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POSITIVE LINEAR MAPS ON C*-ALGEBRAS

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The objective of this paper is to give some concrete distinctions between positive linear maps and completely positive linear maps on C^*-algebras of operators.

Herein, C^*-algebras possess an identity and are written in German type $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$. Capital letters A, B, C stand for operators, script letters $\mathscr{H}$, $\mathscr{K}$ for vector spaces, small letters x, y, z for vectors. Capital Greek letters Φ, Ψ stand for linear maps on C^*-algebras, small Greek letters α, β, γ for complex numbers.

We denote by $\mathfrak{M}_n$ the collection of all $n \times n$ complex matrices. $\mathfrak{M}_n(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{M}_n$ is the C^*-algebra of $n \times n$ matrices over $\mathfrak{A}$. A linear map Φ: $\mathfrak{A} \to \mathfrak{B}$ is *positive* if $\Phi(A)$ is positive for all positive A in $\mathfrak{A}$. We define $\Phi \otimes 1_n$: $\mathfrak{M}_n(\mathfrak{A}) \to \mathfrak{M}_n(\mathfrak{B})$ by

$$\Phi \otimes 1_n((A_{jk})_{1 \leqq j,k \leqq n}) = (\Phi(A_{jk}))_{1 \leqq j,k \leqq n}.$$

We say Φ is *n-positive* if $\Phi \otimes 1_n$: $\mathfrak{M}_n(\mathfrak{A}) \to \mathfrak{M}_n(\mathfrak{B})$ is positive; the set of all such Φ is denoted $\mathbf{P}_n[\mathfrak{A}, \mathfrak{B}]$. Φ is *completely positive* if $\Phi \in \mathbf{P}_n[\mathfrak{A}, \mathfrak{B}]$ for all positive integers n; the set of all such Φ is denoted $\mathbf{P}_\infty[\mathfrak{A}, \mathfrak{B}]$.

It is evident that

$$\mathbf{P}_1[\mathfrak{A}, \mathfrak{B}] \supseteq \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}] \supseteq \mathbf{P}_3[\mathfrak{A}, \mathfrak{B}] \supseteq \ldots \supseteq \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{B}].$$

Stinespring [**4**] and Arveson [**1**] have given examples of positive linear maps that fail to be completely positive. However, all these examples fail to be 2-positive. In Theorem 1, we construct examples of $n - 1$-positive maps that fail to be n-positive.

If $\mathfrak{A}$ or $\mathfrak{B}$ is commutative, then $\mathbf{P}_1[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{B}]$ (see [**4**, **5**; **1**, p. 148]). We establish the converse in Theorem 4, thus giving a characterization of the commutativity of C^*-algebras by means of the 'completeness' of positive linear maps. (The result can be strengthened in the finite-dimensional case, as we explain in the remarks which conclude the paper.)

An extension of the work of Stinespring and Størmer leads to a further generalization, Theorems 7 and 8: If $\mathfrak{C}$ is commutative, then

$$\mathbf{P}_n[\mathfrak{M}_n(\mathfrak{C}), \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{M}_n(\mathfrak{C}), \mathfrak{B}], \mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n(\mathfrak{C})] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{M}_n(\mathfrak{C})].$$

Hence, we get a simplification of the structure of completely positive linear maps on a matrix algebra.

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improvements in the paper. Thanks are also due to the referee for simplifying the proof of Theorem 1.

First we show that n-positivity is different from $(n-1)$-positivity for the linear maps on $\mathfrak{M}_n$. Let $(\alpha_{jk}) \in \mathfrak{M}_n$; we recall that trace $(\alpha_{jk}) = \sum_j \alpha_{jj}$. The map

$$\Phi(A) = \{(n-1)(\text{trace } A)\}I_n - A$$

serves as the simplest example we can manage for

THEOREM 1. $\mathbf{P}_{n-1}[\mathfrak{M}_n, \mathfrak{M}_n] \underset{\neq}{\supset} \mathbf{P}_n[\mathfrak{M}_n, \mathfrak{M}_n]$.

