Canad. Math. Bull. Vol. **31** (1), 1988

NONOCCURENCE OF STABILITY SWITCHING IN SYSTEMS WITH DISCRETE DELAYS

BY

H. I. FREEDMAN* AND K. GOPALSAMY†

ABSTRACT. A two dimensional system of differential equations with a finite number of discrete delays is considered. Conditions are derived for there to be no stability switching for arbitrary such delays.

1. **Introduction.** Many mathematical models of ecological systems involve differential equations with time delays. In a number of models, time delays are considered as parameters and questions of stability of equilibria and periodicity are often discussed.

The question of stability switching is discussed in [1], [2], [3], [6], [8], [10]. In some papers ([4], [7], [8], [9], [11], [14]) criteria are established which when satisfied will imply that a system which is stable in the absence of delays will be stable for all delays. In other works ([10], [15]) it is shown that for certain values of the delay, there occur unstable equilibria with periodic oscillations. For general discussions of stability and instability of such systems we refer to [5], [12], [13], [15].

In the present paper we consider a two dimensional system with a finite number of discrete delays (such systems with two delays are discussed in [8], [15]). We are interested in deriving sufficient conditions for there to be no stability switching of positive equilibrium for arbitrary delays. This question has been considered in the case of a single delay in [7], [14] and for two delays in [8].

In the next section we carry out the analysis and in section 3 we will apply our analysis to two models of ecological systems namely a prey-predator model and a mutualist system. In the final section we give a brief discussion of our results.

Received by the editors May 12, 1986, and, in revised form May 19, 1987.

*Research partially supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. NSERC A4823.

†On leave from Flinders University of South Australia.

AMS Subject Classification: Primary 34K20; Secondary 92A17.

2. **Analysis of a linear system.** In this section we derive criteria for there to be no stability switchings. Our technique will be to obtain conditions for the real parts of the eigenvalues of the characteristic equation of the linear variational system about an equilibrium to have the same sign for arbitrary nonnegative values of all the delays. Hence we consider as a variational system the following:

$$(2.1)\quad \frac{du(t)}{dt} = a_{11}u(t) + a_{12}v(t) + \sum_{j=1}^{m} b_{1j}u(t-\tau_{1j}) + \sum_{j=1}^{m} b_{2j}v(t-\tau_{2j})$$

$$\frac{dv(t)}{dt} = a_{21}u(t) + a_{22}v(t) + \sum_{j=1}^{m} c_{1j}u(t-\xi_{1j}) + \sum_{j=1}^{m} c_{2j}v(t-\xi_{2j})$$

in which τ_{ij}, ξ_{ij} $(i = 1, 2, j = 1, 2, \ldots, m)$ are nonnegative constants. The characteristic equation associated with (2.1) is

$$\begin{aligned}(2.2)\quad p(\lambda) = {} & \lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \\ & - \lambda\left\{\sum_{j=1}^{m} b_{1j}e^{-\lambda\tau_{1j}} + \sum_{j=1}^{m} c_{2j}e^{-\lambda\xi_{2j}}\right\} \\ & + \sum_{i=1}^{m}\sum_{j=1}^{m} b_{1i}c_{2j}e^{-\lambda(\tau_{1i}+\xi_{2j})} + a_{11}\sum_{j=1}^{m} c_{2j}e^{-\lambda\xi_{2j}} \\ & + a_n\sum_{j=1}^{m} b_{1j}e^{-\lambda\tau_{1j}} - a_{21}\sum_{j=1}^{m} b_{2j}e^{-\lambda\tau_{2j}} - a_{12}\sum_{j=1}^{m} c_{1j}e^{-\lambda\xi_{1j}} \\ & - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{1i}b_{2j}e^{-\lambda(\xi_{1i}+\tau_{2j})} = 0.\end{aligned}$$

We let $\lambda = i\omega$ (ω being a real number) in (2.2), separate the real and imaginary parts of (2.2) leading to the two equations in the real unknown parameter ω.

