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Inverse Problems for Partition Functions

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Abstract. Let $p_w(n)$ be the weighted partition function defined by the generating function $\sum_{n=0}^{\infty} p_w(n)x^n = \prod_{m=1}^{\infty}(1-x^m)^{-w(m)}$, where $w(m)$ is a non-negative arithmetic function. Let $P_w(u) = \sum_{n\le u} p_w(n)$ and $N_w(u) = \sum_{n\le u} w(n)$ be the summatory functions for $p_w(n)$ and $w(n)$, respectively. Generalizing results of G. A. Freiman and E. E. Kohlbecker, we show that, for a large class of functions $\Phi(u)$ and $\lambda(u)$, an estimate for $P_w(u)$ of the form $\log P_w(u) = \Phi(u)\bigl\{1 + Ou\bigl(1/\lambda(u)\bigr)\bigr\}$ $(u \to \infty)$ implies an estimate for $N_w(u)$ of the form $N_w(u) = \Phi^*(u)\bigl\{1+O\bigl(1/\log\lambda(u)\bigr)\bigr\}$ $(u \to \infty)$ with a suitable function $\Phi^*(u)$ defined in terms of $\Phi(u)$. We apply this result and related results to obtain characterizations of the Riemann Hypothesis and the Generalized Riemann Hypothesis in terms of the asymptotic behavior of certain weighted partition functions.

0 Introduction

Let $w(n)$ be a non-negative function defined on the set of positive integers, and define $p_w(n)$ by the generating function identity

$$\sum_{n=0}^{\infty} p_w(n)x^n = \prod_{m=1}^{\infty}(1-x^m)^{-w(m)}. \tag{0.1}$$

The usual problem in the asymptotic theory of partition functions in this context is to deduce the asymptotic behavior of the function $p_w(n)$ or that of the summatory function

$$P_w(u) = \sum_{n\le u} p_w(n) \tag{0.2}$$

from that of the function

$$N_w(u) = \sum_{n\le u} w(n). \tag{0.3}$$

In the case when $w(n) = 1$ for all positive integers n, one sees that $p_w(n)$ is the ordinary partition function $p(n)$, and $N_w(u) = u + O(1)$. Hardy and Ramanujan [8] showed that as $n \to \infty$

$$\log p(n) \sim 2\sqrt{\zeta(2)n},$$

and later [9] improved this estimate to an asymptotic formula for $p(n)$ itself.

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More general results of this type have been obtained by many authors, including Brigham [2], Hardy and Ramanujan [8, 9], Ingham [10], Kohlbecker [12], Meinardus [13, 14], Richmond [18, 19] Roth and Szekeres [20], and Schwarz [21, 22]. Additional references can be found in Chapter 6 of Andrews [1].

The converse problem of deducing the asymptotic behavior of $N_w(u)$ from that of $p_w(n)$ has been less investigated. The earliest result of this type is the following result obtained by Erdős in 1942.

Theorem A **(Erdős [3])** *Suppose that $w(n)$ is the characteristic function of a subset S of positive integers. Then, as $u \to \infty$,*

$$N_w(u) = \#\{n \in S : n \leq u\} \sim Au \tag{0.4}$$

for some constant $A > 0$ if and only if, as $n \to \infty$,

$$\log p_w(n) \sim 2\sqrt{A\zeta(2)}n^{1/2}. \tag{0.5}$$

The deduction of (0.4) from (0.5) is an example of a so-called "inverse" result. More generally, an inverse problem in the asymptotic theory of partitions is a problem in which asymptotic information on $N_w(u)$ is to be deduced from the asymptotic behavior of $\log p_w(n)$ or $\log P_w(u)$. Generalizing Theorem A, Kohlbecker proved the following result in 1958.

Theorem B **(Kohlbecker [12])** *Suppose that b is a positive number and that $L(u)$ is a slowly oscillating function, that is, a positive function satisfying $L(cu)/L(u) \to 1$ as $u \to \infty$ for all $c > 0$. For large u let x_u be a positive number such that*

$$u = b\Gamma(1+b)\zeta(1+b)x_u^{-(1+b)}L(1/x_u)$$

and let $L^*(u)$ be a slowly oscillating function defined for large u such that $L^*(u) \sim (1 + 1/b)\{b\Gamma(1+b)\zeta(1+b)L(1/x_u)\}^{1/(1+b)}$. Then, as $u \to \infty$,

$$N_w(u) \sim u^b L(u)$$

if and only if

$$\log P_w(u) \sim u^{b/(1+b)}L^*(u).$$

For the case $b = 0$, an analogous result was obtained by Parameswaran [16]. More general results for this case have been given by Geluk [6, 7], whose main assumption is the condition

$$\lim_{t\to\infty} \limsup_{u\to\infty} \frac{N_w(tu)}{N_w(u)} < \infty.$$

In 1955 Freiman obtained the following inverse result with remainder terms.

Theorem C **(Freiman [4])** *Suppose that, as $u \to \infty$,*

$$\log P_w(u) = Au^a + O(u^{a_1}) \tag{0.6}$$

for some constants $A > 0$ and $0 < a_1 < a < 1$. Then, as $u \to \infty$,

$$N_w(u) = Bu^b + O\Big(\frac{u^b}{\log u}\Big), \tag{0.7}$$

where

$$b = \frac{a}{1-a} \quad \textit{and} \quad B = \frac{A^{b/a}a^b(1-a)}{\Gamma(b/a)\zeta(b/a)}.$$

Kohlbecker's result allows more general main terms than Freiman's result, but it does not give explicit error terms as in (0.8).

We note that the relative error term in (0.7) is much larger than that in estimate (0.6). This raises the question whether one can sharpen (0.7) to an estimate of the form

$$N_w(u) = Bu^b + O_\epsilon(u^{b_1+\epsilon}) \tag{0.8}$$

with $b_1 < b$. However, this is not the case; in his paper [4], Freiman constructed an example showing that the error term in (0.7) is best possible.

In [25], we showed that in the case $w(n) = \Lambda(n)$ (where $\Lambda(n)$ is the von Mangoldt function) the relation (0.6) does implies (0.8). More precisely, Theorem 3 of [25] implies the following inverse result.

Theorem D **(Yang [25])** *Suppose that θ is a positive constant such that for all $\epsilon > 0$, as $n \to \infty$,*

$$\log p_\Lambda(n) = 2\sqrt{\zeta(2)}n^{1/2} + O_\epsilon(n^{(\theta+\epsilon)/2}). \tag{0.9}$$

Then for all $\epsilon > 0$, as $u \to \infty$,

$$N_\Lambda(u) = u + O_\epsilon(u^{\theta+\epsilon}).$$

(In [25], the conclusion is given in terms of zero-free regions for the zeta-function. However, it is well-known that zero-free regions for $\zeta(s)$ are equivalent to estimates for $N_\Lambda(u) = \sum_{n\le u}\Lambda(n)$.)

The main result in this paper is an inverse theorem for partition functions which represents a common generalization of the above results. We will also show that, under additional assumptions on the weight function $w(n)$, conclusion (0.7) in Freiman's theorem can be improved to (0.8).

Notation and Conventions Throughout this paper, we assume that $w(n)$ is a non-negative arithmetic function, and we let $p_w(n)$, $P_w(u)$ and $N_w(u)$ be defined by (0.1), (0.2), and (0.3).

We use the notations $f(u) = O\big(g(u)\big)$ and $f(u) \ll g(u)$ interchangeably to mean that $|f(u)| \le cg(u)$ holds with some constant c for all u in the range under consideration. The constant c here is allowed to depend on other parameters, but any dependence will be explicitly indicated by writing O_k, $\ll_\epsilon$, *etc.* We write $f(u) \asymp g(u)$ if both $f(u) \ll g(u)$ and $g(u) \ll f(u)$ hold.

We will use the notation "$f(u) \searrow 0$ as $u \to \infty$" to mean that "$f(u)$ is monotonic *for sufficiently large u* and $f(u)$ tends to 0 as $u \to \infty$". Similar notations such as "$f(u) \nearrow \infty$ as $u \to \infty$" and "$g(x) \searrow 0$ as $x \to 0$" are to be interpreted analogously.

1 Definitions and Statements of Results

We first define several notations which will be used throughout this paper. For a non-negative and non-decreasing function $\Phi(u)$ defined on $(0,\infty)$ and satisfying $-\int_0^\infty \log(1-e^{-xu})\,d\Phi(u) < \infty$ for all $x > 0$, we define transforms $\hat{\Phi}(x)$ and $\tilde{\Phi}(x)$ by

$$\hat{\Phi}(x) = \int_0^\infty e^{-xu}\,d\Phi(u) \tag{1.1}$$

and

$$\tilde{\Phi}(x) = -\int_0^\infty \log(1-e^{-xu})\,d\Phi(u). \tag{1.2}$$

These two transforms are related by the identities

$$\tilde{\Phi}(x) = \sum_{k=1}^\infty \frac{1}{k}\hat{\Phi}(kx), \quad \hat{\Phi}(x) = \sum_{k=1}^\infty \frac{\mu(k)}{k}\tilde{\Phi}(kx). \tag{1.3}$$

The first of these relations follows from the Maclaurin series for $-\log(1-z)$, while the second follows easily from the identities

$$\sum_{kd=n} \frac{1}{d}\frac{\mu(k)}{k} = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

and

$$r = -\sum_{k=1}^\infty \frac{\mu(k)}{k}\log(1-r^k) \quad (0 < r < 1),$$

where $\mu(k)$ is the Möbius function.

For a non-negative and non-increasing function $\phi(u)$ satisfying $\phi(u) \to 0$ as $u \to \infty$, we define the generalized inverse function $\phi^{\leftarrow}(x)$ for $\phi(u)$ by

$$\phi^{\leftarrow}(x) = \max(0, \inf\{u : \phi(u) \le x\}). \tag{1.4}$$

Finally, we define a class of pairs of functions that generalizes the pairs $u^b L(u)$ and $u^{b/(1+b)} L^*(u)$ occurring in Theorem B.

Definition A pair $\bigl(\Phi(u), \Phi^*(u)\bigr)$ of non-negative and locally integrable functions defined on $(0,\infty)$ is called *admissible* if there exist constants $0 < \alpha < 1$ and $\gamma > \beta > \alpha$ such that the following conditions are satisfied:

(A1) $\Phi(u)$ is differentiable and $\phi(u) = \Phi'(u)$ is non-increasing;

(A2) $$\Phi'(u)u^{1-\alpha} \searrow 0 \quad \text{as } u \to \infty;$$

(A3) $$\Phi^*(u)u^{-\beta} \nearrow \infty \text{ and } \Phi^*(u)u^{-\gamma} \searrow 0 \quad \text{as } u \to \infty;$$

(A4) $$\tilde{\Phi}^*(x) \sim \int_x^\infty \phi^{\leftarrow}(y)\,dy \quad \text{as } x \to 0.$$

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Furthermore, if (A4) holds in the quantitative form

$$\text{(A4}'\text{)} \qquad \tilde{\Phi}^*(x) = \left(1 + O\left(\frac{1}{\lambda(1/x)}\right)\right) \int_x^\infty \phi^{\leftarrow}(y)\,dy,$$

where $\lambda(u) \to \infty$ as $u \to \infty$, then the pair $\big(\Phi(u), \Phi^*(u)\big)$ is called *admissible with accuracy* $\lambda(u)$.

