_{i=0}^{k} \binom{n}{i} c_i^p(A). \tag{2.1}$$

Thus $c_n(A)$ is a polynomial in n with rational coefficients of degree k, whose leading term is $q = c_k^p(A)\binom{n}{k}$. Moreover q satisfies the inequality

$$\frac{1}{k!} \leq q \leq \sum_{j=2}^{k} \frac{(-1)^j}{j!} \to \frac{1}{e}, \quad k \to \infty.$$

In [9] the lower bound was improved for k odd ($q \geq \frac{k-1}{k!}$), and it was shown that these bounds are actually achieved.

It is well known that the symmetric group S_n acts on the vector space P_n by permuting the variables, and P_n is isomorphic to the left regular S_n-representation (see, for instance, [14, Section 2.4]). Clearly, $\Gamma_n \subset P_n$ is an S_n-submodule, and, since $\mathrm{Id}(A)$ is invariant under permutations of the variables, $\Gamma_n/(\Gamma_n \cap \mathrm{Id}(A))$ becomes an S_n-module. Its character, denoted $\chi_n^p(A)$ is called the *proper n-th cocharacter of A*. By complete reducibility $\chi_n^p(A)$ decomposes into irreducibles. Let

$$\chi_n^p(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition λ and m_λ is the corresponding multiplicity.

We recall the following terminology: if M and N are two S_n-modules and

$$\chi_n(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \quad \chi_n(N) = \sum_{\lambda \vdash n} m'_\lambda \chi_\lambda$$

are the corresponding S_n-characters, then we write $\chi_n(M) \leq \chi_n(N)$ if $m_\lambda \leq m'_\lambda$ for all $\lambda \vdash n$.

With this terminology in mind, we clearly have that for any PI-algebra A, $\chi_n^p(A) \leq \chi(\Gamma_n)$, where $\chi(\Gamma_n)$ is the S_n-character of Γ_n.

In what follows we shall actually make use of the representation theory of the general linear group, which is strictly related to that of S_n.

Let $U = \mathrm{span}_F\{x_1, \dots, x_m\}$ and $F\langle x_1, \dots, x_m\rangle$ the free associative algebra in m variables. The group $GL(U) \cong GL_m$ acts naturally on the left on U, and we extend this action diagonally to an action on $F\langle x_1, \dots, x_m\rangle$.

The space $F\langle x_1,\dots,x_m\rangle \cap \mathrm{Id}(A)$ is invariant under this action, hence

$$F_m(A) = \frac{F\langle x_1,\dots,x_m\rangle}{F\langle x_1,\dots,x_m\rangle \cap \mathrm{Id}(A)}$$

inherits a structure of left GL_m-module. If F_m^n denotes the subspace of $F\langle x_1,\dots,x_m\rangle$ of homogeneous polynomials of degree n, then

$$F_m^n(A) = \frac{F_m^n}{F_m^n \cap \mathrm{Id}(A)}$$

is a GL_m-submodule of $F_m(A)$ whose character is denoted by $\psi_n(A)$. Write

$$\psi_n(A) = \sum_{\lambda \vdash n} \overline{m}_\lambda \psi_\lambda,$$

where ψ_λ is the irreducible GL_m-character associated with the partition λ and $\overline{m}_\lambda$ is the corresponding multiplicity.

The S_n-module structure of $P_n/(P_n \cap \mathrm{Id}(A))$ and the GL_m-module structure of $F_m^n(A)$ are related by the following: if $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ is the decomposition of the n-th cocharacter of A, then $m_\lambda = \overline{m}_\lambda$, for all $\lambda \vdash n$ whose corresponding diagram has height at most m (see for instance [6]).

It is also well known that any irreducible submodule of $F_m^n(A)$ corresponding to λ is generated by a non-zero polynomial f_λ, called highest weight vector (*hwv*), of the form

$$f_\lambda = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(x_1,\dots,x_{h_i(\lambda)}) \sum_{\sigma\in S_n} \alpha_\sigma \sigma,$$

where $\alpha_\sigma \in F$, the right action of S_n on $F_m^n(A)$, is defined by place permutation, $h_i(\lambda)$ is the height of the i-th column of the diagram of λ, and

$$St_r(x_1,\dots,x_r) = \sum_{\tau\in S_r} (\mathrm{sgn}\,\tau) x_{\tau(1)} \cdots x_{\tau(r)}$$

is the standard polynomial of degree r. We have the following remark.

Remark If

$$\psi_n(A) = \sum_{\lambda \vdash n} \overline{m}_\lambda \psi_\lambda$$

is the GL_m-character of $F_m^n(A)$, then $\overline{m}_\lambda$ is equal to the maximal number of linearly independent highest weight vectors f_λ in $F_m^n(A)$.

3 Characterizing Minimal Varieties

Recall that if $\mathcal{V}$ is a variety of algebras, then $c_n(\mathcal{V}) = c_n(A)$, where $\mathcal{V} = \mathrm{var}(A)$ and the growth of $\mathcal{V}$ is the growth of the codimensions of A. Also recall that two functions $f(x)$ and $g(x)$ are asymptotically equal, and we write $f(x) \approx g(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

We start with the following definition.

Definition 3.1 A variety $\mathcal{V}$ is minimal of polynomial growth n^k if $c_n(\mathcal{V}) \approx qn^k$ for some $k \geq 1, q > 0$, and for any proper subvariety $\mathcal{U} \subsetneqq \mathcal{V}$ we have that $c_n(\mathcal{U}) \approx q'n^t$ with $t < k$, for some q'.

In the language of algebras, we have the following. Let A be an algebra of polynomial codimension growth with $c_n(A) \approx qn^k, k \geq 1, q > 0$. Then A generates a minimal variety if for any algebra B such that $\mathrm{Id}(A) \subsetneqq \mathrm{Id}(B)$, we have that $c_n(B) \approx q'n^t$, for some q', where $t < k$.

Remark Let $\mathcal{V}$ be a variety of polynomial growth n^k. Then $c_k^p(\mathcal{V}) \neq 0$ and $c_i^p(\mathcal{V}) = 0$, for all $i \geq k + 1$.