It is convenient to regard the elements of $\mathfrak{M}_m(\mathfrak{M}_n)$ as $m \times m$ block matrices with $n \times n$ matrices as entries; then each is also regarded as an $mn \times mn$ matrix with numerical entries. Let E_{jk} be the $n \times n$ matrix with 1 at the j, k component and zeros elsewhere. Then $(E_{jk})_{1 \leqq j,k \leqq n} \in \mathfrak{M}_n(\mathfrak{M}_n)$ is the block matrix having the matrix E_{jk} as its j, k entry, for each j, k. Now we investigate the magnitude of $(E_{jk})_{jk}$ in the following

LEMMA (i) $(n-1)I_{n^2} - (E_{jk})_{1 \leqq j,k \leqq n}$ *is not positive.*

(ii) *For any rank-$(n-1)$-positive projection P in $\mathfrak{M}_n$,*

$$P^{\#}\{(n-1)I_{n^2} - (E_{jk})_{1 \leqq j,k \leqq n}\}P^{\#}$$

is positive, where $P^{\#} = I_n \otimes P$.

Proof. A straight-forward computation shows that

$$(E_{jk})_{jk}{}^2 = n(E_{jk})_{jk},$$

and more generally

$$(E_{jk})_{jk} \cdot A^{\#} \cdot (E_{jk})_{jk} = (\text{trace } A)(E_{jk})_{jk}$$

where A is arbitrary in $\mathfrak{M}_n$ and $A^{\#} = I_n \otimes A$. Now (i) is immediate, since $1/n(E_{jk})_{jk}$ is a projection and

$$\|(E_{jk})_{jk}\| = n > n-1.$$

For (ii) we look at

$$\begin{aligned} \|P^{\#}(E_{jk})_{jk}P^{\#}\| &= \frac{1}{n}\left\|P^{\#}(E_{jk})_{jk} \cdot (E_{jk})_{jk}P^{\#}\right\| \\ &= \frac{1}{n}\left\|(E_{jk})_{jk}P^{\#} \cdot P^{\#}(E_{jk})_{jk}\right\| \\ &= \frac{1}{n}\left\|(E_{jk})_{jk} \cdot P^{\#} \cdot (E_{jk})_{jk}\right\| \\ &= \frac{1}{n}(\text{trace } P)\|(E_{jk})_{jk}\| \\ &= \text{trace } P = n-1 \end{aligned}$$

as rank $P = n - 1$.

Thus we have derived that

$$P^{\#}(n-1)I_{n^2}P^{\#} \geqq P^{\#}(E_{jk})_{jk}P^{\#}.$$

Proof of Theorem 1. $\Phi \otimes 1_n((E_{jk})_{jk}) = (\Phi(E_{jk}))_{jk} = (n-1)I_{n^2} - (E_{jk})_{jk}$ is not positive (Lemma (i)). So we conclude that Φ is not n-positive.

The proof that Φ is $(n-1)$-positive will be written out only in the case $n = 3$; i.e., we will show that

$$\Phi(A) = 2(\text{trace } A)I_3 - A$$

is 2-positive on $\mathfrak{M}_3$.

It suffices to prove that for any rank-1 positive 6×6 matrix X, $\Phi \otimes 1_2(X)$ is positive when regarding X in $\mathfrak{M}_2(\mathfrak{M}_3)$. Let $X = x^*x$ where x is a row matrix $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$, and let

$$X_0 = \left[\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{X}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \in \mathfrak{M}_3(\mathfrak{M}_3).$$

Then $X_0 = L^{\#*}(E_{jk})_{1 \leqq j,k \leqq 3}L^{\#}$ where L is

$$\begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{bmatrix}$$

and $L^{\#} = I_3 \otimes L$. Thus

$$\Phi \otimes 1_3(X_0) = L^{\#*} \cdot \Phi \otimes 1_3(E_{jk})_{jk} \cdot L^{\#} = L^{\#*}\{2I_9 - (E_{jk})_{jk}\}L^{\#}.$$

Since rank $L \leqq 2$, there exist a positive projection P of rank 2 and a matrix N in $\mathfrak{M}_3$ such that $L = PN$. By Lemma (ii) $P^{\#}(2I_9 - (E_{jk})_{jk})P^{\#}$ is positive, so

$$\Phi \otimes 1_3(X_0) = N^{\#*}P^{\#}(2I_9 - (E_{jk})_{jk})P^{\#}N^{\#}$$

is positive. It is equivalent that $\Phi \otimes 1_2(X)$ is positive.

In the general case, the proof is similar; we start with $X = x^*x$ where $x = (\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_n^{(1)}; \ldots; \alpha_1^{(n-1)}, \ldots, \alpha_n^{(n-1)})$ and obtain

$$L = \begin{bmatrix} \alpha_1^{(1)} & \cdots & \alpha_1^{(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ \alpha_n^{(1)} & \cdots & \alpha_n^{(n-1)} & 0 \end{bmatrix}$$

which is of rank at most $n - 1$. This proves Theorem 1.