A simple example of an admissible pair of functions (see the proof of Theorem 2) is

$$\text{(1.5)} \qquad \Phi(u) = Au^a, \quad \Phi^*(u) = Bu^b,$$

where $A > 0$ and $0 < a < 1$ are constants and

$$b = \frac{a}{1-a}, \quad B = \frac{A^{1/(1-a)} a^{a/(1-a)} (1-a)}{\Gamma\big(1/(1-a)\big)\,\zeta\big(1/(1-a)\big)}.$$

We note that this pair of functions is admissible with infinite accuracy in the sense that $\tilde{\Phi}^*(x)$ is equal to the integral in (A4). Another instance of an admissible pair is

$$\Phi(u) = 2\pi\sqrt{\frac{u}{3\log u}}, \quad \Phi^*(u) = \frac{u}{\log u},$$

which appears in the study of the partition function $p(n; \mathcal{P})$, the number of partitions of n into primes (see [8]). In this case one can show that the accuracy is $\log u / \log\log u$. In Section 4 we will describe a method for finding suitable functions $\Phi^*(u)$ for a large class of functions $\Phi(u)$.

Our first result can now be stated as follows.

Theorem 1 *Let* $\big(\Phi(u), \Phi^*(u)\big)$ *be admissible with accuracy* $\lambda(u)$*, where* $\lambda(u)$ *is a differentiable function satisfying the conditions:*

$$\text{(T1)} \qquad \lambda(u) \nearrow \infty \quad \text{as } u \to \infty;$$

$$\text{(T2)} \qquad \frac{\Phi(u)}{\lambda(u)} (\log u)^{-1} \nearrow \infty \quad \text{as } u \to \infty.$$

Suppose that, as $u \to \infty$*,*

$$\text{(1.6)} \qquad \log P_w(u) = \Phi(u)\left\{1 + O\left(\frac{1}{\lambda(u)}\right)\right\}.$$

Then, as $u \to \infty$*,*

$$\text{(1.7)} \qquad N_w(u) = \sum_{n \le u} w(n) = \Phi^*(u)\left\{1 + O\left(\frac{1}{\log\lambda(u)}\right)\right\}.$$

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The proof of Theorem 1 consists of three steps. In the first step we use an Abelian result (Proposition 1) to relate estimates for $\log P_w(u)$ to estimates for the logarithm of the Laplace transform

$$\hat{P}_w(x) = \int_0^\infty e^{-xu}\, dP_w(u) = \prod_{n=1}^\infty (1 - e^{-nx})^{-w(n)}. \tag{1.8}$$

We then observe that

$$\log \hat{P}_w(x) = -\sum_{n=1}^\infty w(n) \log(1 - e^{-nx}) = -\int_0^\infty \log(1 - e^{-xu})\, dN_w(u) = \tilde{N}_w(x),$$

and we show that an estimate for $\tilde{N}_w(x)$ will lead to a corresponding estimate for the Laplace transform $\hat{N}_w(x)$ of $N_w(u)$. Finally, we appeal to a Tauberian theorem of Omey (Proposition 2) to obtain the desired estimate for $N_w(u)$.

By specializing $\bigl(\Phi(u), \Phi^*(u)\bigr)$ to the pair (1.5), we will prove the following generalization of Theorem C.

Theorem 2 *Suppose that, as $u \to \infty$,*

$$\log P_w(u) = Au^a \left\{ 1 + O\left(\frac{1}{\lambda(u)} \right) \right\} \tag{1.9}$$

for some constants $A > 0$ and $0 < a < 1$, where $\lambda(u)$ is a positive differentiable function such that $\lambda(u) \nearrow \infty$ and $u^{-a}\lambda(u) \log u \searrow 0$ as $u \to \infty$. Then, as $u \to \infty$,

$$N_w(u) = \sum_{n \le u} w(n) = Bu^b \left\{ 1 + O\left(\frac{1}{\log \lambda(u)} \right) \right\}, \tag{1.10}$$

where

$$b = \frac{a}{1-a}, \quad B = \frac{A^{1/(1-a)} a^{a/(1-a)} (1-a)}{\Gamma\bigl(1/(1-a)\bigr)\, \zeta\bigl(1/(1-a)\bigr)}. \tag{1.11}$$

Our next theorem shows that, under additional conditions on the Dirichlet series

$$f_w(s) = \sum_{n=1}^\infty \frac{w(n)}{n^s} \tag{1.12}$$

generated by $w(n)$, the conclusion (0.7) in Theorem C can be sharpened to (0.8). Our conditions on $f_w(s)$ are reminiscent of those in Meinardus [13]. However, Meinardus' conditions are significantly more restrictive, requiring $f_w(s)$ to have an analytic continuation, except for a simple pole, to a region of the form $\operatorname{Re} s > -C$ with a positive constant C. In particular, Meinardus' conditions are not satisfied when $w(n)$ is the von Mangoldt function $\Lambda(n)$. By contrast, the conditions in Theorem 3 do apply to this case (see Corollary 1 below).

Theorem 3

(i) *Suppose that, as $u \to \infty$,*

$$\log P_w(u) = Au^a + O(u^{a_1}) \tag{1.13}$$

for some constants $A > 0$ and $0 < a_1 < a < 1$. Then $f_w(s)$ has a representation

$$f_w(s) = \frac{1}{\Gamma(s)\zeta(1+s)}\left\{\frac{D}{s-b} + h_w(s)\right\}, \tag{1.14}$$

where

$$b = \frac{a}{1-a}, \quad D = A^{1/(1-a)}a^{a/(1-a)}(1-a) \tag{1.15}$$

and $h_w(s)$ has an analytic continuation to the half-plane $\{s : \operatorname{Re} s > a_1/(1-a)\}$.

(ii) *In addition to the hypotheses of part (i), suppose that for all $\epsilon > 0$, as $n \to \infty$,*

$$w(n) \ll_\epsilon n^{a_1/(1-a)+\epsilon} \tag{1.16}$$

and that for all $\epsilon, \delta > 0$

$$|f_w(\sigma + it)| \ll_{\delta,\epsilon} t^\epsilon \tag{1.17}$$

uniformly for $\sigma \geq a_1/(1-a) + \delta$ and sufficiently large t. Then we have, for every $\epsilon > 0$, as $u \to \infty$,

$$N_w(u) = \sum_{n \leq u} w(n) = Bu^b + O(u^{b_1+\epsilon}), \tag{1.18}$$

where B and b are defined by (1.11) and $b_1 = a_1/(1-a)$.

Remark The error term in (1.13) corresponds to the choice $\lambda(u) = u^{a-a_1}$ in (1.9). Theorem 3 suggests that, under similar assumptions, hypothesis (1.9) implies an estimate of the form

$$N_w(u) = Bu^b\left\{1 + O\left(\frac{1}{\lambda(u^c)}\right)\right\} \quad (u \to \infty)$$

for more general classes of functions $\lambda(u)$. However, this is not the case. For example, in the case when $w(n) = \Lambda(n)$, it is easy to see that the function $w(n)$ satisfies conditions similar to (1.16) and (1.17). On the other hand, using the methods of [25], one can show that (1.9) holds with

$$\lambda(u) = \exp\left\{\frac{c_1 \log u}{(\log\log u)^{\alpha+\epsilon}}\right\}, \quad \epsilon > 0, \ c_1 = c_1(\epsilon) > 0,$$

if and only if

$$N_\Lambda(u) = u + O_\epsilon\Big(u \exp\big\{-c_2(\log u)^{1/(1+\alpha)-\epsilon}\big\}\Big), \quad \epsilon > 0,\ c_2 = c_2(\epsilon) > 0.$$

As applications of Theorem 3 we will prove the following results which give characterizations of the Riemann Hypothesis and related hypotheses in terms of the asymptotic behavior of certain partition functions. This kind of characterizations, to the author's best knowledge, was first formulated in [25].

Corollary 1 *Let k be a positive integer, and for any integer l with $(k,l) = 1$ let*

$$w_{k,l}(n) = \begin{cases} \Lambda(n), & \text{if } n \equiv l \bmod k, \\ 0, & \text{else.} \end{cases}$$

Then, for any fixed number θ with $1/2 \le \theta < 1$, the Dirichlet L-function $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ has no zeros in the half-plane $\{s : \operatorname{Re} s > \theta\}$ for all characters χ modulo k if and only if, for all l with $(k,l) = 1$ and all $\epsilon > 0$, as $u \to \infty$,

$$\log P_{w_{k,l}}(u) = 2\sqrt{\frac{\zeta(2)}{\varphi(k)}}\, u^{1/2} + O_{\epsilon,k,l}(u^{\theta/2+\epsilon}). \tag{1.19}$$

In particular, the Generalized Riemann Hypothesis is true if and only if, for every $\epsilon > 0$ and all positive integers k, l satisfying $(k,l) = 1$, as $u \to \infty$,

$$\log P_{w_{k,l}}(u) = 2\sqrt{\frac{\zeta(2)}{\varphi(k)}}\, u^{1/2} + O_{\epsilon,k,l}(u^{1/4+\epsilon}).$$

In the special case when $k = l = 1$, this result reduces to the case $\theta = 1/2$ of Theorem D, showing the equivalence between (0.9) and the Riemann Hypothesis.

Corollary 2 *Let $w(n)$ be the characteristic function of the set of positive integers with an even number of prime factors. Then, for any fixed number θ with $1/2 \le \theta < 1$, the Riemann zeta function has no zeros in the half-plane $\{s : \sigma > \theta\}$ if and only if for all $\epsilon > 0$, as $u \to \infty$,*

$$\log P_w(u) = 2\sqrt{\zeta(2)}\, u^{1/2} + O_\epsilon(u^{\theta/2+\epsilon}). \tag{1.20}$$

Corollary 3 *Let k be a positive integer, and let $w(n)$ be defined by*

$$w(n) = \begin{cases} \Lambda(m), & \text{if } n = m^k, \\ 0, & \text{else.} \end{cases}$$

Then the Riemann Hypothesis is true if and only if

$$\log P_w(u) = A_k u^{1/(k+1)} + O_\epsilon(u^{1/(2(k+1))+\epsilon}) \tag{1.21}$$

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as $u \to \infty$ *for every* $\epsilon > 0$, *where*

$$A_k = \left\{ \frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}^{k/(k+1)}(k+1). \tag{1.22}$$

The remainder of the paper is organized as follows. In Section 2 we state the Abelian and Tauberian results mentioned above. In Section 3 we complete the proof of Theorem 1. In Section 4 we prove Theorem 2 and briefly discuss how to find admissible functions for general cases. In Section 5 we prove several auxiliary results which relate the behavior of the Laplace transform $\hat{P}_w(x)$ for $P_w(u)$ to the analytic properties of the Dirichlet series $f_w(s)$ defined by (1.12), and use these results to prove Theorem 3. In Section 6 we prove Corollaries 1–3.

2 Abelian and Tauberian Results

Our first proposition is an elementary Abelian result which generalizes Proposition 1 in [25]. This result also generalizes Theorem 10 in [5], which is a main ingredient in the proof of Geluk's result [7] mentioned above.