This says that in the k-th proper cocharacter of A

$$\chi_k^p(A) = \sum_{\lambda \vdash k} m_\lambda \chi_\lambda,$$

there exists $\mu \vdash k$ such that $m_\mu \neq 0$ and $m_\lambda = 0$, for all $\lambda \vdash k + i,\ i \geq 1$. In other words, $\Gamma_{k+i} \subseteq \mathrm{Id}(\mathcal{V})$, for all $i \geq 1$, and there exists a *hwv* f_μ corresponding to μ such that $f_\mu \notin \mathrm{Id}(\mathcal{V})$.

We shall also use the following terminology.

Definition 3.2 A polynomial g is a consequence of a polynomial f, if g lies in $\langle f \rangle_T$, the T-ideal generated by f. In this case we write $f \rightsquigarrow g$. Accordingly if Q is a T-ideal and $f, g \notin Q$, we say that g is a consequence of $f \pmod{Q}$, and we write $f \rightsquigarrow g \pmod{Q}$ if $g \in \langle Q, f \rangle_T$.

We can now prove the following theorem.

Theorem 3.3 *Let $\mathcal{V}$ be a variety of algebras of polynomial growth n^k and let $\chi_n^p(\mathcal{V}) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ be its proper cocharacter. Then $\mathcal{V}$ is minimal if and only if the following hold.*

(i) *There exists $\lambda \vdash k$ such that $m_\lambda = 1$ and $m_\mu = 0$ for all $\mu \vdash k, \mu \neq \lambda$.*
(ii) *Let $f_\lambda \notin \mathrm{Id}(\mathcal{V})$, $\lambda \vdash k$, be a hwv. Then for every $h < k$ and for every $\mu \vdash h$, if f_μ is a hwv such that $f_\mu \notin \mathrm{Id}(\mathcal{V})$, we must have $f_\mu \rightsquigarrow f_\lambda \pmod{\mathrm{Id}(\mathcal{V})}$.*

Proof We start by assuming that $\mathcal{V}$ is a minimal variety of polynomial growth n^k. By the above remark, $\Gamma_{k+i} \subseteq \mathrm{Id}(\mathcal{V})$ and there exists $\lambda \vdash k$ such that $m_\lambda \neq 0$.

If $m_\lambda > 1$ there exist two *hwv*'s, f'_λ and f''_λ, that are linearly independent mod $\mathrm{Id}(\mathcal{V})$. Since $f'_\lambda \notin \mathrm{Id}(\mathcal{V})$, the T-ideal $Q = \langle f'_\lambda, \mathrm{Id}(\mathcal{V}) \rangle_T$ properly contains $\mathrm{Id}(\mathcal{V})$. Moreover the fact that f'_λ, and f''_λ are linearly independent mod $\mathrm{Id}(\mathcal{V})$ implies that $f''_\lambda \notin Q$. This says that the variety corresponding to Q is a proper subvariety of $\mathcal{V}$ and has polynomial growth n^k, a contradiction.

It is also clear that if for some $\mu \vdash k$ with $\mu \neq \lambda$ we have that $m_\mu \neq 0$, then there exists a *hwv* $f_\mu \notin \mathrm{Id}(\mathcal{V})$ and $\langle f_\mu, \mathrm{Id}(\mathcal{V}) \rangle_T$ corresponds to a proper subvariety of polynomial growth n^k. Thus $m_\lambda = 1$ and $m_\mu = 0$, for all $\mu \vdash k$, $\mu \neq \lambda$, proving (i).

Now let $h < k$ and $\mu \vdash h$ with f_μ *hwv* such that $f_\mu \notin \mathrm{Id}(\mathcal{V})$, *i.e.*, $m_\mu \neq 0$. Suppose that $f_\mu \not\rightsquigarrow f_\lambda \pmod{\mathrm{Id}(\mathcal{V})}$. This says that $f_\lambda \notin Q = \langle \mathrm{Id}(\mathcal{V}), f_\mu \rangle_T$. But then the variety corresponding to Q is a proper subvariety of $\mathcal{V}$ and has polynomial growth n^k. This contradicts the minimality of $\mathcal{V}$.

Conversely, let $\mathcal{V}$ be a variety of algebras satisfying (i) and (ii) and let $\mathcal{W}$ be a proper subvariety of $\mathcal{V}$. For simplicity we denote by $m_\nu(\mathcal{W})$ and $m_\nu(\mathcal{V})$ the multiplicities of χ_ν, $\nu \vdash n$, in $\chi_n^p(\mathcal{W})$ and $\chi_n^p(\mathcal{V})$, respectively.

Since $\mathcal{W}$ is a proper subvariety of $\mathcal{V}$, $\chi_n^p(\mathcal{W}) < \chi_n^p(\mathcal{V})$, *i.e.*, there exists $\mu \vdash h \le k$ such that $m_\mu(\mathcal{W}) < m_\mu(\mathcal{V})$. Now let $\lambda \vdash k$ be such that $m_\lambda(\mathcal{V}) = 1$. If $m_\lambda(\mathcal{W}) = 0$, then $\mathcal{W}$ would have polynomial growth n^t, with $t < k$. Hence we may assume that $m_\lambda(\mathcal{W}) = 1$.

It follows that if $\mu \vdash h$ is such that $m_\mu(\mathcal{W}) < m_\mu(\mathcal{V})$, then $h < k$. This says that there exists a *hwv* f_μ such that $f_\mu \notin \mathrm{Id}(\mathcal{V})$ but $f_\mu \in \mathrm{Id}(\mathcal{W})$. By hypothesis $f_\mu \rightsquigarrow f_\lambda \pmod{\mathrm{Id}(\mathcal{V})}$, and so $f_\lambda \in \langle f_\mu, \mathrm{Id}(\mathcal{V})\rangle_T \subseteq \langle f_\mu, \mathrm{Id}(\mathcal{W})\rangle_T = \mathrm{Id}(\mathcal{W})$. This contradiction completes the proof. ■

We remind the reader that by a result of Kemer ([18]) any variety of polynomial growth can be generated by a finite dimensional algebra.

Before starting our classification of varieties of polynomial growth, we record two results that were proved in [9, 19].

Let UT_k be the algebra of $k \times k$ upper triangular matrices over F and let $E_1 = \sum_{i=1}^{k-1} e_{i,i+1} \in UT_k$ denote the diagonal just above the main diagonal of UT_k, where the e_{ij}'s are the usual matrix units.

