

ON QUANTITATIVE SCHUR AND DUNFORD–PETTIS PROPERTIES

ONDŘEJ F. K. KALENDA and JIŘÍ SPURNÝ✉

(Received 5 November 2014; accepted 8 December 2014; first published online 26 February 2015)

Abstract

We show that the dual to any subspace of $c_0(\Gamma)$ (Γ is an arbitrary index set) has the strongest possible quantitative version of the Schur property. Further, we establish a relationship between the quantitative Schur property and quantitative versions of the Dunford–Pettis property. Finally, we apply these results to show, in particular, that any subspace of the space of compact operators on ℓ_p ($1 < p < \infty$) with the Dunford–Pettis property automatically satisfies both its quantitative versions.

2010 *Mathematics subject classification*: primary 46B25.

Keywords and phrases: quantitative Schur property, quantitative Dunford–Pettis property.

1. Main result

A Banach space X is said to have the *Schur property* if any weakly null sequence in X converges to zero in norm. Equivalently, X has the Schur property if every weakly Cauchy sequence is norm Cauchy. The classical example of a space with the Schur property is the space ℓ_1 of all absolutely summable sequences.

A quantitative version of the Schur property was introduced and studied in [10]. Let us recall the definition: if (x_k) is a bounded sequence in a Banach space X , we set (following [10])

$$\text{ca}(x_k) = \inf_{n \in \mathbb{N}} \text{diam}\{x_k : k \geq n\}$$

and

$$\delta(x_k) = \sup_{x^* \in B_{X^*}} \inf_{n \in \mathbb{N}} \text{diam}\{x^*(x_k) : k \geq n\}.$$

The quantity $\text{ca}(\cdot)$ measures how far the sequence is from being norm Cauchy, while the quantity $\delta(\cdot)$ measures how far it is from being weakly Cauchy. It is easy to check that the quantity $\delta(x_k)$ can be alternatively described as the diameter of the set of all weak* cluster points of (x_k) in X^{**} .

Our research was supported in part by the grant GAČR P201/12/0290. The second author was also supported by The Foundation of Karel Janeček for Science and Research.

© 2015 Australian Mathematical Publishing Association Inc. 0004-9727/2015 \$16.00

Again, following [10], a Banach space X is said to have the C -Schur property (where $C \geq 0$) if

$$ca(x_k) \leq C\delta(x_k) \quad (1.1)$$

for any bounded sequence (x_k) in X . Since obviously $\delta(x_k) \leq ca(x_k)$ for any bounded sequence (x_k) , necessarily $C \geq 1$ (unless X is the trivial space). Moreover, if X has the C -Schur property for some $C \geq 1$, it easily follows that X has the Schur property. Indeed, if (x_k) is weakly Cauchy in X , then $\delta(x_k) = 0$, and thus $ca(x_k) = 0$. The space constructed in [10, Example 1.4] serves as an example of a Banach space with the Schur property but without the C -Schur property for any $C > 0$. On the other hand, $\ell_1(\Gamma)$ possesses the 1-Schur property (see [10, Theorem 1.3]). (Here and below, Γ denotes an index set.)

Our main result is the following generalisation of the theorem quoted above.

THEOREM 1.1. *Let X be a subspace of $c_0(\Gamma)$. Then X^* has the 1-Schur property.*

For the proof of the main result, we will need some lemmas. The first establishes a special property of the norm on $c_0(\Gamma)$ and its subspaces.

LEMMA 1.2. *Let X be a subspace of $c_0(\Gamma)$. Then for any $x^* \in X^*$ and any sequence (x_n^*) in X^* which weak* converges to 0,*

$$\limsup \|x_n^* + x^*\| = \|x^*\| + \limsup \|x_n^*\|.$$

PROOF. Let us first suppose that X is separable. It is obvious that for any $x \in X$ and any weakly null sequence (x_n) in X , have

$$\limsup \|x_n + x\| = \max(\|x\|, \limsup \|x_n\|).$$

The assertion then follows from [11, Theorem 2.6] (applied for $p = \infty$).

The general case follows by a separable reduction argument. Suppose that $x^* \in X^*$ and that (x_n^*) is a weak* null sequence in X^* . Consider the countable set

$$A = \{x^*\} \cup \{x_n^* : n \in \mathbb{N}\} \cup \{x_n^* + x^* : n \in \mathbb{N}\}.$$

We can find a separable subspace $Y \subset X$ such that for each $y^* \in A$ we have $\|y^*\| = \|y^*|_Y\|$. Then the assertion follows immediately from the separable case. \square

The next lemma is a stronger variant of [2, Lemma 1.7] or [11, Lemma 2.3] for the special case of subspaces of $c_0(\Gamma)$.

LEMMA 1.3. *Let X be a subspace of $c_0(\Gamma)$ and (x_n^*) a sequence in X^* weak* converging to x^* . Then for any finite-dimensional subspace $F \subset X^*$,*

$$\liminf \text{dist}(x_n^*, F) \geq \liminf \|x_n^*\| - \|x^*\|.$$

PROOF. Let $c > \liminf \text{dist}(x_n^*, F)$. By passing to a subsequence we may assume that $\text{dist}(x_n^*, F) < c$ for each $n \in \mathbb{N}$. We can thus find a sequence (y_n^*) in F such that $\|x_n^* - y_n^*\| < c$ for each $n \in \mathbb{N}$. Since the sequence (x_n^*) is bounded, the sequence (y_n^*)

is also bounded. Therefore, by passing to a subsequence, we can suppose that the sequence (y_n^*) converges in norm to some $y^* \in F$. Then

$$\begin{aligned} c &\geq \limsup \|x_n^* - y_n^*\| = \limsup \|x_n^* - y^*\| = \limsup \|(x_n^* - x^*) + (x^* - y^*)\| \\ &= \limsup \|x_n^* - x^*\| + \|x^* - y^*\| \geq \limsup \|x_n^*\| - \|x^*\| + \|x^* - y^*\| \\ &\geq \liminf \|x_n^*\| - \|x^*\|. \end{aligned}$$