$$\begin{aligned}(2.3)\quad & \omega^2 - (a_{11}a_{22} - a_{12}a_{21}) \\ & = \omega\left\{\sum_{j=1}^{m} b_{1j}\cos(\omega\tau_{1j} + \pi/2) + \sum_{j=1}^{m} c_{2j}\cos(\omega\xi_{2j} + \pi/2)\right\} \\ & + a_{11}\sum_{j=1}^{m} c_{2j}\cos\omega\xi_{2j} + a_{22}\sum_{j=1}^{m} b_{1j}\cos\omega\tau_{1j} \\ & + \sum_{i=1}^{m}\sum_{j=1}^{m} b_{1i}c_{2j}\cos\{\omega(\tau_{1i} + \tau_{2j})\} - a_{21}\sum_{j=1}^{m} b_{2j}\cos\omega\tau_{2j} \\ & - a_{12}\sum_{j=1}^{m} c_{1j}\cos\omega\xi_{1j} - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{1i}b_{2j}\cos\{\omega(\xi_{1i} + \xi_{2j})\}\end{aligned}$$

(2.4)
$$\begin{aligned}
&\omega(a_{11} + a_{22}) \\
&= -\left[\omega\left\{\sum_{j=1}^{m} b_{1j}\sin(\omega\tau_{1j} + \pi/2) + \sum_{j=1}^{m} c_{2j}\sin(\omega\xi_{2j} + \pi/2)\right\}\right. \\
&+ a_{11}\sum_{j=1}^{m} c_{2j}\sin\omega\xi_{2j} + a_{22}\sum_{j=1}^{m} b_{1j}\sin\omega\tau_{1j} \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{m} b_{1i}c_{2j}\sin\{\omega(\tau_{1i} + \xi_{2j})\} - a_{21}\sum_{j=1}^{m} b_{2j}\sin\omega\tau_{2j} \\
&\left. - a_{12}\sum_{j=1}^{m} c_{1j}\sin\omega\xi_{1j} - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{1i}b_{2j}\sin\{\omega(\xi_{1i} + \tau_{2j})\}\right].
\end{aligned}$$

If we denote by $f(\omega)$ the sum of the squares of the right sides of (2.3) and (2.4) then we have after some algebraic manipulations and simplifications,

(2.5)
$$f(\omega) \leqq \beta^2\omega^2 + 2\beta\gamma\omega + (\gamma + \delta)^2$$

where

$$\beta = \sum_i (|b_{1i}| + |c_{2i}|)$$

$$\gamma = \sum_i (|a_{11}c_{2i}| + |a_{22}b_{1i}| + |a_{12}c_{1i}| + |a_{21}b_{2i}|)$$

$$\delta = \sum_{j,k} \{|b_{1j}c_{2k}| + |b_{2j}c_{1k}|\}.$$

Hence we get that

(2.6)
$$\omega^4 + (\alpha_1^2 - 2\alpha_2)\omega^2 + \alpha_2^2 \leqq \beta^2\omega^2 + 2\beta\gamma\omega + (\gamma + \delta)^2$$

where $\alpha_1 = (a_{11} + a_{22})$ and $\alpha_2 = a_{11}a_{22} - a_{12}a_{21}$. A sufficient condition for there to be no stability switches is that inequality (2.6) not be satisfied for any real ω. This is equivalent to the condition that

(2.7)
$$g(\omega) = \omega^4 + (\alpha_1^2 - 2\alpha_2 - \beta^2)\omega^2 - 2\beta\gamma\omega + \alpha_2^2 - (\gamma + \delta)^2 > 0$$

for all real ω. (2.7) can be written as

(2.8)
$$\omega^4 + (\alpha_1^2 - 2\alpha_2 - \beta^2)\left[\omega - \frac{\beta\gamma}{\alpha_1^2 - 2\alpha_2 - \beta^2}\right]^2 + \alpha_2^2 - (\gamma + \delta)^2 - \frac{\beta^2\gamma^2}{\alpha_1^2 - 2\alpha_2 - \beta^2} > 0$$

for all real ω. From the above the following theorem is clear.