Let N_k be the subalgebra of UT_k defined in [9] as follows

$$N_k = \mathrm{span}\{E, E_1, E_1^2, \dots, E_1^{k-2}; e_{12}, e_{13}, \dots, e_{1k}\},$$

where E denotes the identity $k \times k$ matrix. Notice that if $k = 2$, then $\mathrm{Id}(N_k) = \mathrm{Id}(F)$.

We next state the following result characterizing the polynomial identities and the cocharacter of N_k.

Theorem 3.4 *For any $k \ge 3$, N_k generates a minimal variety of polynomial growth n^{k-1}. Moreover,*

(i) $\mathrm{Id}(N_k) = \langle [x_1, \dots, x_k],\ [x_1, x_2][x_3, x_4]\rangle_T$;
(ii) $\chi_{k-1}^p(N_k) = \chi_{(k-2,1)}$;
(iii) $c_n(N_k) = 1 + \sum_{j=2}^{k-1} (j-1)\binom{n}{j} \approx \frac{k-2}{(k-1)!}\, n^{k-1}, \quad n \to \infty.$

Proof The minimality of $\mathrm{var}(N_k)$ and (ii) were proved in [20, Theorem 11]; (i) and (iii) were proved in [9, Theorem 3.4]. ■

For $t \ge 1$, let G_t denote the Grassmann algebra with 1 on a t-dimensional vector space over F, *i.e.*,

$$G_t = \langle 1, e_1, \dots, e_t \mid e_i e_j = -e_j e_i \rangle.$$

The following result characterizes the polynomial identities and the codimensions of G_t.

Theorem 3.5 *For any $k \ge 1$, G_{2k} generates a minimal variety of polynomial growth n^{2k}. Moreover,*

(i) $\mathrm{Id}(G_{2k}) = \langle [x_1, x_2, x_3],\ [x_1, x_2] \cdots [x_{2k+1}, x_{2k+2}] \rangle_T$;
(ii) $\chi_{2k}(G_{2k}) = \chi_{(1^{2k})}$;
(iii) $c_n(G_{2k}) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}, \quad n \to \infty.$

Proof The minimality of $\mathrm{var}(G_{2k})$ was proved in [20, Theorem 12], and (i) and (iii) were proved in [9, Theorem 3.5]. Since $\mathrm{var}(G_{2k})$ is minimal, by Theorem 3.3 $\chi^p_{2k}(G_{2k}) = \chi_\lambda$, for some $\lambda \vdash 2k$. Since $f_{(1^{2k})} = St_{2k} \notin \mathrm{Id}(G_{2k})$, we get that $\chi_{2k}(G_{2k}) = \chi_{(1^{2k})}$. ■

4 Minimal Varieties of Polynomial Growth $\leq n^4$

In this section we shall classify, up to PI-equivalence, the algebras with 1 generating minimal varieties of polynomial growth $\leq n^4$. In the sequel we shall use the following notation.

Definition 4.1 Two algebras A and B are PI-*equivalent*, and we write $A \sim_{PI} B$ if $\mathrm{Id}(A) = \mathrm{Id}(B)$.

We start by classifying the minimal varieties of polynomial growth n^k, $k \leq 3$.

Theorem 4.2 *Let A be an algebra with* 1 *such that $c_n(A) \leq qn^3$. Then A generates a minimal variety if and only if either $A \sim_{PI} N_3$ or $A \sim_{PI} N_4$.*

Proof Since A is an algebra with 1, by (2.1),

$$c_n(A) = \sum_{i=0}^{3} \binom{n}{i} c^p_i(A).$$

Hence $[x_1, x_2][x_3, x_4] \equiv 0$ is an identity of A and $A \in \mathrm{var}(UT_2)$. The result now follows from [19, Corollary 5.4]. ■

From the above theorem it follows that there are no minimal varieties of linear growth generated by an algebra with 1. But actually a stronger result holds, since it can be proved that there are no algebras with 1 of linear codimension growth (see [8,9]).

Next we are going to determine all minimal varieties $\mathcal{V}$ of polynomial growth n^4. In all cases but one, we shall also determine a finite dimensional generating algebra.

We recall that $\chi(\Gamma_3) = \chi_{(2,1)}$ and that $f_{(2,1)} = [x_2, x_1, x_1]$ is a corresponding *hwv*. Also,

$$\chi(\Gamma_4) = \chi_{(3,1)} + \chi_{(2^2)} + \chi_{(2,1^2)} + \chi_{(1^4)},$$

and it can be easily checked that

$$f_{(3,1)} = [x_2, x_1, x_1, x_1], \qquad f_{(2^2)} = [x_1, x_2]^2,$$
$$f_{(2,1^2)} = \big[x_1, x_3, [x_1, x_2]\big], \qquad f_{(1^4)} = St_4(x_1, x_2, x_3, x_4)$$

are *hwv*'s.

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In the following theorem we give the decomposition of the proper cocharacter of the varieties whose T-ideal is generated by a *hwv* of degree 4. For $\lambda = (3,1), (2^2)$, and $(2,1^2)$ the result is due to Drensky [2, 4]. When $\lambda = (1^4)$, the corresponding result is due to Kemer [17].

Theorem 4.3 *Let* $\mathrm{Id}(\mathcal{V}) = \langle f_\lambda \rangle_T$, *where* $\lambda \vdash 4$. *If*

(i) $\lambda = (3,1)$, *then* $\chi_n^p(\mathcal{V}) = \begin{cases} \chi_{(2,1^3)}, & n = 5, \\ \chi_{(1^{2k})}, & n = 2k > 5, \\ 0, & n = 2k+1 > 5; \end{cases}$

(ii) $\lambda = (2^2)$, *then* $\chi_n^p(\mathcal{V}) = \begin{cases} \chi_{(4,1)} + \chi_{(2,1^3)}, & n = 5, \\ \chi_{(2k-1,1)} + \chi_{(1^{2k})}, & n = 2k > 5, \\ \chi_{(2k,1)}, & n = 2k+1 > 5; \end{cases}$

(iii) $\lambda = (2,1^2)$, *then* $\chi_n^p(\mathcal{V}) = \begin{cases} \chi_{(2k-1,1)} + \chi_{(1^{2k})}, & n = 2k > 5, \\ \chi_{(2k,1)}, & n = 2k+1 \geq 5; \end{cases}$

(iv) $\lambda = (1^4)$, *then* $\chi_n^p(\mathcal{V}) = \chi_n^p(M_2(F)) + \begin{cases} \chi_{(3,2)}, & n = 5, \\ \chi_{(3^2)}, & n = 6, \\ 0, & n > 6. \end{cases}$

Let $\mathcal{V}$ be a minimal variety of polynomial growth n^4, and notice that if $[x_1, x_2] \in \mathrm{Id}(\mathcal{V})$, *i.e.*, $\Gamma_2 \subseteq \mathrm{Id}(\mathcal{V})$, then $c_n(\mathcal{V}) = 1$. Hence we must have that $[x_1, x_2] \notin \mathrm{Id}(\mathcal{V})$.