Proposition 1 *Suppose that* $P(u)$ *is a non-negative and non-decreasing function satisfying* $P(u) = O_\epsilon(e^{\epsilon u})$ *as* $u \to \infty$ *for every positive* ϵ. *Let*

$$\hat{P}(x) = \int_0^\infty e^{-xu}\, dP(u)$$

be the Laplace-Stieltjes transform of $P(u)$. *Suppose that, as* $u \to \infty$,

$$\log P(u) = \int_0^u \phi(v)\, dv + O\left(\int_0^u r(v)\, dv\right), \tag{2.1}$$

where $\phi(v)$ *and* $r(v)$ *are continuous, non-increasing functions defined on* $(0,\infty)$ *and satisfying the following conditions:*

$$\phi(v), r(v) \to 0 \quad \textit{as } v \to \infty; \tag{P1}$$

$$\phi(v)v, r(v)v \to \infty \quad \textit{as } v \to \infty; \tag{P2}$$

$$r(v) = o\big(\phi(v)\big) \quad \textit{as } v \to \infty; \tag{P3}$$

(P4) *there are constants* $c_1 > 1$ *and* $c_2 > 0$ *such that*

$$\phi(c_1 v) - \phi(v) \le -c_2\phi(v) \quad \textit{for all sufficiently large } v.$$

Then, as $x \to 0$,

$$\log \hat{P}(x) = \int_x^\infty \phi^{\leftarrow}(y)\, dy + O\left(\int_0^{\phi^{\leftarrow}(x)} r(v)\, dv\right) \tag{2.2}$$

where $\phi^{\leftarrow}(x) = \max(0,\ \inf\{u : \phi(u) \le x\})$.

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Proof Suppose that (2.1) holds as $u \to \infty$. Without loss of generality, we may assume that $P(0) = 0$. Thus, by the assumptions that $P(u)$ is non-decreasing and non-negative and that $P(u) = O_\epsilon(e^{\epsilon u})$, we have for all $x, u > 0$

$$\hat{P}(x) = \int_0^\infty e^{-xt}\, dP(t) = x \int_0^\infty e^{-xt} P(t)\, dt \tag{2.3}$$
$$\geq x \int_u^\infty e^{-xt} P(t)\, dt \geq x \int_u^\infty e^{-xt} P(u)\, dt = e^{-xu} P(u).$$

By (2.1), it follows that there is a positive constant C such that

$$\log \hat{P}(x) \geq -xu + \int_0^u \phi(v)\, dv - C \int_0^u r(v)\, dv \tag{2.4}$$

for all $x > 0$ and all sufficiently large u. We apply this with $u = u_x = \phi^{\leftarrow}(x)$, where $\phi^{\leftarrow}(x)$ is the generalized inverse function defined by (1.4), so that for sufficiently small x

$$u_x = \phi^{\leftarrow}(x) = \inf\{v : \phi(v) \leq x\}. \tag{2.5}$$

We note that since $\phi(v)$ is continuous and monotonic for sufficiently large v, we have

$$\phi(u_x) = x. \tag{2.6}$$

By considering the graph of $\phi(v)$, we see that

$$-x\phi^{\leftarrow}(x) + \int_0^{\phi^{\leftarrow}(x)} \phi(v)\, dv = \int_x^\infty \phi^{\leftarrow}(y)\, dy. \tag{2.7}$$

It follows from (2.4) and (2.5) that, for all sufficiently small x,

$$\log \hat{P}(x) \geq \int_x^\infty \phi^{\leftarrow}(y)\, dy - C \int_0^{\phi^{\leftarrow}(x)} r(v)\, dv.$$

This proves the lower bound in (2.2). It remains to obtain the corresponding upper bound.

Let c_1 and c_2 be constants appearing in condition (P4), and let u_0 be a constant such that condition (P4) is satisfied for $v \geq u_0$. Given a small positive number x, we let u_x be defined by (2.5) and assume that x is small enough that $u_x \geq c_1 u_0$. We write

$$\hat{P}(x) = x\Big\{ \int_0^{u_0} + \int_{u_0}^{u_x/c_1} + \int_{u_x/c_1}^{c_1 u_x} + \int_{c_1 u_x}^\infty \Big\} e^{-xt} P(t)\, dt \tag{2.8}$$
$$= x\{I_1 + I_2 + I_3 + I_4\}.$$

We will show that, for $j = 1, \ldots, 4$, we have

$$xI_j \ll \exp\Big\{ -xu_x + \int_0^{u_x} \phi(v)\, dv + C_j \int_0^{u_x} r(v)\, dv \Big\} \tag{2.9}$$

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with suitable constants C_j. By (2.5) and (2.7), this implies the desired estimate (2.2).

The integral I_1 is bounded by $\int_0^{u_0} P(t)\,dt$ and is thus $O(1)$. To estimate I_4, we define functions $\psi_0(t)$ and $\psi(t)$ for $t \geq u_0$ by

$$\psi_0(t) = -xt + \int_0^t \phi(v)\,dv, \quad \psi(t) = \psi_0(t) + C\int_0^t r(v)\,dv,$$

where C is the constant implicit in the error term of (2.1). Thus (2.1) implies that

$$P(u) \leq e^{xu+\psi(u)} \tag{2.10}$$

for sufficiently large u. Since $\phi(t)$ is non-increasing, the function $\psi_0'(t) = -x + \phi(t)$ is non-increasing as well. In particular, by (2.6), one has $\psi_0'(t) \geq -x + \phi(u_x) = 0$ for $0 < t < u_x$ and $\psi_0'(t) \leq -x + \phi(u_x) = 0$ for $t > u_x$. Thus, by the assumption that $r(t)$ is non-increasing, we have for all $t \geq c_1 u_x$

$$\psi'(t) = \psi_0'(t) + Cr(t) \leq \psi_0'(c_1 u_x) + Cr(t) = -x + \phi(c_1 u_x) + Cr(u_x).$$

By condition (P4) and (2.6) it follows that for all $t \geq c_1 u_x$

$$\psi'(t) \leq -x + \phi(u_x) - c_2\phi(u_x) + Cr(u_x) = -c_2\phi(u_x) + Cr(u_x).$$

Since, by condition (P3), $r(u_x) = o\big(\phi(u_x)\big)$, this implies that

$$\psi'(t) \leq -\frac{c_2}{2}\phi(u_x) = -\frac{c_2}{2}x$$

for all sufficiently small x and all $t \geq c_1 u_x$. In view of (2.10), it follows that

$$\begin{aligned} I_4 = x\int_{c_1 u_x}^\infty e^{-xt}P(t)\,dt &\leq x\int_{c_1 u_x}^\infty e^{\psi(t)}\,dt \ll \int_{c_1 u_x}^\infty e^{\psi(t)}\big(-\psi'(t)\big)\,dt \\ &= e^{\psi(c_1 u_x)} = \exp\Big\{\psi_0(c_1 u_x) + C\int_0^{c_1 u_x} r(v)\,dv\Big\} \end{aligned}$$

for all sufficiently small x. Since $\psi_0(t)$ is decreasing for $t > u_x$, we have $\psi_0(c_1 u_x) \leq \psi_0(u_x)$. Moreover, the assumption that $r(v)$ is non-increasing implies that

$$\int_0^{c_1 u_x} r(v)\,dv \leq c_1 \int_0^{u_x} r(v)\,dv, \tag{2.11}$$

and we conclude that

$$\begin{aligned} xI_4 &\ll \exp\Big\{\psi_0(u_x) + Cc_1\int_0^{u_x} r(v)\,dv\Big\} \\ &\ll \exp\Big\{-xu_x + \int_0^{u_x}\phi(v)\,dv + Cc_1\int_0^{u_x} r(v)\,dv\Big\}. \end{aligned}$$

Thus (2.9) holds as $x \to 0$ for $j = 4$ with $C_4 = Cc_1$.

By a similar argument one sees that (2.9) also holds for $j = 2$ with $C_2 = Cc_1$. It remains to estimate xI_3.

We first recall that $t = u_x$ maximizes $\psi_0(t) = -xt + \int_0^t \phi(v)\,dv$. Then by (2.10) we have

$$
\begin{aligned}
xI_3 = x\int_{u_x/c_1}^{c_1u_x} e^{-xt}P(t)\,dt &\le x\int_{u_x/c_1}^{c_1u_x} \exp\Big\{\psi_0(t) + C\int_0^t r(v)\,dv\Big\}\,dt \\
&\ll xu_x \exp\Big\{\psi_0(u_x) + C\int_0^{c_1u_x} r(v)\,dv\Big\}
\end{aligned}
$$

for all sufficiently small x. Furthermore, by conditions (P1), (P2) and (2.6), we have $\log xu_x \ll \log u_x = o(\int_0^{u_x} r(v)\,dv)$. Therefore, by (2.11), the last expression is

$$
\ll \exp\Big\{\psi_0(u_x) + (Cc_1 + 1)\Big(\int_0^{u_x} r(v)\,dv\Big)\Big\}.
$$

Hence (2.9) holds as $u \to \infty$ for $j = 3$ with $C_3 = Cc_1 + 1$. This completes the proof of Proposition 1.

We next quote a Tauberian result due to Omey [15] which will play a crucial role in the proof of Theorem 1.

Proposition 2 **(Omey)** *Suppose that $\Psi(u)$ is a non-negative and monotonically increasing function such that $\hat\Psi(x) = \int_0^\infty e^{-xu}\,d\Psi(u) < \infty$ for all $x > 0$. Suppose that $\Theta(u)$ and $\xi(u)$ are positive functions and $A, B, \beta > 0$ and $c_1, c_2 > 1$ are constants such that the following conditions are satisfied:*

(O1) $\quad \Theta(u)u^{-\beta} \searrow 0 \quad$ *as $u \to \infty$;*

(O2) $\quad c_1\Theta(u) \le 2\Theta(2u) \quad$ *for sufficiently large u;*

(O3) $\quad \hat\Theta(x) \le A + B\Theta(1/x) \quad$ *for all $x > 0$;*

(O4) $\quad \xi(u) \to \infty$ *and* $\xi(2u) < c_2\xi(u) \quad$ *for all sufficiently large u.*

Suppose that, as $x \to 0$,

$$
\hat\Psi(x) = \hat\Theta(x) + O\Big(\frac{\Theta(1/x)}{\xi(1/x)}\Big).
$$

Then as $u \to \infty$ one has

$$
\Psi(u) = \Theta(u) + O\Big(\frac{\Theta(u)}{\log \xi(u)}\Big).
$$

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3 Proof of Theorem 1

We first establish some elementary properties of the function $\Phi(u)$ occurring in Theorem 1. We recall that, by convention, the symbols $\searrow$ and $\nearrow$ mean that the functions involved are *eventually* non-increasing and non-decreasing, respectively.

