INTEGRABILITY OF THE BACKWARD DIFFUSION EQUATION IN A COMPACT RIEMANNIAN SPACE

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1. Introduction. Let R be an orientable, compact Riemannian space with the metric $ds^2 = g_{ij}(x)dx^i dx^j$, and consider the *backward diffusion equation*

$$\frac{\partial f(t, x)}{\partial t} = A \cdot f(t, x), \qquad t \geqq 0, \tag{1}$$

$$(Af)(x) = b^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j} + a^i(x)\frac{\partial f}{\partial x^i}.$$

Here $b^{ij}(x)$ is a contravariant tensor such that the quadratic form $b^{ij}(x)\xi_i\xi_j$ is >0 for $\sum_i \xi_i^2 > 0$, and $a^i(x)$ changes, by a coordinate transformation $x \to \bar{x}$, as follows:

$$\bar{a}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^k} a^k(x) + \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^s} b^{ks}(x). \tag{2}$$

These conditions for the coefficients $a^i(x)$ and $b^{ij}(x)$ are connected with the probabilistic interpretation of the equation (1).[1] In preceding notes,[2] the author treated the stochastic integrability of the *forward diffusion equation* (*Fokker-Planck's equation*)

$$\frac{\partial f(t, x)}{\partial t} = A' \cdot f(t, x), \qquad t \geqq 0, \tag{3}$$

$$(A'f)(x) = \frac{1}{\sqrt{g(x)}}\frac{\partial^2}{\partial x^i \partial x^j}(\sqrt{g(x)}\, b^{ij}(x) f(x)) + \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}(-\sqrt{g(x)}\, a^i(x) f(x)), \quad g(x) = \det(g_{ij}(x))$$

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1) A. Kolmogoroff: Zur Theorie der stetigen zufälligen Prozesse, Math. Ann., **108** (1933), 149-160. K. Yosida: An extension of Fokker-Planck's equation, Proc. Japan Acad., **25** (1949), (9), 1-3.

2) K. Yosida: Integration of Fokker-Planck's equation in a compact Riemannian space, Arkiv för Matematik, **1** (1949), 9, 71-75. K. Yosida: Integration of Foker-Planck's equation with boundary condition, Journ. Math. Soc. Japan, (1951), Takagi's Congratulation volume.

by the semi-group theory.[3] The purpose of the present note is to consider the *stochastic integrability* (to be explained below) of (1), also by the semi-group theory. The result[4] may be considered, in a certain sense, a dual of the result in the preceding notes referred to above.

2. The theorems. For the sake of simplicity, we assume that R is an analytic manifold and that $a^j(x)$ and $b^{ij}(x)$ are holomorphic functions of the coordinates $x=(x^1, x^2, \ldots, x^n)$. We consider A as an additive operator whose domain $D(A)$ is the totality of infinitely differentiable functions defined in R, with values in the Banach space $C(R)$ of the totality of continuous functions $f(x)$ defined in R and metrized by the norm $\|f\| = \max_{x\in R} |f(x)|$. The following two simple lemmas are essential for our arguments.

LEMMA 1. For any $f \in D(A)$ and for any positive number m, we have

(4) $$\max_{x\in R} h(x) \geqq f(x) \geqq \min_{x\in R} h(x), \quad h(x) = f(x) - m^{-1}(Af)(x).$$

Proof. Let $f(x)$ reach its maximum (minimum) at $x_0(x_1)$. Then we have

$$h(x_0) = f(x_0) - m^{-1}(Af)(x_0) \geqq f(x_0) \quad (h(x_1) = f(x_1) - m^{-1}(Af)(x_1) \leqq f(x_1)).$$

COROLLARY. For any $f \in D(A)$ and for any positive number, we have

(5) $$\|f - m^{-1}Af\| \geqq \|f\|.$$

LEMMA 2. Let $\{f_n\} \subseteqq D(A)$ and $\{Af_n\}$ converge, as $n \to \infty$, strongly[5] to 0 and h respectively. Then we have $h(x) \equiv 0$.

Proof. For any $k(x) \in D(A)$, we have $(dx = \sqrt{g(x)}\,dx^1 dx^2 \ldots dx^n$, $g(x) = \det(g_{ij}(x)))$

$$\int_R h(x)k(x)dx = \lim_{n\to\infty} \int_R (Af_n)k(x)dx = \lim_{n\to\infty} \int_R f_n(x)(A'k)(x)dx = 0,$$

and thus $h(x)$ must $\equiv 0$.

COROLLARY. The smallest closed extension $\bar{A}$ of the operator A exists. $\bar{A}$ is defined as follows:

(6) $\bar{A}f$ is defined and $= h$ if there exists $\{f_n\} \subseteqq D(A)$ such that $\{f_n\}$ and $\{Af_n\}$ converge, as $n \to \infty$, strongly to f and h.