By Theorem 3.3, since $\mathcal{V}$ is minimal of growth n^4, we have $\chi_4^p(\mathcal{V}) = \chi_\lambda$, where $\lambda = (3,1)$ or (2^2) or $(2,1^2)$ or (1^4), and we examine the four cases separately.

Suppose first that $\chi_4^p(\mathcal{V}) = \chi_{(2^2)}$. Then $f_{(2^2)} = [x_1, x_2]^2 \notin \mathrm{Id}(\mathcal{V})$, and since $[x_1, x_2, x_3] \rightsquigarrow f_{(2^2)}$, we get that 4 is the least degree of a polynomial identity of $\mathcal{V}$. Hence, since by Theorem 4.3, Γ_5 and $[x_1, x_2][x_3, x_4][x_5, x_6]$ belong to the T-ideal generated by $f_{(3,1)}$, $f_{(2,1^2)}$, and $f_{(1^4)}$, we get

$$\mathrm{Id}(\mathcal{V}) = \Big\langle [x_2, x_1, x_1, x_1], \big[x_1, x_3, [x_1, x_2]\big], St_4(x_1, x_2, x_3, x_4) \Big\rangle_T.$$

Notice that $c_n(\mathcal{V}) = \sum_{i=0}^4 \binom{n}{i} c_i^p(A) = 1 + \binom{n}{2} + 2\binom{n}{3} + 2\binom{n}{4}$.

A finite dimensional algebra generating $\mathcal{V}$ is exhibited in the following theorem.

Theorem 4.4 *Let*

$$M = \left\{ \begin{pmatrix} a & b & d & e & f \\ 0 & a & c & g & h \\ 0 & 0 & a & c & i \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c, d, e, f, g, h, i \in F \right\}.$$

Then M generates a minimal variety of growth n^4 and $\chi_4^p(\mathcal{V}) = \chi_{(2^2)}$.

Proof It is easily checked that $[x_2, x_1, x_1, x_1], St_4, [x_1, x_3, [x_1, x_2]]$ are identities of M, and so, by Theorem 4.3, Γ_5 and $[x_1, x_2][x_3, x_4][x_5, x_6]$ also belong to $\mathrm{Id}(M)$. Moreover, since $f_{(2^2)} = [x_1, x_2][x_1, x_2] \notin \mathrm{Id}(M)$, we get that $\chi_4^p(M) = \chi_{(2^2)}$ and $c_n(M) \approx qn^4$. Hence, $\mathrm{Id}(M) \supseteq \mathrm{Id}(\mathcal{V})$, and so, by minimality of $\mathcal{V}$, M generates $\mathcal{V}$. ■

Now suppose that $\chi_4^p(\mathcal{V}) = \chi_{(2,1^2)}$. Then, $f_{(2,1^2)} = [x_1, x_3, [x_1, x_2]] \notin \mathcal{V}$, and since $[x_1, x_2, x_3] \rightsquigarrow [x_1, x_3, [x_1, x_2]]$, then also $[x_1, x_2, x_3] \notin \mathcal{V}$. Hence,

$$\mathrm{Id}(\mathcal{V}) = \left\langle [x_2, x_1, x_1, x_1], [x_1, x_2]^2, St_4 \right\rangle_T.$$

Notice that in this case, $c_n(\mathcal{V}) = \sum_{i=0}^4 \binom{n}{i} c_i^p(A) = 1 + \binom{n}{2} + 2\binom{n}{3} + 3\binom{n}{4}$.

Now let $\chi_4^p(\mathcal{V}) = \chi_{(3,1)}$. Then $f_{(3,1)} = [x_1, x_2, x_1, x_1] \notin \mathcal{V}$ and from Theorems 3.3 and 3.4 we deduce that $\mathcal{V} = \mathrm{var}(N_5)$. Hence

$$\mathrm{Id}(\mathcal{V}) = \mathrm{Id}(N_5) = \left\langle [x_1, x_2][x_3, x_4], [x_1, x_2, x_3, x_4, x_5] \right\rangle_T,$$

$$c_n(A) = \sum_{i=0}^{4} \binom{n}{i} c_i^p(A) = 1 + \binom{n}{2} + 2\binom{n}{3} + 3\binom{n}{4}.$$

Finally, in case $\chi_4^p(\mathcal{V}) = \chi_{(1^4)}$, $St_4 \notin \mathcal{V}$ and from Theorems 3.3 and 3.5 we deduce that $\mathcal{V} = \mathrm{var}(G_4)$. Hence,

$$\mathrm{Id}(\mathcal{V}) = \mathrm{Id}(G_4) = \left\langle [x_1, x_2, x_3], [x_1, x_2][x_3, x_4][x_5, x_6] \right\rangle_T.$$

We summarize the results obtained in the following theorem.

Theorem 4.5 *Let A be a unitary algebra such that $c_n(A) \approx qn^4$, for some $q > 0$. Then A generates a minimal variety if and only if $\mathrm{Id}(A)$ coincides with one of the following T-ideals:*

(i) $\langle [x_1, x_2][x_3, x_4], [x_1, x_2, x_3, x_4, x_5] \rangle_T$,
(ii) $\langle [x_1, x_2, x_3], [x_1, x_2][x_3, x_4][x_5, x_6] \rangle_T$,
(iii) $\langle [x_2, x_1, x_1, x_1], [x_1, x_3, [x_1, x_2]], St_4 \rangle_T$,
(iv) $\langle [x_2, x_1, x_1, x_1], [x_1, x_2]^2, St_4 \rangle_T$.