From the above theorem, we may perceive that, in general, a positive linear map will usually not be completely positive. However, Stinespring and

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Størmer prove, in the special case that $\mathfrak{A}$ or $\mathfrak{B}$ is commutative, that $\mathbf{P}_1[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{B}]$. We will show that this can never happen in non-commutative C^*-algebras. In other words, if and only if $\mathfrak{A}$ or $\mathfrak{B}$ is commutative, will positivity be the same thing as complete positivity.

We shall adopt Berberian's extension (see [**2**] for details) in our proof.

Let $\mathscr{M}$ be the space of all bounded sequences of complex numbers endowed with supremum norm. Let glim be a generalized Banach limit defined on $\mathscr{M}$; i.e., glim is a linear functional defined on $\mathscr{M}$ such that for any real sequence (α_j),

$$\liminf (\alpha_j) \leqq \text{glim } (\alpha_j) \leqq \limsup (\alpha_j).$$

For a fixed Hilbert space $\mathscr{H}$, we define a positive Hermitian bilinear form on $\mathscr{H}^\infty$, the space of all bounded sequences in $\mathscr{H}$, by

$$\langle (x_j), (y_j) \rangle = \text{glim } (\langle x_j, y_j \rangle)$$

where $\langle x_j, y_j \rangle$ is the inner-product of x_j and y_j in $\mathscr{H}$.

The quotient space of $\mathscr{H}^\infty$ modulo the subspace of all (x_j) such that $\langle (x_j), (x_j) \rangle = 0$ is an inner-product space. Let $\mathscr{H}^\circ$ be the completion. Denote the coset containing (x_j) by $[(x_j)]$. $\mathscr{H}$ can be imbedded in $\mathscr{H}^\circ$ by identifying each x with $[(x)]$. For each $A \in \mathscr{B}(\mathscr{H})$, we assign $A^\circ \in \mathscr{B}(\mathscr{H}^\circ)$ such that

$$A^\circ[(x_j)] = [(Ax_j)].$$

We can see this determines a $*$-isomorphism of $\mathscr{B}(\mathscr{H})$ into $\mathscr{B}(\mathscr{H}^\circ)$. Furthermore,

$$\Pi(A) = \Pi(A^\circ) = \Pi_0(A^\circ)$$

where Π_0 is the point spectrum and Π is the approximate point spectrum.

LEMMA 2. *If* $\mathfrak{A}$ *is not commutative, then*

$$\mathbf{P}_1[\mathfrak{A}, \mathfrak{M}_2] \underset{\neq}{\supset} \mathbf{P}_2[\mathfrak{A}, \mathfrak{M}_2].$$

Proof. If $\mathfrak{A}$ is not commutative, there exist Hermitian operators A_1, A_2, A_3 in $\mathfrak{A}$ such that

$$A_1 = i(A_2A_3 - A_3A_2) \neq 0.$$

Let $\mathscr{H}$ be the underlying space of $\mathfrak{A}$. By Berberian's extension, we can extend each $A \in \mathscr{B}(\mathscr{H})$ to $A^\circ \in \mathscr{B}(\mathscr{H}^\circ)$. Thus A_1° is a Hermitian operator and has a non-trivial eigenspace $\mathscr{S}$ corresponding to a non-zero eigenvalue λ. A_1° restricted to $\mathscr{S}$ is a non-zero scalar operator, and hence cannot be of the form $XY - YX$ for $X, Y \in \mathscr{B}(\mathscr{S})$ [**3**, p. 126]. From $A_1^\circ = i(A_2^\circ A_3^\circ - A_3^\circ A_2^\circ)$ we derive that $\mathscr{S}$ is not a common invariant subspace under A_2° and A_3°. Without loss of generality, we assume $A_2^\circ\mathscr{S} \not\subseteq \mathscr{S}$; i.e., there exist non-zero vectors x, y in $\mathscr{H}^\circ$, such that

$$(A_1^\circ - \lambda)x = 0, \quad (A_1^\circ - \lambda)A_2^\circ x = y.$$

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Define Ψ: $\mathfrak{A} \to \mathfrak{M}_2$ by

$$\Psi(A) = \begin{bmatrix} \langle A^\circ x, x\rangle\langle A^\circ y, x\rangle \\ \langle A^\circ x, y\rangle\langle A^\circ y, y\rangle \end{bmatrix}.$$