https://doi.org/10.4153/CMB-1988-008-0 Published online by Cambridge University Press

THEOREM. *If*

(2.9) $$\text{(i) } \alpha_1^2 - 2\alpha_2 - \beta^2 > 0$$

(2.10) $$\text{(ii) } (\alpha_1^2 - 2\alpha_2 - \beta^2)[\alpha_2^2 - (\gamma + \delta)^2] > \beta^2\delta^2$$

then the trivial solution of the system (2.1) *has the same stability for all nonnegative values of the delay parameters.*

3. **Applications.**

EXAMPLE 1. Predator-Prey systems.

We consider a predator-prey system modeled by

(3.1) $$\frac{dx_1(t)}{dt} = B_1(x_1(t - \tau_{11})) - D_1(x_1(t), x_2(t - \tau_{12}))$$

$$\frac{dx_2(t)}{dt} = B_2(x_1(t - \tau_{21}), x_2(t - \tau_{22})) - D_2(x_2(t))$$

where $\tau_{ij}(i, j = 1, 2)$ are nonnegative constants and B_1, the prey birth function, D_1 the prey death function (due to natural causes and predation), B_2 the predator birth function and D_2 the predator death function (due to lack of its food, the prey) are nonnegative scalar valued continuous functions of their arguments having continuous partial derivatives. Let $(\tilde{x}_1, \tilde{x}_2)$ denote a steady state of (3.1). A linear variational system corresponding to (3.1) about $(\tilde{x}_1, \tilde{x}_2)$ is of the form (2.1), where $m = 2$ and where

(3.2) $$a_{11} = -\frac{\partial D_1}{\partial x_1};\ a_{12} = 0;\ b_{11} = \frac{\partial B_1}{\partial x_1};\ b_{12} = -\frac{\partial D_1}{\partial x_2};$$

$$b_{21} = 0;\ b_{22} = 0;\ a_{22} = -\frac{\partial D_2}{\partial x_2};$$

$$c_{11} = 0;\ c_{12} = 0;\ c_{21} = \frac{\partial B_2}{\partial x_1};\ c_{22} = \frac{\partial B_2}{\partial x_2}$$

all partial derivatives in (3.2) being evaluated at $(\tilde{x}_1, \tilde{x}_2)$.

A set of sufficient conditions for the nonoccurence of stability switching of the steady state $(\tilde{x}_1, \tilde{x}_2)$ is

(3.3) $$\text{(i) } \left(\frac{\partial D_1}{\partial x_1}\right)^2 + \left(\frac{\partial D_2}{\partial x_2}\right)^2 > \left(\left|\frac{\partial B_1}{\partial x_1}\right| + \left|\frac{\partial B_2}{\partial x_2}\right|\right)^2 \text{ at } (\tilde{x}_1, \tilde{x}_2).$$

(3.4) $$\text{(ii) } \left[\left(\frac{\partial D_1}{\partial x_1}\right)^2 + \left(\frac{\partial D_2}{\partial x_2}\right)^2 - \left(\left|\frac{\partial B_1}{\partial x_1}\right| + \left|\frac{\partial B_2}{\partial x_2}\right|\right)^2\right]\left[\left\{\left(\frac{\partial D_1}{\partial x_1}\frac{\partial D_2}{\partial x_2}\right)^2 - \left|\frac{\partial D_2}{\partial x_2}\frac{\partial B_1}{\partial x_1}\right| + \left|\frac{\partial D_1}{\partial x_1}\frac{\partial B_2}{\partial x_2}\right| + \left|\frac{\partial B_1}{\partial x_1}\frac{\partial B_2}{\partial x_2}\right| + \left|\frac{\partial D_1}{\partial x_2}\frac{\partial B_2}{\partial x_1}\right|\right\}^2\right]$$

$$> \left\{ \left|\frac{\partial B_1}{\partial x_1}\frac{\partial B_2}{\partial x_2}\right| \left|-\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2}\right| + \left(\left|\frac{\partial B_1}{\partial x_1}\right| + \left|\frac{\partial B_2}{\partial x_2}\right|\right) \right.$$

$$\left. \times \left(\left|\frac{\partial B_1}{\partial x_1}\frac{\partial B_2}{\partial x_2}\right| + \left|\frac{\partial D_1}{\partial x_2}\frac{\partial B_2}{\partial x_1}\right|\right)\right\}^2 \text{ at } (\tilde{x}_1, \tilde{x}_2).$$