Lemma 1 *Let $\phi(u)$ be a continuous and non-increasing function. Let $\Phi(u) = \int_0^u \phi(v)\,dv$, let $\phi^{\leftarrow}(x)$ be the generalized inverse function given by (1.4) and set $\Phi^{\leftarrow}(x) = \int_x^\infty \phi^{\leftarrow}(y)\,dy$. Suppose that there exists a constant $0 < \alpha < 1$ such that*

$$\phi(u)u^{1-\alpha} \searrow 0 \quad \text{as } u \to \infty. \tag{3.1}$$

Then we have

$$\Phi(u)u^{-\alpha} \searrow 0 \quad \text{as } u \to \infty \tag{3.2}$$

and

$$\phi^{\leftarrow}(x)x^{1/(1-\alpha)} \searrow 0 \text{ and } \Phi^{\leftarrow}(x)x^{\alpha/(1-\alpha)} \searrow 0 \quad \text{as } x \to 0. \tag{3.3}$$

Proof By (3.1) we have $\phi(u) = o(u^{\alpha-1})$ as $u \to \infty$. Consequently, we have $\Phi(u) = \int_0^u \phi(v)\,dv = o(u^\alpha)$ as $u \to \infty$. To complete the proof of (3.2), it suffices to show that $u^{-\alpha}\Phi(u)$ is non-increasing for sufficiently large u.

By (3.1), there exists a positive constant u_0 such that $\phi(u)u^{1-\alpha}$ is non-increasing for $u \ge u_0$. Then one has $\phi(u)u^{1-\alpha} \le \phi(v)v^{1-\alpha}$ for all $u \ge v \ge u_0$. Multiplying by $v^{\alpha-1}$ and integrating from u_0 to u, we obtain

$$\frac{1}{\alpha}\phi(u)u^{1-\alpha}(u^\alpha - u_0^\alpha) \le \Phi(u) - \Phi(u_0).$$

It follows that

$$u^{\alpha+1}\frac{d}{du}\big(\Phi(u)u^{-\alpha}\big) = \phi(u)u - \alpha\Phi(u) \le \phi(u)u^{1-\alpha}u_0^\alpha - \alpha\Phi(u_0),$$

which, by (3.1), is strictly negative when u is sufficiently large. This proves assertion (3.2).

To prove (3.3), we first observe that (3.1) implies that $c^{1-\alpha}\phi(cu) \le \phi(u)$ for $c \ge 1$ and all sufficiently large u. Replacing c with $c^{1/(1-\alpha)}$ and setting $x = \phi(u)$, we obtain $\phi\big(c^{1/(1-\alpha)}\phi^{\leftarrow}(x)\big) \le c^{-1}x$ for $c \ge 1$ and all sufficiently small x. Note that $\phi^{\leftarrow}(x)$ is non-increasing for sufficiently small x since $\phi(u)$ is non-increasing for sufficiently large u. Therefore we have $\phi^{\leftarrow}(c^{-1}x) \le c^{1/(1-\alpha)}\phi^{\leftarrow}(x)$, or equivalently, $(c^{-1}x)^{1/(1-\alpha)}\phi^{\leftarrow}(c^{-1}x) \le x^{1/(1-\alpha)}\phi^{\leftarrow}(x)$. Thus the function $x^{1/(1-\alpha)}\phi^{\leftarrow}(x)$ is non-decreasing when x is sufficiently small. We now show that $x^{1/(1-\alpha)}\phi^{\leftarrow}(x) \to 0$ as $x \to 0$. Given a real number $\epsilon \in (0,1)$, let u_ϵ be a positive number such that $\phi(u)u^{1-\alpha} \le \phi(u_\epsilon)u_\epsilon^{1-\alpha} \le \epsilon$ for all $u \ge u_\epsilon$. Then we have $\phi(u_\epsilon) \le \epsilon u_\epsilon^{\alpha-1}$ and thus $u_\epsilon = \phi^{\leftarrow}\big(\phi(u_\epsilon)\big) \ge \phi^{\leftarrow}(\epsilon u_\epsilon^{\alpha-1})$. Hence, for any $0 < x \le \epsilon u_\epsilon^{\alpha-1}$ one has

$$x^{1/(1-\alpha)}\phi^{\leftarrow}(x) \le (\epsilon u_\epsilon^{\alpha-1})^{1/(1-\alpha)}\phi^{\leftarrow}(\epsilon u_\epsilon^{\alpha-1}) \le \epsilon^{1/(1-\alpha)}.$$

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It follows that $x^{1/(1-\alpha)}\phi^{\leftarrow}(x) \to 0$ as $x \to 0$, which implies that

$$\Phi^{\leftarrow}(x) = o\Big(\int_x^\infty y^{-1/(1-\alpha)}\,dy\Big) = o(x^{\alpha/(1-\alpha)}).$$

Finally, using an argument similar to that used in the proof of (3.2), we see that $\Phi^{\leftarrow}(x)x^{a/(1-a)}$ is monotonic for sufficiently small x. This completes the proof of (3.3).

Lemma 2 *Let $\phi(u)$, $\Phi(u)$, $\phi^{\leftarrow}(x)$ and $\Phi^{\leftarrow}(x)$ be given as in Lemma 1. Suppose that there is a constant $0 < \alpha < 1$ such that (3.1) holds. Then we have, as $x \to \infty$,*

$$\Phi\big(\phi^{\leftarrow}(x)\big) \asymp \Phi^{\leftarrow}(x).$$

Proof It suffices to show that $\Phi(u) \asymp \Phi^{\leftarrow}\big(\phi(u)\big)$ as $u \to \infty$. By considering the graph of $\phi(u)$, one sees that, for all $u \geq 0$,

$$\Phi^{\leftarrow}\big(\phi(u)\big) = \int_{\phi(u)}^\infty \phi^{\leftarrow}(y)\,dy = -\phi(u)u + \Phi(u) \tag{3.4}$$

and hence $\Phi^{\leftarrow}\big(\phi(u)\big) \leq \Phi(u)$. It remains to show that $\Phi^{\leftarrow}\big(\phi(u)\big) \gg \Phi(u)$ as $u \to \infty$. By Lemma 1, $\Phi(u)u^{-\alpha}$ is decreasing for all sufficiently large u. Differentiating $\Phi(u)u^{-\alpha}$ and then multiplying by $u^{1+\alpha}$, we obtain $\phi(u)u - \alpha\Phi(u) \leq 0$. By (3.4), it follows that

$$(1-\alpha)\Phi(u) \leq -\phi(u)u + \Phi(u) = \Phi^{\leftarrow}\big(\phi(u)\big)$$

for all sufficiently large u. Thus, $\Phi^{\leftarrow}\big(\phi(u)\big) \gg \Phi(u)$ and proof of the lemma is complete.

Lemma 3 *Suppose that $\Psi(u)$ is a non-negative and non-decreasing function defined on $(0,\infty)$ with a finite transform $\tilde{\Psi}(x) = -\int_0^\infty \log(1-e^{-xu})\,d\Psi(u)$ for all $x > 0$. Suppose that there are constants $0 < \beta < \gamma$ such that $\Psi(u)u^{-\beta} \nearrow \infty$ and $\Psi(u)u^{-\gamma} \searrow 0$ as $u \to \infty$. Then we have, as $x \to 0$,*

$$\tilde{\Psi}(x) \asymp \Psi(1/x).$$

Proof Let $u_0 > 0$ be a constant such that $\Psi(u)u^{-\beta}$ is non-decreasing and $\Psi(u)u^{-\gamma}$ is non-increasing for $u \geq u_0$. We write

$$\tilde{\Psi}(x) = \int_0^\infty \big(-\log(1-e^{-xu})\big)\,d\Psi(u) = \int_0^{u_0} + \int_{u_0}^\infty = I_1 + I_2.$$

For $x \leq 1/(2u_0)$ and $u \leq u_0$, we have $-\log(1-e^{-xu}) \asymp \log(xu)^{-1}$. It follows that

$$I_1 \asymp \int_0^{u_0} \log(xu)^{-1}\,d\Psi(u) \asymp \log x^{-1}.$$

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On the other hand, integrating by parts, we obtain

$$I_2 = \tilde{\Psi}(u_0)\log(1 - e^{-xu_0}) + \int_{u_0}^{\infty} \frac{x\Psi(u)}{e^{xu} - 1}\, du = O(\log x^{-1}) + J.$$

By the assumption that $\Psi(u)u^{-\beta}$ is non-decreasing for $u \geq u_0$, we see that, as $x \to 0$,

$$J \geq \Psi(u_0)u_0^{-\beta} \int_{u_0}^{\infty} \frac{xu^{\beta}}{e^{xu} - 1}\, du \gg x^{-\beta}.$$

Therefore, one has $\log x^{-1} = o(J)$. Hence, to prove the lemma, it suffices to show that $J \asymp \Psi(1/x)$.

Since $\Psi(u)$ is non-decreasing, as $x \to 0$ we have

$$J \geq \int_{1/x}^{\infty} \frac{x\Psi(u)}{e^{xu} - 1}\, du \geq \Psi(1/x) \int_{1/x}^{\infty} xe^{-xu}\, du \gg \Psi(1/x).$$

To obtain an analogous upper bound, we write

$$J = \int_{u_0}^{1/x} + \int_{1/x}^{\infty} = J_1 + J_2.$$

Since $\Psi(u)u^{-\beta}$ is non-decreasing and $\Psi(u)u^{-\gamma}$ is non-increasing for $u \geq u_0$, we have, for sufficiently small x,

$$\frac{J_1}{\Psi(1/x)} = \int_{u_0}^{1/x} \frac{x}{e^{xu} - 1} \cdot \frac{\Psi(u)}{\Psi(1/x)}\, du \leq \int_{u_0}^{1/x} \frac{x(ux)^{\beta}}{e^{xu} - 1}\, du \leq \int_{0}^{\infty} \frac{v^{\beta}}{e^{v} - 1}\, dv = C_{\beta}$$

and

$$\frac{J_2}{\Psi(1/x)} = \int_{1/x}^{\infty} \frac{x}{e^{xu} - 1} \cdot \frac{\Psi(u)}{\Psi(1/x)}\, du \leq \int_{1/x}^{\infty} \frac{x(ux)^{\gamma}}{e^{xu} - 1}\, du \leq \int_{0}^{\infty} \frac{v^{\gamma}}{e^{v} - 1}\, dv = C_{\gamma},$$

where C_β and C_γ are constants depending only on β and γ. It follows that $J = J_1 + J_2 \ll \Psi(1/x)$. This proves the lemma.

The next result will be used in the second step of the proof of Theorem 1.