Let Θ be the transpose map: $\mathfrak{M}_2 \to \mathfrak{M}_2$. Obviously, $\Theta \circ \Psi$ is positive. It is not 2-positive because

$$(\Theta \circ \Psi) \otimes 1_2 \cdot \left[\begin{array}{c:c} (A_1 - \lambda)^2 & (A_1 - \lambda)A_2 \\ \hdashline A_2(A_1 - \lambda) & A_2{}^2 \end{array}\right] = \left[\begin{array}{cc:cc} 0 & 0 & 0 & \|y\|^2 \\ 0 & * & 0 & * \\ \hdashline 0 & 0 & * & * \\ \|y\|^2 & * & * & * \end{array}\right],$$

of which the associated quadratic form applied to the column vector $a = [1, 0, 0, -\epsilon]$,

$$\langle \cdot (a), a\rangle = -2\epsilon\|y\|^2 + \epsilon^2 *,$$

is non-positive if ϵ is a sufficiently small positive number.

LEMMA 3. *If* $\mathfrak{B}$ *is not commutative, then*

$$\mathbf{P}_1[\mathfrak{M}_2, \mathfrak{B}] \underset{\neq}{\supset} \mathbf{P}_2[\mathfrak{M}_2, \mathfrak{B}].$$

Proof. Let $\mathscr{K}$ be the underlying space of $\mathfrak{B}$. By Berberian's extension, we can extend each $B \in \mathscr{B}(\mathscr{K})$ to $B^\circ \in \mathscr{B}(\mathscr{K}^\circ)$.

By the same manner as in the first paragraph of the proof of Lemma 2, we get Hermitian operators B_1, B_2 in $\mathfrak{B}$, non-zero vectors u, v in $\mathscr{K}^\circ$, and a real number μ such that

$$(B_2{}^\circ - \mu)u = 0, \quad (B_2{}^\circ - \mu)B_1{}^\circ u = v.$$

Define Φ: $\mathfrak{M}_2 \to \mathfrak{B}$ by

$$\Phi \cdot \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \alpha B_1{}^2 + \beta B_1(B_2 - \mu) + \gamma(B_2 - \mu)B_1 + \delta(B_2 - \mu)^2.$$

It is evident that Φ is positive. Let Θ be the transpose map: $\mathfrak{M}_2 \to \mathfrak{M}_2$. Then $\Phi \circ \Theta$ is not 2-positive because

$$(\Phi \circ \Theta) \otimes 1_2 \cdot \left[\begin{array}{cc:cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c:c} B_1{}^2 & (B_2 - \mu)B_1 \\ \hdashline B_1(B_2 - \mu) & (B_2 - \mu)^2 \end{array}\right]$$

which is not positive, since

$$\left\langle \begin{bmatrix} (B_1{}^\circ)^2 & (B_2{}^\circ - \mu)B_1{}^\circ \\ B_1{}^\circ(B_2{}^\circ - \mu) & (B_2{}^\circ - \mu)^2 \end{bmatrix} (-\epsilon v \oplus u),\ -\epsilon v \oplus u \right\rangle = \epsilon^2\|B_1{}^\circ v\|^2 - 2\epsilon\|v\|^2$$

is not positive when ϵ is a sufficiently small positive number.

THEOREM 4. *If* $\mathbf{P}_1[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, *then either* $\mathfrak{A}$ *or* $\mathfrak{B}$ *is commutative.*

Proof. Assume $\mathfrak{A}$, $\mathfrak{B}$ are not commutative. We use the same notations as in

Lemma 2 and Lemma 3. It is evident that $\Phi \circ \Theta \circ \Psi$ is positive. It is not 2-positive because

$$(\Phi \circ \Theta \circ \Psi) \otimes 1_2 \cdot \begin{bmatrix} (A_1 - \lambda)^2 & (A_1 - \lambda)A_2 \\ A_2(A_1 - \lambda) & A_2{}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \rho(B_2 - \mu)^2 & \zeta(B_2 - \mu)^2 - \|y\|^2 B_1(B_2 - \mu) \\ \zeta^*(B_2 - \mu)^2 + \|y\|^2(B_2 - \mu)B_1 & T \end{bmatrix}$$

(T is an operator in $\mathfrak{B}$, ρ is a real number, ζ is a complex number) which is not positive, since Berberian's extension applied to

$$\langle \cdot (u \oplus - \epsilon v), u \oplus - \epsilon v\rangle = -2\epsilon\|y\|^2 \|v\|^2 + \epsilon^2\langle T^\circ v, v\rangle$$

is not positive if ϵ is a sufficiently small positive number.