EXAMPLE 2. A mutualist system.

We consider a mutualist system modeled by

$$\begin{aligned} \frac{dx_1(t)}{dt} &= B_1(x_1(t-\tau_{11})) - D_1(x_1(t)) \\ &\quad + M_1(x_1(t-\tau_{11}), x_2(t-\tau_{12})) \\ \frac{dx_2(t)}{dt} &= B_2(x_2(t-\tau_{22})) - D_2(x_2(t)) \\ &\quad + M_2(x_1(t-\tau_{21}), x_2(t-\tau_{22})) \end{aligned} \tag{3.5}$$

where B_1, B_2 denote the birth rates, D_1, D_2 denote the death rates and M_1, M_2 denote effects of mutualism between the species. As before it is assumed that B_1, B_2, D_1, D_2, M_1, M_2 are nonnegative valued continuous functions of their respective arguments having continuous partial derivatives. Assuming the existence of a steady state for (3.5) and proceeding as before one can show that a set of sufficient conditions for the nonoccurence of stability switching in (3.5) is

$$\text{(i)} \left(\frac{\partial D_1}{\partial x_2}\right)^2 + \left(\frac{\partial D_2}{\partial x_2}\right)^2 > \left(\frac{\partial B_1}{\partial x_1} + \frac{\partial M_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial M_2}{\partial x_2}\right)^2 \tag{3.6}$$

$$\begin{aligned} \text{(ii)} &\left(\frac{\partial B_1}{\partial x_2} + \frac{\partial M_1}{\partial x_1}\right)^2\left(\frac{\partial B_2}{\partial x_2} + \frac{\partial M_2}{\partial x_2}\right)^2\left[\left|\frac{\partial D_2}{\partial x_2} - \frac{\partial D_1}{\partial x_1}\right| + \frac{\partial M_1}{\partial x_2}\right. \\ &\left. + \frac{\partial M_2}{\partial x_1}\left(\frac{\partial B_1}{\partial x_2} + \frac{\partial M_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial M_2}{\partial x_2}\right)\right]^2 \\ &< \left[\left(\frac{\partial D_1}{\partial x_1}\right)^2 + \left(\frac{\partial D_2}{\partial x_2}\right)^2 - \left(\frac{\partial B_1}{\partial x_2} + \frac{\partial M_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial M_2}{\partial x_2}\right)^2\right] \\ &\times \left[\left(\frac{\partial D_1}{\partial x_2}\right)^2\left(\frac{\partial D_1}{\partial x_2}\right)^2 - \left\{\frac{\partial D_2}{\partial x_2}\left(\frac{\partial B_1}{\partial x_1} + \frac{\partial M_1}{\partial x_1}\right) + \frac{\partial D_1}{\partial x_1}\left(\frac{\partial B_2}{\partial x_2} + \frac{\partial M_2}{\partial x_2}\right)\right.\right. \\ &\left.\left. + \left(\frac{\partial B_1}{\partial x_1} + \frac{\partial M_1}{\partial x_1}\right)\left(\frac{\partial B_2}{\partial x_2} + \frac{\partial M_2}{\partial x_2}\right)\frac{\partial M_1}{\partial x_2}\frac{\partial M_2}{\partial x_1}\right\}^2\right] \end{aligned} \tag{3.7}$$

the partial derivatives being evaluated at the steady state.