Proposition 3 *Let $w(n)$ and $N_w(u) = \sum_{n \leq u} w(n)$ be given as in the statement of Theorem 1. Suppose that $\Psi(u)$ is a non-negative and non-decreasing function with a finite transform $\tilde{\Psi}(x) < \infty$ for all $x > 0$, and suppose that $\gamma > \beta > 0$ are constants such that, as $u \to \infty$,*

$$\Psi(u)u^{-\beta} \nearrow \infty \quad \text{and} \quad \Psi(u)u^{-\gamma} \searrow 0. \tag{3.5}$$

Suppose that $R(x)$ is a non-negative function and $\delta > 0$ is a constant such that, as $x \to 0$,

$$R(x)x^{\delta} \nearrow \infty \tag{3.6}$$

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and

$$\tilde{N}_w(x) = \tilde{\Psi}(x) + O\bigl(R(x)\bigr). \tag{3.7}$$

Then we have, as $x \to 0$,

$$\hat{N}_w(x) = \hat{\Psi}(x) + O\bigl(R(x) + x^{\beta}\Psi(1/x)\bigr). \tag{3.8}$$

Proof Let $u_0 > 0$ be a constant such that $\Psi(u)u^{-\gamma}$ is monotonic for $u \geq u_0$, and let $x_0 > 0$ and $C > 0$ be constants such that

$$|\tilde{N}_w(x) - \tilde{\Psi}(x)| \leq CR(x) \tag{3.9}$$

for $0 < x \leq x_0$ and such that $R(x)x^{\delta}$ is monotonic in this range. Without loss of generality, we may assume that

$$u_0 = \frac{1}{2x_0}. \tag{3.10}$$

We define $N_0(u)$ and $\Psi_0(u)$ by

$$N_0(u) = \begin{cases} N_w(u), & \text{if } u \geq u_0, \\ 0, & \text{if } 0 < u < u_0, \end{cases} \qquad \Psi_0(u) = \begin{cases} \Psi(u), & \text{if } u \geq u_0, \\ 0, & \text{if } 0 < u < u_0. \end{cases}$$

We then have, for all $x > 0$,

$$\hat{N}_w(x) = \hat{N}_0(x) + \int_0^{u_0} e^{-xu}\, dN_w(u),$$
$$\hat{\Psi}(x) = \hat{\Psi}_0(x) + \int_0^{u_0} e^{-xu}\, d\Psi(u)$$

and

$$\tilde{N}_w(x) = \tilde{N}_0(x) - \int_0^{u_0} \log(1 - e^{-xu})\, dN_w(u),$$
$$\tilde{\Psi}(x) = \tilde{\Psi}_0(x) - \int_0^{u_0} \log(1 - e^{-xu})\, d\Psi(u).$$

Using (1.3), it follows that

$$\begin{aligned} |\hat{N}_w(x) - \hat{\Psi}(x)| &\leq \int_0^{u_0} e^{-xu}\, dN_w(u) + \int_0^{u_0} e^{-xu}\, d\Psi(u) + |\hat{N}_0(x) - \hat{\Psi}_0(x)| \\ &\leq I_1 + I_2 + \sum_{k=1}^{\infty} \frac{1}{k} |\tilde{N}_0(kx) - \tilde{\Psi}_0(kx)| \\ &= I_1 + I_2 + \sum_{k \leq x_0/x} + \sum_{k > x_0/x} = I_1 + I_2 + S_1 + S_2. \end{aligned}$$

We will estimate the four terms in the last expression separately.

Trivially, we have $I_{1,2} \ll 1$. To estimate S_1, we write

$$\begin{aligned} S_1 &\le \sum_{k\le x_0/x} \frac{1}{k}|\tilde{N}_w(kx) - \tilde{\Psi}(kx)| + \sum_{k\le x_0/x} \frac{1}{k}\int_0^{u_0} \big(-\log(1-e^{-kxu})\big)\, dN_w(u) \\ &\quad + \sum_{k\le x_0/x} \frac{1}{k}\int_0^{u_0} \big(-\log(1-e^{-kxu})\big)\, d\Psi(u) \\ &= S_{11} + S_{12} + S_{13}. \end{aligned}$$

By (3.6) and (3.9), one sees that

$$S_{11} \ll \sum_{k\le x_0/x} \frac{1}{k} R(kx) \ll \sum_{k\le x_0/x} k^{-1-\delta} R(x) \ll R(x).$$

Moreover, using (3.10) and the fact that $N_w(u) = 0$ for $0 \le u < 1$, we obtain, as $x \to 0$,

$$S_{12} \ll \sum_{k\le x_0/x} \frac{1}{k}\int_{1-}^{u_0} \log(kxu)^{-1}\, dN_w(u) \ll \sum_{k\le x_0/x} \frac{1}{k}\log(kx)^{-1} \ll (\log x^{-1})^2.$$

Finally, as $x \to 0$ we have

$$\begin{aligned} S_{13} &\ll \sum_{k\le x_0/x} \frac{1}{k}\int_0^{u_0} \log(kxu)^{-1}\, d\Psi(u) \\ &= \sum_{k\le x_0/x} \frac{1}{k}\int_0^{u_0} \big(\log(kx/x_0)^{-1} + \log(x_0u)^{-1}\big)\, d\Psi(u) \\ &\ll (\log x^{-1})^2 - \log x^{-1}\int_0^{u_0} \log(1-e^{-x_0u})\, d\Psi(u) \ll (\log x^{-1})^2. \end{aligned}$$

Thus,

$$S_1 \le S_{11} + S_{12} + S_{13} \ll R(x) + (\log x^{-1})^2 \ll R(x), \tag{3.11}$$

where in the last step we used the bound $R(x) \gg x^{-\delta}$, which follows from (3.6). It remains to estimate S_2.

Integrating $\tilde{\Psi}_0(x) = -\int_{u_0}^{\infty} \log(1-e^{-kxu})\, d\Psi(u)$ by parts, we obtain

$$\begin{aligned} \sum_{k>x_0/x} \frac{1}{k}\tilde{\Psi}_0(kx) &= \sum_{k>x_0/x} \frac{\Psi(u_0)}{k}\log(1-e^{-kxu_0}) + \sum_{k>x_0/x}\int_{u_0}^{\infty} \frac{x}{e^{kxu}-1}\Psi(u)\, du \\ &\le \sum_{k>x_0/x}\int_{u_0}^{1/x} \frac{x}{e^{kxu}-1}\Psi(u)\, du + \sum_{k>x_0/x}\int_{1/x}^{\infty} \frac{x}{e^{kxu}-1}\Psi(u)\, du \\ &= S_{21} + S_{22}. \end{aligned}$$

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Since, by the first part of (3.5), $\Psi(u)u^{-\beta} \leq \Psi(1/x)x^{\beta}$ for $u_0 \leq u \leq 1/x$, we have

$$S_{21} \leq \Psi(1/x) \sum_{k>x_0/x} \int_{u_0}^{1/x} \frac{x(ux)^{\beta}}{e^{kxu}-1}\, du \ll \Psi(1/x) \sum_{k>x_0/x} k^{-\beta-1} \ll x^{\beta}\Psi(1/x).$$

A similar argument, using the second part of (3.5), gives $S_{22} \ll x^{\gamma}\Psi(1/x)$. On the other hand, we have

$$\begin{aligned} \sum_{k>x_0/x} \frac{1}{k} \tilde{N}_0(kx) &= - \sum_{k>x_0/x} \frac{1}{k} \sum_{n \geq u_0} w(n) \log(1-e^{-knx}) \\ &\ll \sum_{k>x_0/x} \frac{1}{k} \sum_{n\geq u_0} w(n) e^{-knx} \ll \sum_{n \geq u_0} w(n) e^{-nx_0} \\ &\leq \tilde{N}_w(x_0) \ll 1. \end{aligned}$$

Therefore, one has $S_2 \ll x^{\beta}\Psi(1/x) + x^{\gamma}\Psi(1/x) + 1 \ll x^{\beta}\Psi(1/x)$. Combining this and (3.11), we obtain conclusion (3.8) of Proposition 3.

Proof of Theorem 1 Let $\Phi(u)$, $\Phi^*(u)$ and $\lambda(u)$ be given as in the theorem, and suppose that $p_w(n)$ is a weighted partition function such that $P_w(u) = \sum_{n\leq u} p_w(n)$ satisfies (1.6) as $u \to \infty$. We first apply Proposition 1 with $P(u) = P_w(u)$, $\phi(u) = \Phi'(u)$ and $r(u) = \big(\Phi(u)/\lambda(u)\big)'$. Condition (P1) of Proposition 1 follows from (A2) in the definition of admissibility and (T1). To see that condition (P2) is satisfied, we note that, by (T2), the function $\Phi(u)/\big(\lambda(u)\log u\big)$ is non-decreasing for sufficiently large u and thus

$$\left(\frac{\Phi(u)}{\lambda(u)}\right)'(\log u)^{-1} - \frac{\Phi(u)}{u\lambda(u)}(\log u)^{-2} \geq 0$$

for sufficiently large u. Hence by (T2) we have, as $u \to \infty$,

$$r(u)u = \left(\frac{\Phi(u)}{\lambda(u)}\right)' u \geq \left(\frac{\Phi(u)}{\lambda(u)}\right)(\log u)^{-1} \to \infty. \tag{3.12}$$

Moreover, by the definition of $r(u)$ and (T1),

$$0 \leq r(u) = \frac{\phi(u)}{\lambda(u)} - \frac{\Phi(u)\lambda'(u)}{\lambda(u)^2} \leq \frac{\Phi(u)}{\lambda(u)} = o\big(\phi(u)\big). \tag{3.13}$$

Therefore $\phi(u)u \to \infty$ as $u \to \infty$, and condition (P2) is satisfied. Finally, condition (P3) follows from (3.13), and condition (P4) follows from (A2) with $c_1 = 2$ and $c_2 = 1 - 2^{\alpha-1}$, so the hypotheses of Proposition 1 are satisfied. Applying conclusion (2.2) of Proposition 1, we obtain

$$\tilde{N}(x) = \log \hat{P}_w(x) = \int_x^{\infty} \phi^{\leftarrow}(y)\, dy + O\left(\frac{\Phi\big(\phi^{\leftarrow}(x)\big)}{\lambda\big(\phi^{\leftarrow}(x)\big)}\right).$$

Furthermore, by (3.12) and (3.13), we have $\phi(u)u \ge 1$ for all sufficiently large u. Thus, by (T1), we see that $\lambda\big(\phi^{\leftarrow}(x)\big) \ge \lambda(1/x)$ and hence as $x \to 0$ one has

$$\tilde{N}_w(x) = \int_x^\infty \phi^{\leftarrow}(y)\,dy + O\bigg(\frac{\Phi\big(\phi^{\leftarrow}(x)\big)}{\lambda(1/x)}\bigg).$$

By Lemma 2, it follows that, as $x \to 0$,

$$\tilde{N}_w(x) = \bigg(1 + O\bigg(\frac{1}{\lambda(1/x)}\bigg)\bigg)\int_x^\infty \phi^{\leftarrow}(y)\,dy.$$

By (A4′), this is equivalent to, as $x \to 0$,

$$\tilde{N}_w(x) = \tilde{\Phi}^*(x) + O\bigg(\frac{\Phi^*(1/x)}{\lambda(1/x)}\bigg).$$

We now apply Proposition 3 with $\Psi(u) = \Phi^*(u)$ and $R(x) = \Phi^*(1/x)/\lambda(1/x)$. Hypothesis (3.5) of Proposition 3 follows immediately from (A3). To show that hypothesis (3.6) is satisfied, we choose $\delta = \beta - \alpha$ and write

$$R(x)x^\delta = \Phi^*(1/x)x^\beta\bigg\{\frac{\log x^{-1}}{\Phi(1/x)x^\alpha}\bigg\}\bigg\{\frac{\Phi(1/x)}{\lambda(1/x)}(-\log x)^{-1}\bigg\}.$$

By condition (A3) we have $\Phi^*(1/x)x^\beta \nearrow \infty$ as $x \to 0$. By (A2) and (3.2) in Lemma 1 we have $(\log x^{-1})/\big(\Phi(1/x)x^\alpha\big) \nearrow \infty$ as $x \to 0$, and by (T2) we have $\Phi(1/x)(-\log x)^{-1}/\lambda(1/x) \nearrow \infty$ as $x \to 0$. Hence we see that $R(x)x^\delta \nearrow \infty$ as $x \to 0$. Thus all hypotheses of Proposition 3 hold, and we conclude that as $x \to \infty$

$$\hat{N}_w(x) = \hat{\Phi}^*(x) + O\bigg(\frac{\Phi^*(1/x)}{\lambda(1/x)} + x^\beta\Phi^*(1/x)\bigg).$$

Finally, in the last step of proof we apply Proposition 2 with $\Psi(u) = N_w(u)$, $\Theta(u) = \Phi^*(u)$, and $\xi(u) = \min\big(\lambda(u), u^\gamma\big)$. Conditions (O1) and (O2) follow from condition (A3) and Lemma 1, condition (O3) follows from (3.15), and condition (O4) follows from conditions (T1) and (3.2) in Lemma 1. Therefore the conclusion of Proposition 2 holds, and one has, as $u \to \infty$,

$$N(u) = \Phi^*(u)\bigg\{1 + O\bigg(\frac{1}{\log\lambda(u)} + \frac{1}{\log u}\bigg)\bigg\}.$$

Since, by condition (T2) and (3.2), $\lambda(u) = o\big(\Phi(u)(\log u)^{-1}\big) = o(u^\alpha)$, we have $\log\lambda(u) \ll \log u$. It follows that

$$N(u) = \Phi^*(u)\bigg\{1 + O\bigg(\frac{1}{\log\lambda(u)}\bigg)\bigg\},$$

which is the claimed result.