In the first three cases we have that $A \sim_{PI} N_5$ or $A \sim_{PI} G_4$, or $A \sim_{PI} M$, respectively.

5 Minimal Varieties of Polynomial Growth n^5

In this section we shall classify the minimal varieties of polynomial growth n^5. We start by recalling the decomposition of the S_5-character of Γ_5:

$$\chi(\Gamma_5) = \chi_{(4,1)} + 2\chi_{(3,2)} + 2\chi_{(3,1^2)} + 2\chi_{(2^2,1)} + 2\chi_{(2,1^3)}.$$

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It can be easily checked that

$$f_{(4,1)} = [x_2, x_1, x_1, x_1, x_1],$$

$$f'_{(3,1^2)} = [x, z, x][x, y] - [x, y, x][x, z], \tag{5.1}$$

$$f''_{(3,1^2)} = [x, y][x, z, x] - [x, z][x, y, x],$$

$$f'_{(3,2)} = [x, y, x][x, y], \quad f''_{(3,2)} = [x, y][x, y, x], \tag{5.2}$$

$$f'_{(2^2,1)} = [x, y, z][x, y] + [y, x, y][x, z] + [x, y, x][y, z], \tag{5.3}$$

$$f''_{(2^2,1)} = [x, y][x, y, z] + [x, z][y, x, y] + [y, z][x, y, x], \tag{5.4}$$

$$f'_{(2,1^3)} = [x, y, x][z, t] - [x, z, x][y, t] + [y, z, x][x, t] + [x, t, x][y, z] - [y, t, x][x, z] + [z, t, x][x, y], \tag{5.5}$$

$$f''_{(2,1^3)} = [z, t][x, y, x] - [y, t][x, z, x] + [x, t][y, z, x] + [y, z][x, t, x] - [x, z][y, t, x] + [x, y][z, t, x] \tag{5.6}$$

are linearly independent *hwv*'s.

Given a minimal variety $\mathcal{V}$ of polynomial growth n^5, we shall study five distinct cases according as $\chi_5^p(\mathcal{V}) = \chi_\lambda$, where λ is one of the five partitions appearing in the decomposition of $\chi(\Gamma_5)$. We shall classify the corresponding varieties and we shall see that in four out of five cases there are infinitely many distinct varieties.

Let $\mathcal{V}$ be a minimal variety and assume that $\chi_5^p(\mathcal{V}) = \chi_\lambda$, where χ_λ is one of the four characters appearing with multiplicity 2 in the decomposition of Γ_5. If f'_λ and f''_λ are linearly independent, since $\mathcal{V}$ is minimal, one of f'_λ and f''_λ, say f''_λ, is not an identity of $\mathcal{V}$. Hence there exist $\alpha, \beta \in F$, not both zero and unique up to a scalar such that $\alpha f'_\lambda + \beta f''_\lambda \equiv 0$ is an identity of $\mathcal{V}$.

Then define the following T-ideals:

$$Q_5 = \langle \Gamma_6, f'_\mu, f''_\mu, \alpha f'_\lambda + \beta f''_\lambda \mid \mu \vdash 5, \mu \neq \lambda \rangle_T,$$

$$Q_4 = \left\langle Q_5, f_\nu \mid f_\nu, \nu \vdash 4, \ hwv \text{ such that } f_\nu \not\rightsquigarrow f''_\lambda \pmod{Q_5} \right\rangle_T.$$

Also, set $Q_3 = \langle Q_4, f_{(2,1)} \rangle_T$ if $f_{(2,1)} \not\rightsquigarrow f''_\lambda \pmod{Q_4}$, and $Q_3 = Q_4$ otherwise. Finally, set $Q_2 = \langle Q_3, f_{(1^2)} \rangle_T$ if $f_{(1^2)} \not\rightsquigarrow f''_\lambda \pmod{Q_3}$, and $Q_2 = Q_3$ otherwise. Notice that $Q_2 = Q_3$, since all proper polynomials of positive degree follow from $f_{(1^2)}$.

We claim that $\mathrm{Id}(\mathcal{V}) = Q_2$.

In fact we shall prove that the variety $\mathcal{Q}$ determined by Q_2 is minimal of growth n^5.

First, $Q_5 \subseteq Q_2$ and $f''_\lambda \notin Q_2$ says that $\chi_6^p(\mathcal{Q}) = 0$ and $\chi_5^p(\mathcal{Q}) = \chi_\lambda$. Moreover, if $f_\mu, \mu \vdash h \leq 4$ is a *hwv* such that $f_\mu \notin Q_2$, then since $Q_h \subseteq Q_2$, $f_\mu \notin Q_h$. Hence by the definition of Q_h, $f_\mu \rightsquigarrow f''_\lambda \pmod{Q_{h+1}}$. This means that $f''_\lambda \in \langle f_\mu, Q_{h+1} \rangle_T \subseteq \langle f_\mu, Q_2 \rangle_T$, *i.e.*, $f_\mu \rightsquigarrow f''_\lambda \pmod{Q_2}$. We have shown that the requirements of Theorem 3.3 are fulfilled, and the claim is proved.

In light of the above, if $\mathcal{V}$ is a minimal variety of growth n^5 such that $\chi_5^p(\mathcal{V}) = \chi_\lambda$, with $\lambda \in \{(3, 2), (3, 1^2), (2^2, 1), (2, 1^3)\}$, then $\mathcal{V}$ is uniquely determined, up to a scalar, by a linear combination $\alpha f'_\lambda + \beta f''_\lambda \equiv 0$ of two linearly independent *hwv*'s that is an identity of $\mathcal{V}$.

Then it is worth introducing the notation $\mathrm{Id}(\mathcal{V}) = Q_{\alpha f'_\lambda + \beta f''_\lambda}$.

Throughout, we let $\mathcal{V}$ be a minimal variety of polynomial growth n^5, and we examine the following five distinct cases.

Case 1. $\chi_5^p(\mathcal{V}) = \chi_{(4,1)}$.