4. **Discussion.** In this paper we have considered two dimensional systems with arbitrary finite number of time delays. We have derived criteria in terms of the characteristic equation of the linearized system which when they hold will imply that a given equilibrium will not change its stability as a function of the delay parameters.

It is well known [4], [5], [11], [12] that in many examples in models of ecosystems, stability of equilibria does indeed change as a function of the delay parameters. On the other hand in [7] and [8], criteria for no switching were given for one and two time delays respectively for predator-prey systems. The novelty of this paper is that we have established criteria for no stability switching where there are arbitrarily many delays.

We have applied our criteria to a predator-prey model and a mutualist model with four delays. The key criteria for no stability switching inequalities (3.3), (3.4), (3.6), (3.7) are difficult if not impossible to interpret biologically. We note from [7] that even in the case of one delay, biological interpretation of such criteria is difficult.

We expect a similar technique will work in higher dimensional systems, however the computations involved become increasingly more tedious and lengthy. We also expect that in the case of distributed delays, one could develop similar criteria for the nonswitching of the stability of the equilibrium states. Unfortunately the same techniques used in this paper do not seem to work in the case of distributed delays and we leave this investigation for future work.

References

1. S. P. Blythe, R. M. Nisbet, W. S. C. Gurney and N. MacDonald, *Stability switches in distributed delay models*, J. Math. Anal. Appl. **109** (1985), pp. 388-396.

2. K. L. Cooke, *Stability of delay differential equations with applications in biology and medicine*, in *Mathematics in Biology and Medicine*, Lecture Notes in Biomathematics No. 57, Springer-Verlag, Heidelberg (1985), pp. 439-446.

3. K. L. Cooke and Z. Grossman, *Discrete delay, distributed delay and stability switches*, J. Math. Anal. Appl. **86** (1982), pp. 592-627.

4. K. L. Cooke and P. van den Driessche, *On zeroes of some transcendental equations*, Funkcialaj Ekvacioj, Vol. **29** (1986), pp. 77-90.

5. J. M. Cushing, *Integrodifferential equations and delay models in population dynamics*, Lecture Notes in Biomathematics No. 20, Springer-Verlag, Heidelberg, 1977.

6. H. I. Freedman, J. F. Addicott and B. Rai, *Nonobligate and obligate models of mutualism*, in *Population Biology Proceedings, Edmonton 1982*, Lecture Notes in Biomathematics No. 52, Springer-Verlag, Heidelberg (1983), pp. 349-354.

7. H. I. Freedman and V. S. H. Rao, *The trade-off between mutual interference and time delays in predator-prey systems*, Bull. Math. Biol. **45** (1983), pp. 991-1004.

8. H. I. Freedman and V. S. H. Rao, *Stability criteria for a system involving two time delays*, SIAM J. Appl. Math. **46** (1986), pp. 552-560.

9. K. Gopalsamy, *Delayed responses and stability in two-species systems*, J. Austral. Math. Soc. Ser. B. **25** (1984), pp. 473-500.

10. K. Gopalsamy and B. D. Aggarwala, *Limit cycles in two species competition with time delays*, J. Austral. Math. Soc. Ser. B. **22** (1980), pp. 148-160.

11. J. K. Hale, E. F. Infante and F.-S. P. Tsen, *Stability in linear delay equations*, J. Math. Anal. Appl., **105** (1985), pp. 533-555.

12. N. MacDonald, *Time lags in biological models*, Lecture Notes in Biomathematics No. 27, Springer-Verlag, Heidelberg, 1978.

13. J. M. Mahaffy, *A test for stability of linear differential-delay equations*, Quart. Appl. Math. **40** (1982), pp. 193-202.

14. L. Nunney, *The effect of long time delays in predator-prey systems*, Theor. Pop. Biol. **27** (1985), pp. 202-221.

15. R. D. Nussbaum, *Differential-delay equations with two time lags*, Mem. Amer. Math. Soc. No. 205, Amer. Math. Soc., Providence, 1978.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, CANADA T6G 2G1