4 Proof of Theorem 2

Let $P_w(u)$ and $\lambda(u)$ be given as in the statement of Theorem 2. Suppose that (1.9) holds as $u \to \infty$ with some constants $A > 0$ and $0 < a < 1$. Let $\Phi(u) = Au^a$ and set $\Phi^*(u) = Bu^b$, where B and b are constants given by (1.11). The pair $\big(\Phi(u), \Phi^*(u)\big)$ clearly satisfies conditions (A1)–(A3) in the definition of admissibility. Furthermore, we have $\phi(u) = \Phi'(u) = Aau^{-(1-a)}$, $\phi^{\leftarrow}(y) = (Aay^{-1})^{1/(1-a)}$ and hence

$$\int_x^\infty \phi^{\leftarrow}(y)\,dy = \int_x^\infty (Aay^{-1})^{1/(1-a)}\,dy = Dx^{-b},$$

where $D = A^{1/(1-a)}a^{a/(1-a)}(1-a)$. We then observe that, by definition (1.11) of b and B, one has $B = D/\big(\Gamma(1+b)\zeta(1+b)\big)$ and thus

$$\begin{aligned}\tilde{\Phi}^*(x) &= -B\int_0^\infty \log(1-e^{-xu})\,du^b = B\sum_{k=1}^\infty \frac{1}{k}\int_0^\infty e^{-kxu}\,du^b \\ &= B\Gamma(1+b)\sum_{k=1}^\infty k^{-1-b}x^{-b} = B\Gamma(1+b)\zeta(1+b)x^{-b} \\ &= Dx^{-b} = \int_x^\infty \phi^{\leftarrow}(y)\,dy.\end{aligned}$$

Therefore, condition (A4′) is satisfied and the pair of functions $\big(\Phi(u), \Phi^*(u)\big)$ is admissible with accuracy $\lambda(u)$. (In fact, it is admissible with arbitrary accuracy since the main terms on the left and the right of (A4′) are equal.) We now apply Theorem 1 with $\Phi(u)$, $\Phi^*(u)$, and $\lambda(u)$ given as above. Conditions (T1) and (T2) in Theorem 1 follow immediately from the assumption on $\lambda(u)$ in the statement of Theorem 2, and (1.6) follows from (1. 9). Hence, Theorem 1 applies, and as $u \to \infty$ we have

$$N(u) = Bu^b\left\{1 + O\left(\frac{1}{\log\lambda(u)}\right)\right\},$$

completing the proof of Theorem 2.

Remark More generally, if $\Phi(u)$ is a function such that $\log \hat{P}_w(x)$ has an asymptotic expansion with main term $\Phi^{\leftarrow}(x) = x^{-b}\sum_{k=0}^n c_k(\log x^{-1})^{n-k}$, where n is a positive integer, $b, c_0 > 0$, and the numbers c_k are arbitrary constants, we can find a suitable function $\Phi^*(u)$ as follows. By the Mellin inversion formula, for $\sigma, x > 0$ we have

$$e^{-x} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)x^{-s}\,ds. \tag{4.1}$$

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Hence, whenever $\int_0^\infty u^{-\sigma}\,d\Phi^*(u)$ converges, one has

$$\begin{aligned}\tilde{\Phi}^*(x) &= -\int_0^\infty \log(1-e^{-xu})\,d\Phi^*(u)\\ &= \frac{1}{2\pi i}\sum_{k=1}^\infty \frac{1}{k}\int_0^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)(kxu)^{-s}\,ds\,d\Phi^*(u)\\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\zeta(1+s)x^{-s}\Big(\int_0^\infty u^{-s}\,d\Phi^*(u)\Big)\,ds.\end{aligned}$$

On the other hand, we have

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}\sum_{k=0}^n \frac{c_k}{(s-b)^{n-k+1}}\,ds = x^{-b}\sum_{k=0}^n c_k(\log x^{-1})^{c-k}.$$

Thus, in order for (A4) to be satisfied, we should choose $\Phi^*(u)$ so that

$$\int_0^\infty u^{-s}\,d\Phi^*(u) \sim \frac{1}{\Gamma(s)\zeta(1+s)}\sum_{k=0}^n \frac{c_k}{(s-b)^{c-k+1}}$$

as $s \to b+$. By the Mellin inversion formula this condition translates to

$$\Phi^*(u) \sim \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{u^s}{\Gamma(1+s)\zeta(1+s)}\sum_{k=0}^n \frac{c_k}{(s-b)^{n-k+1}}\,ds \sim u^b\sum_{k=0}^n d_k(\log u)^{n-k},$$

where

$$d_k = \frac{1}{(n-k)!}\sum_{j=0}^k c_{k-j}\frac{d^j}{ds^j}\frac{1}{\Gamma(1+s)\zeta(1+s)}\bigg|_{s=b}.$$

5 Proof of Theorem 3

As usual, let $w(n)$ be a non-negative weight function. We first give a representation of the Dirichlet series $f_w(s) = \sum_{n=1}^\infty w(n)n^{-s}$ as an integral involving the Laplace transform $\hat{P}_w(x)$ of $P_w(u) = \sum_{n\le u} p_w(n)$.

Lemma 4 *Let σ_a be the abscissa of absolute convergence for $f_w(s)$. Then we have, for* $\operatorname{Re} s > \max(0,\sigma_a)$,

$$\int_0^\infty x^{s-1}\log\hat{P}_w(x)\,dx = \Gamma(s)\zeta(s+1)f_w(s).$$

Proof Suppose that $\operatorname{Re} s > \max(0, \sigma_a)$. Then by (1.8) we have

$$\begin{aligned}\int_0^\infty x^{s-1} \log \hat{P}_w(x)\, dx &= \int_0^\infty x^{s-1} \sum_{n=1}^\infty w(n)\bigl(-\log(1-e^{-nx})\bigr)\, dx \\ &= \int_0^\infty x^{s-1} \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{w(n)}{k} e^{-knx}\, dx \\ &= \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{\Gamma(s) w(n)}{n^s k^{s+1}} = \Gamma(s)\zeta(s+1) f_w(s),\end{aligned}$$

where interchanging of order of integration and summation is justified by absolute convergence of the double series involved.

The next lemma is a version of Landau's theorem on singularities of Dirichlet series.

***Lemma 5* (Landau)**

(i) *Let $g(n)$ be a function defined on the set of positive integers and of constant sign for all sufficiently large n. Suppose that the Dirichlet series $\sum_{n=1}^\infty g(n)n^{-s}$ has finite abscissa of absolute convergence σ_a. Then $s = \sigma_a$ is a singularity of the function represented by the Dirichlet series.*

(ii) *Let $g(x)$ be an integrable function on $[0,1]$ and of constant sign for all sufficiently small x. Suppose that the Dirichlet integral $\int_0^1 g(x)x^{s-1}\, dx$ has finite abscissa of absolute convergence σ_a. Then $s = \sigma_a$ is a singularity of the function represented by the Dirichlet integral.*

Proof Part (i) is the classical version of Landau's theorem (see, *e.g.*, Ingham [11, p. 88]). To obtain part (ii), we note that the given integral can be written as $\int_1^\infty h(u)u^{-s}\, du$ with $h(u) = g(1/u)/u$ and $x = 1/u$. By the integral version of Landau's theorem (Ingham [11, p. 88]) the last integral has a singularity at $s = \sigma_a$.

Proof of Theorem 3 Suppose that (1.13) holds as $u \to \infty$ with some constants $A > 0$ and $0 < a_1 < a < 1$. We apply Proposition 1 with $P(u) = P_w(u)$, $\phi(v) = Aav^{a-1}$ and $r(v) = v^{a_1 - 1}$ and note that $\phi^{\leftarrow}(y) = (Aa)^{1/(1-a)} y^{-1/(1-a)}$. Hypothesis (2.1) follows from (1.13), and it is easy to see that conditions (P1)–(P4) of Proposition 1 are satisfied. Thus conclusion (2.2) of Proposition 1 holds, and as $x \to 0$ we have

$$\begin{aligned}\log \hat{P}_w(x) &= \int_x^\infty \phi^{\leftarrow}(y)\, dy + O\Bigl(\int_0^{\phi^{\leftarrow}(x)} r(v)\, dv\Bigr) \qquad (5.1)\\ &= (Aa)^{1/(1-a)} \int_x^\infty y^{-1/(1-a)}\, dy + O\Bigl(\int_0^{(Aax^{-1})^{1/(1-a)}} v^{a_1-1}\, dv\Bigr) \\ &= A^{1/(1-a)} a^{a/(1-a)} (1-a) x^{-a/(1-a)} + O(x^{-a_1/(1-a)}) \\ &= Dx^{-b} + O(x^{-b_1}),\end{aligned}$$

where $b = a/(1-a)$, $b_1 = a_1/(1-a)$ and $D = A^{1/(1-a)} a^{a/(1-a)}(1-a)$. Therefore there exists a constant $C > 0$ such that, for $x \in (0,1)$,

$$|\log \hat{P}_w(x) - Dx^{-b}| \le Cx^{-b_1}.$$

On the other hand, for $x \ge 1$ we have the bound

$$\begin{aligned} \log \hat{P}_w(x) &= -\sum_{n=1}^{\infty} w(n)\log(1-e^{-nx}) \\ &\le -e^{-x/2}\sum_{n=1}^{\infty} w(n)\log(1-e^{-nx/2}) \\ &\le e^{-x/2}\log \hat{P}_w(1/2) \ll e^{-x/2}. \end{aligned} \tag{5.2}$$

Setting

$$E(x) = \log \hat{P}_w(x) - Dx^{-b}$$

and applying Lemma 4, we then obtain

$$\begin{aligned} \Gamma(s)\zeta(1+s) f_w(s) &= \int_0^{\infty} x^{s-1}\log \hat{P}_w(x)\,dx \\ &= \int_1^{\infty} x^{s-1}\log \hat{P}_w(x)\,dx + D\int_0^1 x^{s-1-b}\,dx \\ &\quad + \int_0^1 x^{s-1}E(x)\,dx \\ &= I_1(s) + \frac{D}{s-b} + I_2(s), \end{aligned} \tag{5.3}$$

provided that $\operatorname{Re} s > \max(0, \sigma_a)$. To prove part (i) of Theorem 3, we thus need to show that the function

$$h_w(s) = I_1(s) + I_2(s) = \int_1^{\infty} x^{s-1}\log \hat{P}_w(x)\,dx + \int_0^1 x^{s-1}E(x)\,dx$$

is analytic in the half-plane $S = \{s : \operatorname{Re} s > b_1\}$. Using the upper bound (5.2), we see that the integral $I_1(s)$ converges uniformly on every compact subset of the complex plane and hence defines an entire function. Furthermore, since, by (5.1), $E(x) \ll x^{-b_1}$ for $0 < x < 1$, the integral $I_2(s)$ converges uniformly on every compact subset of the half-plane S and thus represents an analytic function in S. Therefore, $h_w(s)$ is analytic in S. This proves part (i) of Theorem 3.