Then $f_{(4,1)} = [x_2, x_1, x_1, x_1, x_1] \notin \mathcal{V}$ and by Theorems 3.3 and 3.4 we deduce that $\mathcal{V} = \mathrm{var}(N_6)$. Hence,

$$\mathrm{Id}(\mathcal{V}) = \mathrm{Id}(N_6) = \big\langle [x_1, x_2][x_3, x_4], [x_1, x_2, x_3, x_4, x_5, x_6] \big\rangle_T.$$

Case 2. $\chi_5^p(\mathcal{V}) = \chi_{(3,1^2)}$.

We consider the two linearly independent *hwv*'s corresponding to the partition $(3, 1^2)$ given in (5.1):

$$f'_{(3,1^2)} = [x, z, x][x, y] - [x, y, x][x, z], \ f''_{(3,1^2)} = [x, y][x, z, x] - [x, z][x, y, x].$$

Since $\chi_5^p(\mathcal{V}) = \chi_{(3,1^2)}$, we deduce that one of them is not an identity of $\mathcal{V}$.

Clearly

$$f_{(1^2)}, f_{(2,1)} \rightsquigarrow f'_{(3,1^2)}, f''_{(3,1^2)}, \tag{5.7}$$

(here we mean that each polynomial on the left-hand side implies each polynomial on the right-hand side).

Also, by Theorem 4.3,

$$f_{(2^2)}, f_{(3,1)}, f_{(2,1^2)} \rightsquigarrow f'_{(3,1^2)}, f''_{(3,1^2)}. \tag{5.8}$$

Finally, we consider $f_{(1^4)} = St_4$. By [17] we deduce that

$$f_{(1^4)} \rightsquigarrow f'_{(3,1^2)} - f''_{(3,1^2)} \text{ and } f_{(1^4)} \not\rightsquigarrow f''_{(3,1^2)}, f'_{(3,1^2)} + \alpha f''_{(3,1^2)} \quad \text{for any } \alpha \neq -1. \tag{5.9}$$

If we now collect the results obtained in (5.7)–(5.9) and combine them with Theorem 3.3 and the above remark, we get the following.

Theorem 5.1 *Suppose that either $\chi_5^p(\mathcal{V}) = \chi_{(4,1)}$ or $\chi_5^p(\mathcal{V}) = \chi_{(3,1^2)}$. Then $\mathcal{V}$ is a minimal variety of growth n^5 if and only if* $\mathrm{Id}(\mathcal{V})$ *coincides with one of the following T-ideals:*

(i) $\mathrm{Id}(N_6) = \langle [x_1, x_2][x_3, x_4], [x_1, x_2, x_3, x_4, x_5, x_6] \rangle_T$,

(ii) $Q_{f'_{(3,1^2)} + f''_{(3,1^2)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(3,1^2)} + f''_{(3,1^2)} \mid \mu \vdash 5, \mu \neq (3,1^2), (4,1) \rangle_T$,

(iii) $Q_{f'_{(3,1^2)} - f''_{(3,1^2)}} =$
$\langle f_{(1^4)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(3,1^2)} - f''_{(3,1^2)} \mid \mu \vdash 5, \mu \neq (3,1^2), (4,1) \rangle_T$,

(iv) $Q_{f''_{(3,1^2)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f''_{(3,1^2)} \mid \mu \vdash 5, \mu \neq (3,1^2), (4,1) \rangle_T$,

(v) $Q_{f'_{(3,1^2)} + \alpha f''_{(3,1^2)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(3,1^2)} + \alpha f''_{(3,1^2)} \mid \mu \vdash 5, \mu \neq (3,1^2), (4,1) \rangle_T$,
where $\alpha \in F$ is any scalar such that $\alpha \neq \pm 1$.

Case 3. $\chi_5^p(\mathcal{V}) = \chi_{(3,2)}$.

Take the two linearly independent *hwv*'s corresponding to the partition $(3,2)$ given in (5.2):

$$f'_{(3,2)} = [x,y,x][x,y], \quad f''_{(3,2)} = [x,y][x,y,x].$$

Since $\chi_5^p(\mathcal{V}) = \chi_{(3,2)}$, one of them is not an identity.

Clearly $f_{(1^2)}, f_{(2,1)} \rightsquigarrow f'_{(3,2)}, f''_{(3,2)}$.

Also, by Theorem 4.3 and [17] we deduce that

$$f_{(2^2)}, f_{(3,1)}, f_{(2,1^2)} \rightsquigarrow f'_{(3,2)}, f''_{(3,2)}$$

and

$$f_{(1^4)} \not\rightsquigarrow f''_{(3,2)}, \quad f'_{(3,2)} + \alpha f''_{(3,2)} \quad \text{for any } \alpha \in F.$$

By collecting the results obtained so far and combining them with Theorem 3.3 as in the previous case, we get the following theorem.

Theorem 5.2 *Let $\chi_5^p(\mathcal{V}) = \chi_{(3,2)}$. Then $\mathcal{V}$ is a minimal variety of growth n^5 if and only if* $\mathrm{Id}(\mathcal{V})$ *coincides with one of the following T-ideals:*

(i) $Q_{f'_{(3,2)}+f''_{(3,2)}} = \langle f_{(1^4)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(3,2)} + f''_{(3,2)} \mid \mu \vdash 5, \mu \neq (3,2), (4,1)\rangle_T$,

(ii) $Q_{f'_{(3,2)}-f''_{(3,2)}} = \langle f_{(1^4)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(3,2)} - f''_{(3,2)} \mid \mu \vdash 5, \mu \neq (3,2), (4,1)\rangle_T$,

(iii) $Q_{f''_{(3,2)}} = \langle f_{(1^4)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f''_{(3,2)} \mid \mu \vdash 5, \mu \neq (3,2), (4,1)\rangle_T$,

(iv) $Q_{f'_{(3,2)}+\alpha f''_{(3,2)}} = \langle f_{(1^4)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(3,2)} + \alpha f''_{(3,2)} \mid \mu \vdash 5, \mu \neq (3,2), (4,1)\rangle_T$, *where $\alpha \in F$ is any scalar such that $\alpha \neq \pm 1$.*

Case 4. $\chi_5^p(\mathcal{V}) = \chi_{(2^2,1)}$.