Therefore $\mathbf{P}_1[\mathfrak{A}, \mathfrak{B}] \underset{\neq}{\supset} \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$. This leads to a contradiction.

From Theorem 1, we see that for linear maps on $\mathfrak{M}_n$, $(n - 1)$-positivity is different from complete positivity. It will not be surprising that n-positivity coincides with complete positivity.

THEOREM 5. $\mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{M}_n]$.

Proof. We will first establish that

$$\mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n] = \mathbf{P}_{n+1}[\mathfrak{A}, \mathfrak{M}_n].$$

Assume $\Phi \in \mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n]$. Let (A_{pq}) be a positive element in $\mathfrak{M}_{n+1}(\mathfrak{A})$. We wish to prove that if $x_1, \ldots x_{n+1}$ are vectors in n-dimensional complex space, then

$$\sum\nolimits_{1 \leqq p,q \leqq n+1} \langle \Phi(A_{pq})x_q, x_p\rangle \geqq 0.$$

Since $\{x_1, \ldots x_{n+1}\}$ are vectors in n-dimensional complex space, they are linearly dependent. We may assume that x_{n+1} is linearly dependent on $x_1, \ldots x_n$; i.e.,

$$x_{n+1} = \alpha_1 x_1 + \ldots + \alpha_n x_n$$

for some complex numbers α_j. From

$$\begin{bmatrix} I & & & & \alpha_1{}^*I \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & I & \alpha_n{}^*I \\ & & & & 0 \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1,n+1} \\ \vdots & & \vdots \\ A_{n+1,1} & \cdots & A_{n+1,n+1} \end{bmatrix} \begin{bmatrix} I & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & I & \\ \alpha_1 I & \cdots & & \alpha_n I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & \cdots & C_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ C_{n1} & \cdots & C_{nn} & \cdot \\ 0 & \cdots & & 0 \end{bmatrix}$$

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where

$$C_{jk} = A_{jk} + \alpha_j^* A_{n+1,k} + \alpha_k A_{j,n+1} + \alpha_j^* \alpha_k A_{n+1,n+1},$$

we know $(C_{jk})_{1 \leqq j,k \leqq n}$ is positive. As Φ is n-positive,

$$\sum_{1 \leqq j,k \leqq n} \langle \Phi(C_{jk}) x_k, x_j \rangle \geqq 0.$$

Substitute the definition of C_{jk} and rearrange terms. We get

$$\sum_{1 \leqq p,q \leqq n+1} \langle \Phi(A_{pq}) x_q, x_p \rangle \geqq 0$$

as required. So $\mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n] = \mathbf{P}_{n+1}[\mathfrak{A}, \mathfrak{M}_n]$.

Replacing n by $n + 1$, $\mathbf{P}_{n+1}[\mathfrak{A}, \mathfrak{M}_{n+1}] = \mathbf{P}_{n+2}[\mathfrak{A}, \mathfrak{M}_{n+1}]$. Now, we regard $\mathfrak{M}_n$ as a subalgebra of $\mathfrak{M}_{n+1}$ naturally and obtain $\mathbf{P}_{n+1}[\mathfrak{A}, \mathfrak{M}_n] = \mathbf{P}_{n+2}[\mathfrak{A}, \mathfrak{M}_n]$. Repeating the argument, $\mathbf{P}_{n+m}[\mathfrak{A}, \mathfrak{M}_n] = \mathbf{P}_{n+m+1}[\mathfrak{A}, \mathfrak{M}_n]$ $(m = 0, 1, 2, \ldots)$ and we conclude that $\mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{M}_n]$.

THEOREM 6. $\mathbf{P}_n[\mathfrak{M}_n, \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{M}_n, \mathfrak{B}]$.

Proof. We will establish $\mathbf{P}_n[\mathfrak{M}_n, \mathfrak{B}] = \mathbf{P}_{n+1}[\mathfrak{M}_n, \mathfrak{B}]$ first.