Part (ii) of the theorem can be deduced from part (i) by a standard contour integration argument (see, *e.g.* [23, p. 167]). For completeness, we provide details here. Without loss of generality, we may assume that u is of the form $u = [u] + 1/2$. Since the Dirichlet series $f_w(s)$ has non-negative coefficients $w(n)$, a simple pole at $s = b$, and no pole to the right of $s = b$, Lemma 5 implies that the abscissa of absolute convergence σ_a of $f_w(s)$ is b. Thus, by an effective version of the Perron formula (see [23, Theorem II.2.2]), we have

$$N_w(u) = \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \frac{f_w(s)}{s} u^s\, ds + O\Big(u^\kappa \sum_{n=1}^{\infty} \frac{w(n)}{n^\kappa\big(1 + T|\log(u/n)|\big)}\Big), \tag{5.4}$$

where $\kappa = b+1/\log u$ and T is a positive number to be chosen later. The contribution of the ranges $n > 2u$ and $n < u/2$ to the O-term is

$$\ll \frac{u^\kappa}{T}\sum_{n=1}^{\infty} w(n)n^{-\kappa} \ll \frac{u^b f_w(b + 1/\log u)}{T} \ll \frac{u^b \log u}{T}.$$

When $u/2 \le n \le 2u$ we use condition (1.16) and obtain

$$u^\kappa \sum_{u/2\le n\le 2u} \frac{w(n)}{n^\kappa\big(1 + T|\log(u/n)|\big)} \ll_\epsilon u^{a_1/(1-a)+\epsilon} \sum_{u/2\le n\le 2u} \frac{1}{1 + T|n/u - 1|}$$
$$\ll_\epsilon u^{b_1+\epsilon}\int_{u/2}^{2u} \frac{dv}{1 + T|v/u - 1|} \ll_\epsilon \frac{u^{b_1+1+\epsilon}\log T}{T},$$

where we set $b_1 = a_1/(1 - a)$. Therefore the O-term in (5.4) is bounded by

$$u^\kappa \sum_{n=1}^{\infty} \frac{w(n)}{n^\kappa(1 + T|\log(u/n)|)} \ll_\epsilon \frac{u^b \log u}{T} + \frac{u^{b_1+1+\epsilon}\log T}{T}. \tag{5.5}$$

We now move the path of integration to the path consisting of three line segments $\gamma_1 = \{\sigma - iT : \kappa \ge \sigma \ge b_1 + \delta\}$, $\gamma_2 = \{b_1 + \delta + it : -T \le t \le T\}$ and $\gamma_3 = \{\sigma + iT : b_1 + \delta \le \sigma \le \kappa\}$, where $0 < \delta \le \epsilon$. The residue of the pole at $s = b$ contributes $Du^b/\Gamma(1 + b)\zeta(1 + b) = Bu^b$. By condition (1.17), the contribution from γ_1 and γ_3 is

$$\int_{\gamma_{1,3}} \ll_\epsilon u^b T^{\epsilon-1}, \tag{5.6}$$

and the contribution from γ_2 is

$$\int_{\gamma_2} \ll_\epsilon u^{b_1+\epsilon}\int_0^T t^{\epsilon-1}\, dt = u^{b_1+\epsilon}T^\epsilon. \tag{5.7}$$

Combining (5.4)–(5.7) we obtain

$$N_w(u) = Bu^b + O_\epsilon\Big(\frac{u^b \log u}{T} + \frac{u^{b_1+1+\epsilon}\log T}{T} + u^b T^{\epsilon-1} + u^{b_1+\epsilon}T^\epsilon\Big).$$

We now choose $T = \max(u, u^{b-b_1})$. It follows that for any fixed $\epsilon > 0$, as $u \to \infty$,

$$N_w(u) = Bu^b + O_\epsilon(u^{b_1+\epsilon}),$$

which is the claimed result.

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6 Proof of the Corollaries

We first prove an identity relating the Laplace transform $\hat{P}_w(x)$ of $P_w(u) = \sum_{n\le u} p_w(n)$ to the Dirichlet series $f_w(s) = \sum_{n=1}^{\infty} w(n)n^{-s}$.

Lemma 6 *Let σ_a be the abscissa of absolute convergence of $f_w(s)$. We then have, for all $x > 0$ and all $\sigma > \max(0, \sigma_a)$,*

$$\log \hat{P}_w(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\zeta(1+s) f_w(s)x^{-s}\, ds.$$

Proof Using (4.1), we have, for all $x > 0$ and $\sigma > \max(0, \sigma_a)$,

$$\begin{aligned}\log \hat{P}_w(x) &= -\sum_{n=1}^{\infty} w(n)\log(1-e^{-nx}) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{w(n)}{k}e^{-knx}\\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{w(n)}{k}(knx)^{-s}\, ds\\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\zeta(1+s)f_w(s)x^{-s}\, ds,\end{aligned}$$

where interchanging of summation and integration is justified by absolute convergence of the series involved.

We next quote a Tauberian result from [25].

Proposition 4 **(Yang [25, Proposition 3])** *Let $P(u)$ be a non-negative and non-decreasing function defined on $(0,\infty)$. Let $\hat{P}(x)$ be the Laplace transform of $P(u)$. Suppose that, for some constants D and $b > 0$, as $x \to 0$,*

$$\log \hat{P}(x) = Dx^{-b} + O\big(R(x^{-1})\big), \tag{6.1}$$

where $R(u)$ is a positive differentiable function satisfying the following conditions:

$$R(u)u^{-b} \searrow 0 \text{ and } R(u)(\log u)^{-1} \nearrow \infty \quad \text{as } u \to \infty; \tag{R1}$$

$$R(u) \gg u^{b/2} \quad \text{as } u \to \infty. \tag{R2}$$

Suppose further that the function $G(x) = \log \hat{P}(x)$ satisfies, as $x \to 0$,

$$G'(x) = -Dbx^{-b-1} + O\big(x^{-1}R(x^{-1})\big) \tag{6.2}$$

and

$$G''(x) \gg x^{-b-2}. \tag{6.3}$$

Then as $u \to \infty$ one has

$$\log P(u) = Au^a + O\big(R(u^{1/(1+b)})\big),$$

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where A and a are constants determined by

$$a = \frac{b}{1+b}, \quad A = \left(1 + \frac{1}{b}\right)(Db)^{1/(1+b)}.$$

Proof of Corollary 1 Fix a positive integer k and let $w_{k,l}(n)$ be defined as in the statement of Corollary 1. For simplicity we write $P_{k,l}(u)$ for $P_{w_{k,l}}(u)$ and $f_{k,l}(u)$ for $f_{w_{k,l}}(u)$, respectively. Let $\theta \in [1/2, 1)$ be given and suppose first that (1.19) holds as $u \to \infty$ for all l with $(k, l) = 1$ and all $\epsilon > 0$. We apply part (i) of Theorem 3 with $w(n) = w_{k,l}(n)$, $A = 2\sqrt{\zeta(2)/\varphi(k)}$, $a = 1/2$ and $a_1 = \theta/2 + \epsilon$, where ϵ is a fixed positive number with $0 < \epsilon < (1-\theta)/2$. The quantities b and D in Theorem 3 become

$$b = \frac{a}{1-a} = 1, \quad D = A^{1/(1-a)} a^{a/(1-a)} (1-a) = \frac{A^2}{4} = \frac{\zeta(2)}{\varphi(k)},$$

and the theorem gives

$$f_{k,l}(s) = \frac{1}{\Gamma(s)\zeta(1+s)} \left\{ \frac{\zeta(2)}{\varphi(k)(s-1)} + h_{k,l}(s) \right\},$$

where $h_{k,l}(s)$ is a function having an analytic continuation to the half-plane

$$S = \{s : \operatorname{Re} s > a_1/(1-a)\} = \{s : \operatorname{Re} s > \theta + 2\epsilon\}.$$

It follows that, for all Dirichlet characters χ modulo k,

$$(6.4) \qquad -\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s} = \sum_{\substack{l=1\\(k,l)=1}}^{k-1} \chi(l) f_{k,l}(s)$$
$$= \frac{\zeta(2)c(\chi)}{\Gamma(s)\zeta(1+s)(s-1)} + \sum_{\substack{l=1\\(k,l)=1}}^{k-1} \chi(l) h_{k,l}(s),$$

where $c(\chi) = 1$ if χ is the principal character modulo k and $c(\chi) = 0$ otherwise. Since $\zeta(2)\big(\Gamma(s)\zeta(1+s)(s-1)\big)^{-1}$ is analytic in the half-plane $\{s : \operatorname{Re} s > 0\}$ except for a simple pole at $s = 1$ with residue 1, (6.4) implies that the function

$$-\frac{L'(s,\chi)}{L(s,\chi)} - \frac{c(\chi)}{s-1} = \frac{\zeta(2)c(\chi)}{\Gamma(s)\zeta(1+s)(s-1)} - \frac{c(\chi)}{s-1} + \sum_{\substack{l=1\\(k,l)=1}}^{k-1} \chi(l) h_{k,l}(s)$$

is analytic in $S = \{s : \operatorname{Re} s > \theta + 2\epsilon\}$. Since ϵ can be taken arbitrarily small, we conclude that $L(s,\chi)$ has no zeros in the half-plane $\{s : \operatorname{Re} s > \theta\}$.

Conversely, suppose that $\theta \in [1/2, 1)$ is such that, for all characters χ modulo k, $L(s,\chi)$ does not vanish in the half-plane $\{s : \operatorname{Re} s > \theta\}$. Let ϵ with $0 < \epsilon < 1 - \theta$ be given. We fix an integer l with $(k,l)=1$ and apply Proposition 4 with $P(u)=P_{k,l}(u)$, $b=1$, $D=\zeta(2)/\varphi(k)$ and $R(u)=u^{\theta+\epsilon}$. Then conditions (R1) and (R2) hold, and we will show that conditions (6.1)–(6.3) are also satisfied.