We consider the two linearly independent *hwv*'s corresponding to the partition $(2^2,1)$ given in (5.3) and (5.4):

$$f'_{(2^2,1)} = [x,y,z][x,y] + [y,x,y][x,z] + [x,y,x][y,z],$$

$$f''_{(2^2,1)} = [x,y][x,y,z] + [x,z][y,x,y] + [y,z][x,y,x].$$

Since $\chi_5^p(\mathcal{V}) = \chi_{(2^2,1)}$, one of them is not an identity of $\mathcal{V}$.

Clearly $f_{(1^2)} \rightsquigarrow f'_{(2^2,1)}, f''_{(2^2,1)}$.

Also, by Theorem 4.3,

$$f_{(2,1)}, f_{(2^2)}, f_{(3,1)}, f_{(2,1^2)} \rightsquigarrow f'_{(2^2,1)}, f''_{(2^2,1)}$$

and, by [17]

$$f_{(1^4)} \rightsquigarrow f'_{(2^2,1)} + f''_{(2^2,1)}$$

$$f_{(1^4)} \not\rightsquigarrow f''_{(2^2,1)}, \; f'_{(2^2,1)} + \alpha f''_{(2,2,1)}, \text{ for any } \alpha \neq 1.$$

By collecting the results obtained so far and combining them with Theorem 3.3, we get the following theorem.

https://doi.org/10.4153/CJM-2013-016-5 Published online by Cambridge University Press

Theorem 5.3 *Let $\chi_5^p(\mathcal{V}) = \chi_{(2^2,1)}$. Then $\mathcal{V}$ is a minimal variety of growth n^5 if and only if* $\mathrm{Id}(\mathcal{V})$ *coincides with one of the following T-ideals*

(i) $Q_{f'_{(2^2,1)}+f''_{(2^2,1)}} =$
$$\langle f_{(1^4)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(2^2,1)} + f''_{(2^2,1)} \mid \mu \vdash 5, \mu \neq (2^2,1),(4,1)\rangle_T,$$

(ii) $Q_{f'_{(2^2,1)}-f''_{(2^2,1)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(2^2,1)} - f''_{(2^2,1)} \mid \mu \vdash 5, \mu \neq (2^2,1),(4,1)\rangle_T$,

(iii) $Q_{f''_{(2^2,1)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f''_{(2^2,1)} \mid \mu \vdash 5, \mu \neq (2^2,1),(4,1)\rangle_T$,

(iv) $Q_{f'_{(2^2,1)}+\alpha f''_{(2^2,1)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(2^2,1)} + \alpha f''_{(2^2,1)} \mid \mu \vdash 5, \mu \neq (2^2,1),(4,1)\rangle_T$, *where $\alpha \in F$ is any such that $\alpha \neq \pm 1$.*

Case 5. $\chi_5^p(\mathcal{V}) = \chi_{(2,1^3)}$.

We take now the two linearly independent *hwv*'s corresponding to the partition $(2,1^3)$ given in (5.5) and (5.6):

$$\begin{aligned} f'_{(2,1^3)} &= [x,y,x][z,t] - [x,z,x][y,t] + [y,z,x][x,t] + [x,t,x][y,z] \\ &\quad - [y,t,x][x,z] + [z,t,x][x,y], \\ f''_{(2,1^3)} &= [z,t][x,y,x] - [y,t][x,z,x] + [x,t][y,z,x] + [y,z][x,t,x] \\ &\quad - [x,z][y,t,x] + [x,y][z,t,x]. \end{aligned}$$

Clearly $f_{(1^2)} \rightsquigarrow f'_{(2,1^3)}, f''_{(2,1^3)}$.

By [2] we deduce that

$$f_{(2^2)} \rightsquigarrow f'_{(2,1^3)} + f''_{(2,1^3)} \text{ and } f_{(2^2)} \not\rightsquigarrow f''_{(2,1^3)},\ f'_{(2,1^3)} + \alpha f''_{(2,1^3)}, \text{ for any } \alpha \neq 1;$$
$$f_{(3,1)} \rightsquigarrow f'_{(2,1^3)} - f''_{(2,1^3)} \text{ and } f_{(3,1)} \not\rightsquigarrow f''_{(2,1^3)},\ f'_{(2,1^3)} + \alpha f''_{(2,1^3)}, \text{ for any } \alpha \neq -1;$$
$$f_{(2,1^2)}, f_{(1^4)} \rightsquigarrow f'_{(2,1^3)}, f''_{(2,1^3)}.$$

By collecting the results obtained so far and combining them with Theorem 3.3, we get the following theorem.

Theorem 5.4 *Let $\chi_5^p(\mathcal{V}) = \chi_{(2,1^3)}$. Then $\mathcal{V}$ is a minimal variety of growth n^5 if and only if* $\mathrm{Id}(\mathcal{V})$ *coincides with one of the following T-ideals*

(i) $Q_{f'_{(2,1^3)}+f''_{(2,1^3)}} =$
$$\langle f_{(2^2)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(2,1^3)} + f''_{(2,1^3)} \mid \mu \vdash 5, \mu \neq (2,1^3),(4,1)\rangle_T,$$

(ii) $Q_{f'_{(2,1^3)}-f''_{(2,1^3)}} =$
$$\langle g, f_{(3,1)}, \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f'_{(2,1^3)} - f''_{(2,1^3)} \mid \mu \vdash 5, \mu \neq (2,1^3),(4,1)\rangle_T,$$

(iii) $Q_{f''_{(2,1^3)}} = \langle \Gamma_6, f_{(4,1)}, f'_\mu, f''_\mu, f''_{(2,1^3)} \mid \mu \vdash 5, \mu \neq (2,1^3),(4,1)\rangle_T$,

(iv) $Q_{f'_{(2,1^3)}+\alpha f''_{(2,1^3)}} = \langle \Gamma_6, f_{(4,1)} f'_\mu, f''_\mu, f'_{(2,1^3)} + \alpha f''_{(2,1^3)} \mid \mu \vdash 5, \mu \neq (2,1^3),(4,1)\rangle_T$,

where $\alpha \in F$ is any such that $\alpha \neq \pm 1$, where

$$g = f_{(2,1)} \in Q_{f'_{(2,1^3)}-f''_{(2,1^3)}} \text{ if } f_{(2,1)} \not\rightsquigarrow f'_{(2,1^3)}, f''_{(2,1^3)} \pmod{\widetilde{Q}_{f'_{(2,1^3)}-f''_{(2,1^3)}}}$$

and $g = 0$ *otherwise.*

Here $\widetilde{Q}_{f'_{(2,1^3)} - f''_{(2,1^3)}}$ *is the T-ideal generated by the generators of* $Q_{f'_{(2,1^3)} - f''_{(2,1^3)}}$ *except* g.