Let $\Phi \in \mathbf{P}_n[\mathfrak{M}_n, \mathfrak{B}]$; we wish to prove that for any positive

$$(A_{pq}) \in \mathfrak{M}_{n+1}(\mathfrak{M}_n),$$

$(\Phi(A_{pq}))_{1 \leqq p,q \leqq n+1}$ is positive. We may assume that (A_{pq}) is a rank-1 positive $n(n + 1) \times n(n + 1)$ matrix. Hence, $Q_p = (A_{p1}, A_{p2}, \ldots A_{p,n+1})$ is an $n \times n(n + 1)$ matrix with pairwise dependent rows. So $\{Q_1, \ldots Q_{n+1}\}$ must be linearly dependent. Without loss of generality, we let

$$Q_{n+1} = \alpha_1 Q_1 + \ldots + \alpha_n Q_n$$

for certain complex numbers α_j; i.e.,

$$A_{n+1,q} = \alpha_1 A_{1q} + \ldots + \alpha_n A_{nq}$$

for all q. Consequently,

$$A_{p,n+1} = \alpha_1^* A_{p1} + \ldots + \alpha_n^* A_{pn}$$

for all p. Therefore

$$(\Phi(A_{pq}))_{1 \leqq p,q \leqq n+1}$$

$$= \begin{bmatrix} I & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & I & \\ \alpha_1^* I & \ldots & & \alpha_n^* I & 0 \end{bmatrix} \begin{bmatrix} \Phi(A_{11}) & \ldots & \Phi(A_{1n}) & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \Phi(A_{n1}) & \ldots & \Phi(A_{nn}) & \cdot \\ 0 & \ldots & \cdot & 0 \end{bmatrix} \begin{bmatrix} I & & & & \alpha_1 I \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & I & \alpha_n I \\ & & & & 0 \end{bmatrix}$$

The middle matrix is positive since Φ is n-positive. Therefore $(\Phi(A_{pq}))_{1 \leqq p,q \leqq n+1}$ is positive. So $\mathbf{P}_n[\mathfrak{M}_n, \mathfrak{B}] = \mathbf{P}_{n+1}[\mathfrak{M}_n, \mathfrak{B}]$.

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Replacing n by $n + 1$, $\mathbf{P}_{n+1}[\mathfrak{M}_{n+1}, \mathfrak{B}] = \mathbf{P}_{n+2}[\mathfrak{M}_{n+1}, \mathfrak{B}]$. Now, we regard $\mathfrak{M}_n$ as a quotient space of $\mathfrak{M}_{n+1}$ naturally and obtain

$$\mathbf{P}_{n+1}[\mathfrak{M}_n, \mathfrak{B}] = \mathbf{P}_{n+2}[\mathfrak{M}_n, \mathfrak{B}].$$

Repeating the argument, $\mathbf{P}_{n+m}[\mathfrak{M}_n, \mathfrak{B}] = \mathbf{P}_{n+m+1}[\mathfrak{M}_n, \mathfrak{B}]$ $(m = 0, 1, 2, \ldots)$ and we conclude that $\mathbf{P}_n[\mathfrak{M}_n, \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{M}_n, \mathfrak{B}]$.

The generalizations of the above theorems are valid for matrices over a commutative C^*-algebra. These can also be viewed as direct generalizations of Stinespring and Størmer's results.

THEOREM 7. *If* $\mathfrak{C}$ *is commutative, then* $\mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n(\mathfrak{C})] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{M}_n(\mathfrak{C})]$.

Proof. We may take $\mathfrak{C}$ as the set of all continuous functions defined on a compact Hausdorff space $\mathscr{S}$. Let $\Phi \in \mathbf{P}_n[\mathfrak{A}, \mathfrak{M}_n(\mathfrak{C})]$. If $(A_{pq})_{1 \leqq p,q \leqq m}$ is positive in $\mathfrak{M}_m(\mathfrak{A})$ and

$$\Phi(A_{pq}) = (f_{pqjk})_{1 \leqq j,k \leqq n},$$

we wish to prove that

$$(f_{pqjk})_{\substack{1 \leqq p,q \leqq m \\ 1 \leqq j,k \leqq n}}$$

is positive. For any $s \in \mathscr{S}$, define Ψ_s: $\mathfrak{M}_n(\mathfrak{C}) \to \mathfrak{M}_n$ by

$$\Psi_s((f_{jk})) = (f_{jk}(s)).$$

Obviously, Ψ_s is completely positive. Hence $\Psi_s \circ \Phi$: $\mathfrak{A} \to \mathfrak{M}_n$ is n-positive, and thus completely positive by Theorem 5. So

$$(f_{pqjk}(s))_{p,q;j,k} = (\Psi_s \circ \Phi) \otimes 1_m((A_{pq})_{pq})$$

is positive. Since s is arbitrary in $\mathscr{S}$, $(f_{pqjk})_{p,q;j,k}$ is positive as required.