Using the orthogonality relation for Dirichlet characters

$$\sum_{\chi \bmod k} \chi(m)\overline{\chi(l)} = \begin{cases} \varphi(k), & \text{if } m \equiv l \bmod k, \\ 0, & \text{else,} \end{cases}$$

we obtain

$$\begin{aligned} f_{k,l}(s) &= \sum_{n \equiv l \bmod k} \frac{\Lambda(n)}{n^s} = \frac{1}{\varphi(k)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \sum_{\chi \bmod k} \chi(n)\overline{\chi(l)} \\ &= -\frac{1}{\varphi(k)} \sum_{\chi \bmod k} \overline{\chi(l)} \frac{L'(s,\chi)}{L(s,\chi)}. \end{aligned}$$

On the other hand, Lemma 6 shows that for $\sigma > 1$ and $x > 0$

$$\begin{aligned} \log \hat{P}_{k,l}(x) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s)\zeta(1+s) f_{k,l}(s) x^{-s}\, ds \\ &= -\frac{1}{2\pi i \varphi(k)} \sum_{\chi \bmod k} \overline{\chi(l)} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s)\zeta(1+s) \frac{L'(s,\chi)}{L(s,\chi)} x^{-s}\, ds. \end{aligned} \tag{6.5}$$

We then move the path of integration in the last integral to the vertical line with $\operatorname{Re} s = \theta + \epsilon$. This is possible by the assumption that $L(s,\chi)$ has no zeros for $\operatorname{Re} s > \theta$ and by the bounds

$$|\zeta(1+\sigma+it)| \ll 1, \quad |\Gamma(\sigma+it)| \ll |t|^2 e^{-\pi|t|/2} \quad (\theta+\epsilon \le \sigma \le 2), \tag{6.6}$$

and

$$\left| \frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} - \frac{c(\chi)}{1-s} \right| \ll_\epsilon \log\bigl(k(|t|+2)\bigr) \quad (\theta+\epsilon \le \sigma \le 2). \tag{6.7}$$

The latter bound follows easily from the partial fraction decomposition ([17, p. 225])

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{\substack{\rho: L(\rho,\chi)=0 \\ |\operatorname{Im}\rho - \operatorname{Im} s| \le 1}} \frac{1}{s-\rho} - \frac{c(\chi)}{s-1} + \frac{c_1(\chi)}{s} + \frac{c_2(\chi)}{s+1} + O\Bigl(\log\bigl(k(|t|+2)\bigr)\Bigr),$$

where $c_1(\chi)$ and $c_2(\chi)$ are suitable constants, and from the zero density estimate ([17, p. 220])

$$\#\{\rho : L(\rho,\chi)=0, \quad 0 < \operatorname{Re}\rho < 1, |\operatorname{Im}\rho - t| \le 1\} \ll \log\bigl(k(|t|+2)\bigr).$$

The residue of the pole of of the integrand at $s = 1$ contributes $\zeta(2)x^{-1}/\varphi(k)$ to (6.5). Taking into account bounds (6.6) and (6.7), we obtain, as $x \to 0$,

$$\log \hat{P}_{k,l}(x) = \frac{\zeta(2)}{\varphi(k)}x^{-1} - \frac{1}{2\pi i\varphi(k)} \sum_{\chi \bmod k} \overline{\chi(l)} \int_{\theta+\epsilon-i\infty}^{\theta+\epsilon+i\infty} \Gamma(s)\zeta(1+s)\frac{L'(s,\chi)}{L(s,\chi)}x^{-s}\,ds$$
$$= \frac{\zeta(2)}{\varphi(k)}x^{-1} + O(x^{-(\theta+\epsilon)}).$$

Hence condition (6.1) in Proposition 4 is satisfied. A similar argument shows that for $\sigma > 1$, as $x \to 0$,

$$\frac{d}{dx}\log \hat{P}_{k,l}(x) = \frac{1}{2\pi i\varphi(k)} \sum_{\chi \bmod k} \overline{\chi(l)} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s+1)\zeta(s+1)\frac{L'(s,\chi)}{L(s,\chi)}x^{-s-1}\,ds$$
$$= -\frac{\zeta(2)}{\varphi(k)}x^{-2} + O(x^{-(\theta+1+\epsilon)})$$

and

$$\frac{d^2}{dx^2}\log \hat{P}_{k,l}(x) = -\frac{1}{2\pi i\varphi(k)} \sum_{\chi \bmod k} \overline{\chi(l)} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s+2)\zeta(s+1)\frac{L'(s,\chi)}{L(s,\chi)}x^{-s-2}\,ds$$
$$= \frac{2\zeta(2)}{\varphi(k)}x^{-3} + O(x^{-(\theta+2+\epsilon)})$$

Thus conditions (6.2) and (6.3) are satisfied, and the proposition gives, as $u \to \infty$,

$$\log P_{k,l}(u) = Au^a + O(u^{(\theta+\epsilon)/(1+b)}) = Au^a + O(u^{(\theta+\epsilon)/2}),$$

where

$$a = \frac{b}{1+b} = \frac{1}{2}, \quad A = \left(1+\frac{1}{b}\right)(Db)^{1/(1+b)} = 2\sqrt{\frac{\zeta(2)}{\varphi(k)}}.$$

This establishes the desired relation (1.19) and completes the proof of Corollary 1.

Proof of Corollary 2 Let $w(n)$ be the characteristic function of the set of positive integers with an even number of prime factors. Suppose first that (1. 20) holds for every $\epsilon > 0$. In the half-plane $\{s : \operatorname{Re} s > 1\}$ we have

$$f_w(s) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1+(-1)^{\Omega(n)}}{n^s} = \frac{1}{2}\Big(\prod_p (1-p^{-s})^{-1} + \prod_p (1+p^{-s})^{-1}\Big)$$
$$= \frac{1}{2}\left(\zeta(s) + \frac{\zeta(2s)}{\zeta(s)}\right), \tag{6.8}$$

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where as usual $\Omega(n)$ is the number of prime factors of n, counted with multiplicities. Applying part (i) of Theorem 3 with $A = \sqrt{2\zeta(2)}$, $a = 1/2$, $a_1 = \theta/2 + \epsilon$, $b = a/(1-a) = 1$ and $D = A^{1/(1-a)}a^{a/(1-a)}(1-a) = \zeta(2)/2$, we obtain

$$f_w(s) = \frac{1}{\Gamma(s)\zeta(1+s)}\left\{\frac{\zeta(2)}{2(s-1)} + h_w(s)\right\},$$

where $h_w(s)$ is a complex function having an analytic continuation to the half-plane $\{s : \operatorname{Re} s > \theta + 2\epsilon\}$. Combining this and (6.8) yields

$$\frac{1}{2}\left(\zeta(s) + \frac{\zeta(2s)}{\zeta(s)}\right) - \frac{1}{2(s-1)} = \frac{1}{\Gamma(s)\zeta(1+s)}\frac{\zeta(2)}{2(s-1)} - \frac{1}{2(s-1)} + \frac{h_w(s)}{\Gamma(s)\zeta(1+s)}.$$

Since the expression on the right-hand side has no poles with $\operatorname{Re} s > \theta + 2\epsilon$ and ϵ can be taken arbitrarily small, we conclude that $\zeta(s)$ has no zeros in the half-plane $\{s : \operatorname{Re} s > \theta\}$.

Conversely, suppose that $\theta \in [1/2, 1)$ is a positive constant such that $\zeta(s)$ has no zeros in the half-plane $\{s : \operatorname{Re} s > \theta\}$. Let $0 < \delta \le \epsilon < 1 - \theta$ be given. We apply Proposition 4 with $P(u) = P_w(u)$, $b = 1$, $D = \zeta(2)/2$ and $R(u) = u^{\theta+\epsilon}$. It is obvious that $R(u)$ satisfies conditions (R1) and (R2). We show that conditions (6.1)–(6.3) hold as well. Arguing as in the proof of Corollary 1, we first deduce from Lemma 6 that, for $\sigma > 1$ and $x > 0$,

$$\log \hat{P}_w(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s)\zeta(1+s)}{2}\left(\zeta(s) + \frac{\zeta(2s)}{\zeta(s)}\right)x^{-s}\,ds.$$

We then move the path of integration to the line $\{s : \operatorname{Re} s = \theta + \delta\}$. Noting that the residue of the integrand at $s = 1$ is $\zeta(2)x^{-1}/2$ and using the bounds (6.5),

$$\left|\zeta\big(2(\sigma+it)\big)\right| \ll \log|t| \quad (\sigma \ge \theta, |t| \to \infty), \tag{6.9}$$

and

$$|\zeta(\sigma+it)|,\ 1/|\zeta(\sigma+it)| \ll_{\epsilon,\delta} t^{\epsilon} \quad (\sigma \ge \theta+\delta,\ |t| \to \infty), \tag{6.10}$$

we obtain, as $x \to 0$,

$$\log \hat{P}_w(x) = \frac{\zeta(2)}{2}x^{-1} + O_\epsilon(x^{-(\theta+\epsilon)}).$$

(For a proof of (6.9), see [24, p. 49]. A proof of (6.10) for the case when $\theta = 1/2$ can be found in [24, p. 337]; the proof for the general case is similar.) Hence (6.1) in Proposition 4 is satisfied. A similar argument shows that (6.2) and (6.3) are also satisfied. Hence the conclusion of Proposition 4 holds, and we have, as $u \to \infty$,

$$\log P_w(u) = Au^a + O_\epsilon(u^{(\theta+\epsilon)/(1+b)}) = Au^a + O_\epsilon(u^{(\theta+\epsilon)/2}),$$

where

$$a = \frac{b}{1+b} = \frac{1}{2}, \quad A = \left(1 + \frac{1}{b}\right)(Db)^{1/(1+b)} = \sqrt{2\zeta(2)}.$$

This completes the proof of Corollary 2.

Proof of Corollary 3 The proof of Corollary 3 is very similar to that of Corollaries 1 and 2. We therefore give a brief sketch of the argument.

Let $w(n)$ be given as in the statement of Corollary 3. We first observe that the Dirichlet series generated by $w(n)$ is

$$f_w(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n^k)}{n^{ks}} = -\frac{\zeta'(ks)}{\zeta(ks)}. \tag{6.11}$$

Suppose that the Riemann Hypothesis is true. Applying Lemma 6 and using standard contour integration arguments, we obtain

$$\log \hat{P}_w(x) = \frac{1}{k}\Gamma\left(\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)x^{-1/k} + O_\epsilon(x^{-(1/(2k)+\epsilon)}).$$

We then apply Proposition 4 with $b = 1/k$ and obtain, as $u \to \infty$,

$$\log P_w(u) = Au^a + O_\epsilon(u^{a/2+\epsilon}),$$

where $a = b/(1+b) = 1/(k+1)$ and

$$A = (k+1)\left\{\frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}.$$

Conversely, suppose that (1.21) holds as $u \to \infty$. We apply part (i) of Theorem 3 and obtain

$$f_w(s) = \frac{1}{\Gamma(s)\zeta(1+s)}\left\{\frac{D}{s-b} + h_w(s)\right\},$$

where $b = a/(1-a) = 1/k$,

$$D = A^{1/(1-a)}a^{a/(1-a)}(1-a) = \Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right),$$

and $h_w(s)$ is a complex function having an analytic continuation to the half-plane $\{s : \operatorname{Re} s > 1/(2k)\}$. Hence $f_w(s)$ is analytic in this half-plane except for a pole at $s = b = 1/k$ with residue $\Gamma(1+1/k)/\Gamma(1/k) = 1/k$. In view of the representation (6.11) this implies that $\zeta(s)$ has no zeros in the half-plane $\{s : \operatorname{Re} s > 1/2\}$, that is, that the Riemann Hypothesis holds.

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