As the referee has pointed out, the list of generators of the T-ideals $Q_{\alpha f'_\lambda + \beta f''_\lambda}$, can be shortened by applying the results of Popov given in [21].

6 Minimal Varieties of Higher Growth

The procedure of the previous section can be generalized to higher growth.

Let $\mathcal{V}$ be a minimal variety of polynomial growth n^k, $k \geq 2$. Let B_k be the space of homogeneous proper polynomials of degree k. Decompose $B_k = \oplus_{\mu \vdash k} B^\mu$, where B^μ is the sum of the irreducible submodules of B_k with character χ_μ and μ runs over all partitions of k corresponding to proper polynomials.

According to Theorem 3.3, $\chi_k^p(\mathcal{V}) = \chi_\lambda$, for some character χ_λ and let $\chi(\Gamma_k) = \cdots + m_\lambda \chi_\lambda + \cdots$, *i.e.*, m_λ is the multiplicity of χ_λ in $\chi(\Gamma_k)$; in other words, B^λ is the direct sum of m_λ irreducible submodules, each generated by a *hwv* that is a proper polynomial. Fix these m_λ linearly independent h.v.w.'s say, $f_1, \dots, f_{m_\lambda}$, and let H^λ denote the vector space generated by them.

Since $\dim H^\lambda \cap \operatorname{Id}(\mathcal{V}) = m_\lambda - 1$, there is an $m_\lambda \times (m_\lambda - 1)$ matrix (α_{ij}), with entries in F of rank $m_\lambda - 1$ such that the polynomials

$$f_1' = \sum_{i=1}^{m_\lambda} \alpha_{i1} f_i, \dots, f'_{m_\lambda - 1} = \sum_{i=1}^{m_\lambda} \alpha_{i, m_\lambda - 1} f_i$$

span $H^\lambda \cap \operatorname{Id}(\mathcal{V})$.

Let $f'' \notin \operatorname{span}\{f_1', \dots, f'_{m_\lambda - 1}\}$ be another proper polynomial that is a h.v.w. corresponding to λ. Clearly $f'' \notin \operatorname{Id}(\mathcal{V})$ as $\dim H^\lambda \cap \operatorname{Id}(\mathcal{V}) = m_\lambda - 1$.

Define the following sequence of T-ideals

$$Q_k = \langle \Gamma_{k+1}, H^\mu, f_1', \dots, f'_{m_\lambda - 1} \mid \mu \vdash k, \mu \neq \lambda \rangle_T$$

and for $1 \leq i \leq k-2$,

$$Q_{k-i} = \big\langle Q_{k-i+1}, f_\nu \mid \nu \vdash k-i, \text{ and } f_\nu \text{ } hwv \text{ with } f_\nu \not\rightsquigarrow f'' \pmod{Q_{k-i+1}} \big\rangle_T.$$

Hence

$$Q_k \subseteq Q_{k-1} \subseteq \cdots \subseteq Q_2,$$

and we claim that $\operatorname{Id}(\mathcal{V}) = Q_2$.

Let $\mathcal{Q}$ be the variety determined by Q_2. Now, $\Gamma_{k+1} \subseteq Q_k \subseteq Q_2$ says that $\chi^p_{k+1}(\mathcal{Q}) = 0$. Also, by the construction of the T-ideals Q_{k-i}, $f'' \notin Q_k$ implies that $f'' \notin Q_{k-1}$ and by induction $f'' \notin Q_2$. Thus, since $f_1', \dots, f'_{m-1} \in Q_2$, we get that $\chi_k^p(\mathcal{Q}) = \chi_\lambda$.

Next we check that the second condition of Theorem 3.3 is satisfied.

Let $f_\mu, \mu \vdash h \leq k-1$ be a *hwv* such that $f_\mu \notin Q_2$. Then since $Q_h \subseteq Q_2$, $f_\mu \notin Q_h$. Hence by the definition of Q_h, $f_\mu \rightsquigarrow f'' \pmod{Q_{h+1}}$. This means that $f'' \in \langle f_\mu, Q_{h+1} \rangle_T \subseteq \langle f_\mu, Q_2 \rangle_T$, *i.e.*, $f_\mu \rightsquigarrow f'' \pmod{Q_2}$. We have proved that the

https://doi.org/10.4153/CJM-2013-016-5 Published online by Cambridge University Press

requirements of Theorem 3.3 are fulfilled and that $\mathfrak{Q}$ is a minimal variety coinciding with $\mathcal{V}$.

Write

$$\chi(\Gamma_k) = \sum_{\lambda \vdash k} m_\lambda \chi_\lambda.$$

The above construction point out that a minimal variety $\mathcal{V}$ of polynomial growth n^k is determined by

(a) a partition $\lambda \vdash k$,

(b) a subspace $\operatorname{span}\{f_1, \dots, f_{m_\lambda - 1}\} \subseteq B^\lambda$, where $f_1, \dots, f_{m_\lambda-1} \in B^\lambda \cap \operatorname{Id}(\mathcal{V})$ are linearly independent h.w.v's.

Hence we introduce the notation $\operatorname{Id}(\mathcal{V}) = Q_{f_1,\dots,f_{m_\lambda-1}}$.

We have proved the following theorem.

Theorem 6.1 *Let $\mathcal{V}$ be a minimal variety of polynomial growth n^k, $k \geq 5$. If $\chi(\Gamma_k) = \sum_{\mu \vdash k} m_\mu \chi_\mu$ is the decomposition of the S_k-character of Γ_k into irreducibles, then there exist a partition $\lambda \vdash n$ and $m_\lambda - 1$ linearly independent h.w.v's $f_1, \dots, f_{m_\lambda-1} \in B^\lambda \cap \operatorname{Id}(\mathcal{V})$ such that $\operatorname{Id}(\mathcal{V}) = Q_{f_1,\dots,f_{m_\lambda-1}}$.*

Acknowledgment We are deeply grateful to the referee for his very valuable comments that improved and shortened the paper.

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