THEOREM 8. *If* $\mathfrak{C}$ *is commutative, then* $\mathbf{P}_n[\mathfrak{M}_n(\mathfrak{C}), \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{M}_n(\mathfrak{C}), \mathfrak{B}]$.

We may assume $n \geqq 2$, and $\mathfrak{C} = C(\mathscr{S})$ = the set of all continuous functions defined on a compact Hausdorff space $\mathscr{S}$. Denote by $E_{jk}(f) \in \mathfrak{M}_n(C(\mathscr{S}))$ the matrix with $f \in C(\mathscr{S})$ at the j, k component and zeros elsewhere, and by $I_n(f) \in \mathfrak{M}_n(C(\mathscr{S}))$ the diagonal matrix with f along the diagonal. As in the special case proved by Stinespring, we must use integral representations.

LEMMA. *If* $\Psi \in \mathbf{P}_n[\mathfrak{M}_n(C(\mathscr{S})), \mathfrak{M}_m]$, *then there exist a regular positive Borel measure* $\mathbf{m}$ *on* $\mathscr{S}$ *and* $\mathbf{m}$*-measurable matrix-valued functions* $G_{jk} \in \mathfrak{M}_m(\mathbf{m}(\mathscr{S}))$ $(1 \leqq j, k \leqq n)$, *such that*

(i) *for all* f *in* $C(\mathscr{S})$,

$$\Psi E_{jk}(f) = \int_{\mathscr{S}} f G_{jk}\, d\mathbf{m},$$

(ii) $(G_{jk}(s))_{jk}$ *is positive in* $\mathfrak{M}_n(\mathfrak{M}_m)$ *a.e.* $(\mathbf{m})$.

Proof. Let $\{x_1, \ldots x_m\}$ be the canonical orthonormal basis of the underlying space of $\mathfrak{M}_m$. By the Riesz-Markoff theorem, there exists a regular positive

Borel measure $\mathbf{m}$ on $\mathscr{S}$ such that for all f in $C(\mathscr{S})$

$$\sum_p \langle \Psi I_n(f)x_p, x_p\rangle = \int_{\mathscr{S}} f\, d\mathbf{m}.$$

Since

$$\begin{bmatrix} E_{jj}(|f|) & E_{jk}(f) \\ E_{kj}(f^*) & E_{kk}(|f|) \end{bmatrix}$$

is positive, its image under $\Psi \otimes 1_2$ is positive, too; thus

$$\begin{bmatrix} \langle \Psi E_{jj}(|f|)x_p, x_p\rangle & \langle \Psi E_{jk}(f)x_q, x_p\rangle \\ \langle \Psi E_{kj}(f^*)x_p, x_q\rangle & \langle \Psi E_{kk}(|f|)x_q, x_q\rangle \end{bmatrix}$$

is positive. From the elementary fact

$$\begin{bmatrix} \alpha_1 & \beta \\ \beta^* & \alpha_2 \end{bmatrix} \text{ positive in } \mathfrak{M}_2 \Rightarrow |\beta| \leqq \tfrac{1}{2}(\alpha_1 + \alpha_2),$$

we derive that

$$\begin{aligned} |\langle \Psi E_{jk}(f)x_q, x_p\rangle| &\leqq \tfrac{1}{2}\{\langle \Psi E_{jj}(|f|)x_p, x_p\rangle + \langle \Psi E_{kk}(|f|)x_q, x_q\rangle\} \\ &\leqq \textstyle\sum_p \langle \Psi I_n(|f|)x_p, x_p\rangle \\ &= \textstyle\int_{\mathscr{S}} |f|\, d\mathbf{m}. \end{aligned}$$

By the Riesz and Radon-Nikodym theorems, there exists an $\mathbf{m}$-measurable function g_{jkpq} such that for all f in $C(\mathscr{S})$

$$\langle \Psi E_{jk}(f)x_q, x_p\rangle = \int_{\mathscr{S}} f \cdot g_{jkpq}\, d\mathbf{m}.$$