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[3] E. Hille: Functional Analysis and Semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, Journ. Math. Soc. Japan, **1** (1949), 1, 15-21, and K. Yosida: An operator-theoretical treatment of temporally homogeneous Markoff process, ibid., **1** (1949), 1, 224-235.

[4] Cf. another approach by K. Itô: Stochastic differential equations on a differentiable manifold, Nagoya Math. J., **1** (1950), 35-48.

[5] By the topology defined by the norm $\|f\|$, viz. by the uniform convergence on R.

From these two lemmas we have the

THEOREM 1. *The inverse* $I_m = (-m^{-1}\bar{A})^{-1}$ *exists* (I=*the identity operator*) *for* $m>0$ *and* I_m *is a positive, contraction operator, leaving the constant functions invariant:*

(7) if $h(x)$ in the domain $D(\bar{A})$ of $\bar{A}$ be non-negative, then $(I_m h)(x)$ is also non-negative and $\|I_m h\| \leqq \|h\|$; $I_m \cdot 1 = 1$.

By the semi-group theory, the coincidence of the domains $D(I_m)$ of I_m with $C(R)$ is the necessary and sufficient condition for the existence of the one-parameter semi-group of linear operators in $C(R)$ with the properties:

(8) $T_t T_s = T_{t+s}$ $(t, s \geqq 0)$, $T_0 = I$;
T_t are positive, contraction operators, leaving constant functions invariant; strong $\lim_{t \to t_0} T_t f = T_{t_0} f$, $f \in C(R)$;
strong $\lim_{\delta \to 0} \frac{T_{t+\delta} - T_t}{\delta} f = \bar{A} T_t f = T_t \bar{A} f$ for f in $D(\bar{A})$, which is surely dense in $C(R)$.

This T_t is, in fact, defined by

(9) $T_t f = \text{strong} \lim_{m \to \infty} (I - t\, m^{-1}\bar{A})^{-m} f$, $f \in C(R)$.

The existance of this semi-group T_t may be considered as the *stochastic integrability* of (1). The coincidence of the domains $D(I_m)$ with $C(R)$ is equivalent to the denseness of the ranges $R(I - m^{-1}A) = \{f - m^{-1}Af;\ f \in D(A)\}$, $m>0$, in $C(R)$. Hence we have the

COROLLARY. (1) is stochastically integrable if and only if positive numbers m do not belong to the residual spectra of the operator A.

When the dimension n of the space R is $\geqq 2$, we have the

THEOREM 2. *The backward diffusion equation* (1) *is stochastically integrable if the compact space* R *is of dimension* $\geqq 2$.

Proof. Let the range $R(I - m^{-1}A)$ be not dense in $C(R)$. Then there exists a measure φ, countably additive for Borel sets of R such that

(10) $0 <$ total variation of φ in $R < \infty$,

(11) $\int_R (f(x) - m^{-1}(Af)(x))\varphi(dx) = 0$ for $f \in D(A)$,

since the conjugate space of $C(R)$ is the space of measures, countably additive for Borel sets and of bounded total variations. If we define the *distribution* (in the sense of L. Schwartz[6]) by

(12) $H(f) = \int_R f(x)\varphi(dx)$, $f \in D(A)$,

[6] L. Schwartz: Théorie des distributions, 1, Paris (1950).

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H satisfies, by (11), the *differential equation* (*in the sense of the distribution*)

(13) $A'H = mH.$

By the *elliptic character* of the differential operator A', there must exist[7] an infinitely differentiable function $h(x)$ such that

(14) $(A'h)(x) = mh(x), \qquad H(f) = \int_R f(x)h(x)dx.$

By (10), we have

(15) $h(x) \not\equiv 0.$

Let $k(x)$ be $=1$, -1 or $=0$ according as $h(x)>0$, <0 or $=0$. Then we have

(16) $$0 = \int_R |h(x) - m^{-1}(A'h)(x)|dx \geqq \int_R (h(x) - m^{-1}(A'h)(x))k(x)dx$$
$$= \int_R |h(x)|dx - m^{-1}\sum_i \int_{P_i} (A'h)(x)dx + m^{-1}\sum_j \int_{N_j} (A'h)(x)dx,$$

where $P(N)$ are connected domains in which $h(x)>0$ (<0) such that $h(x)=0$ on their boundaries $\partial P\,(\partial N)$. By the integral theorem of Green's type we have

(17) $$\int_P (A'h)(x)dx = \int_{\partial P} \frac{\partial h}{\partial n} dS,$$

where n and dS respectively denote outer normal and positive measure on ∂P. Hence we have $\int_P (A'h)(x)dx \leqq 0$, and similarly $\int_N (A'h)(x)dx \geqq 0$. Therefore we obtain, from (14)–(15), a contradiction $0 \geqq \int_R |h(x)|dx > 0$.

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[7] L. Schwartz: loc. cif., p. 136,

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