Let

$$G_{jk} = (g_{jkpq})_{pq} \in \mathfrak{M}_m(\mathbf{m}(\mathscr{S})).$$

Then it is immediate that

$$\Psi E_{jk}(f) = \int_{\mathscr{S}} f G_{jk}\, d\mathbf{m}.$$

Let $h \in C(\mathscr{S})$ be positive. Then $(E_{jk}(h))_{jk}$ is positive in $\mathfrak{M}_n(\mathfrak{M}_n(\mathfrak{C}))$, so its image under $\Psi \otimes 1_n$ is positive; i.e.,

$$(\Psi E_{jk}(h))_{jk} = \left(\int_{\mathscr{S}} h G_{jk}\, d\mathbf{m}\right)_{jk} \geqq 0.$$

By varying the positive function h, we get

$$(G_{jk}(s))_{jk} \geqq 0 \text{ a.e. } (\mathbf{m})$$

Proof of Theorem 8. Assume $\Phi \in \mathbf{P}_n[\mathfrak{M}_n(\mathfrak{C}), \mathfrak{B}]$. We wish to prove that for any positive integer m, if $y_1, \ldots y_m$ are vectors in the underlying space of $\mathfrak{B}$ and

$$(f_{jkpq})_{\substack{1 \leqq j,k \leqq n \\ 1 \leqq p,q \leqq m}}$$

is positive in $\mathfrak{M}_m(\mathfrak{M}_n(\mathfrak{C}))$, then

$$\sum_{pq} \langle \Phi(f_{jkpq})_{jk} y_q, y_p\rangle \geqq 0.$$

Let $\mathscr{K}$ be the space spanned by $\{y_1, \ldots y_m\}$. Let Ψ be the effect of Φ followed by the compression of $\mathfrak{B}$ into $\mathscr{B}(\mathscr{K})$ and then the imbedding into $\mathfrak{M}_m$. It is

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evident that Ψ is n-positive. By the Lemma, there exist $\mathbf{m}$ and G_{jk} with the prescribed properties. Let

$$G_{jkpq} = G_{jk} \quad (1 \leqq p, q \leqq m).$$

Then

$$(G_{jkpq}(s))_{jkpq} \geqq 0 \quad \text{a.e. } (\mathbf{m}).$$

Hence

$$\begin{aligned}(f_{jkpq}(s) \cdot G_{jk}(s))_{jkpq} &= (f_{jkpq}(s) \cdot G_{jkpq}(s))_{jkpq} \\ &\geqq 0 \quad \text{a.e. } (\mathbf{m}).\end{aligned}$$

So

$$(\textstyle\sum_{jk} f_{jkpq}(s) \cdot G_{jk}(s))_{pq} \geqq 0 \quad \text{a.e. } (\mathbf{m}).$$

Therefore

$$(\Psi(f_{jkpq})_{jk})_{pq} = (\textstyle\int_{\mathscr{S}} \sum_{jk} f_{jkpq} \cdot G_{jk}\, \mathrm{d}\mathbf{m})_{pq} \geqq 0.$$

It follows that

$$\textstyle\sum_{pq} \langle \Phi(f_{jkpq})_{jk} y_q, y_p \rangle = \sum_{pq} \langle \Psi(f_{jkpq})_{jk} y_q, y_p \rangle \geqq 0$$

as required. Thus Theorem 8 is established.

Referring to Theorems 5–8, $\mathbf{P}_n = \mathbf{P}_{n+1} \Rightarrow \mathbf{P}_n = \mathbf{P}_\infty$ naturally. In the general case, we pose

Conjecture 1. $\mathbf{P}_n[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_{n+1}[\mathfrak{A}, \mathfrak{B}] \Rightarrow \mathbf{P}_n[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_\infty[\mathfrak{A}, \mathfrak{B}]$.

Hopefully, the above conjecture will be a corollary of a 'generalization of Theorem 4' which we state as

Conjecture 2. If $\mathbf{P}_n[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_{n+1}[\mathfrak{A}, \mathfrak{B}]$, then, either $\mathfrak{A}$ is a quotient space or $\mathfrak{B}$ is a subalgebra of $\mathfrak{M}_n(\mathfrak{C})$ for certain commutative $\mathfrak{C}$.

We remark that in the finite-dimensional case, every C^*-algebra is of the form $\mathfrak{M}_{n_1} \oplus \mathfrak{M}_{n_2} \oplus \ldots \oplus \mathfrak{M}_{n_m}$; hence Conjecture 2 in this case is valid by virtue of Theorem 1.

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