The first equality follows from the fact that the sequence (y_n^*) converges to y^* in the norm; the third one follows from Lemma 1.2. □

The next lemma is a refinement of constructions from [10, Lemma 2.1] and [2, Theorem 1.1]. In its proof we will use the following notation: if $x \in c_0(\Gamma)$ or $x \in \ell_1(\Gamma)$ and $A \subset \Gamma$, then $x|_A$ denotes an element defined as

$$(x|_A)(\gamma) = \begin{cases} x(\gamma) & \gamma \in A, \\ 0 & \gamma \in \Gamma \setminus A. \end{cases}$$

LEMMA 1.4. *Let X be a subspace of $c_0(\Gamma)$, $c > 0$ and (y_n) be a sequence in $\ell_1(\Gamma) = c_0(\Gamma)^*$ such that:*

- (y_n) weak* converges to 0 in $\ell_1(\Gamma)$;
- $\|y_n|_X\| > c$ for each $n \in \mathbb{N}$.

Then for any $\eta > 0$ there is a subsequence (y_{n_k}) such that each weak cluster point of $(y_{n_k}|_X)$ in X^{***} has norm at least $c - \eta$.*

PROOF. For $n \in \mathbb{N}$, set $\varphi_n = y_n|_X$. Let $\varepsilon \in (0, c/6)$. Without loss of generality, we may assume that $\varepsilon < 1$. Select strictly positive numbers (ε_k) such that $\sum_{k=1}^\infty \varepsilon_k < \varepsilon$.

We inductively construct elements $x_k \in X$, indices $n_1 < n_2 < \dots$ and finite sets $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma$ such that, for each $k \in \mathbb{N}$:

- (a) $\|x_k\| \leq 1$, $x_k|_{\Gamma_{k-1}} = 0$ and $\|x_k|_{\Gamma \setminus \Gamma_k}\| < \varepsilon_k$;
- (b) $|\varphi_{n_k}(x_k)| > c - \varepsilon$ and $|\varphi_{n_k}(\sum_{i=1}^{k-1} x_i)| \leq \varepsilon \cdot \|\sum_{i=1}^{k-1} x_i\|$;
- (c) if we denote $y_{n_k}^1 = y_{n_k}|_{\Gamma_k}$ and $y_{n_k}^2 = y_{n_k}|_{\Gamma \setminus \Gamma_k}$, then $\|y_{n_k}^2\| < \varepsilon_k$.

In the first step, we set $\Gamma_0 = \emptyset$ and $n_1 = 1$. Since $\|\varphi_{n_1}\| > c$, there is $x_1 \in B_X$ with $|\varphi_{n_1}(x_1)| > c$. Choose a finite set $\Gamma_1 \subset \Gamma$ satisfying

$$\|x_1|_{\Gamma \setminus \Gamma_1}\| < \varepsilon_1 \quad \text{and} \quad \|y_{n_1}|_{\Gamma \setminus \Gamma_1}\| < \varepsilon_1.$$

Since the second requirement in (b) is vacuous, the first step is finished.

Assume now that we have found indices $n_1 < \dots < n_k$, finite sets $\emptyset = \Gamma_0 \subset \dots \subset \Gamma_k$ and elements x_1, \dots, x_k satisfying (a), (b) and (c). We define an operator $R_k : X \rightarrow c_0(\Gamma)$ by

$$R_k x = x|_{\Gamma_k}, \quad x \in X.$$

Then $\text{Ker } R_k$ is of finite codimension and thus $F_k = (\text{Ker } R_k)^\perp$ is a finite-dimensional space in X^* . Let $m \in \mathbb{N}$ be chosen such that, for each $n \geq m$:

- $|\varphi_n(\sum_{i=1}^{k-1} x_i)| \leq \varepsilon \cdot \|\sum_{i=1}^{k-1} x_i\|$; and
- $\text{dist}(\varphi_n, F_k) > c - \varepsilon$.

(The first requirement can be fulfilled because (φ_n) converges weak* to 0 and the second by Lemma 1.3.) Let $n_{k+1} = m$ and

$$x_{k+1} \in (F_k)_\perp = \text{Ker } R_k$$

be chosen such that $\|x_{k+1}\| \leq 1$ and $\varphi_{n_{k+1}}(x_{k+1}) > c - \varepsilon$. (We use the fact that $X^*/F_k = ((F_k)_\perp)^*$.) We find a finite set $\Gamma_{k+1} \supset \Gamma_k$ satisfying

$$\|x_{k+1}|_{\Gamma \setminus \Gamma_{k+1}}\| < \varepsilon_{k+1} \quad \text{and} \quad \|y_{n_{k+1}}|_{\Gamma \setminus \Gamma_{k+1}}\| < \varepsilon_{k+1}.$$

This finishes the construction.

For $J \in \mathbb{N}$, let

$$u_J = \sum_{i=1}^J x_i.$$

It follows from (a) that, for each $k \in \mathbb{N}$ and $J > k$,

$$\left\| \sum_{i=1}^k x_i \right\| < 1 + \varepsilon, \quad \left\| \sum_{i=1}^{k-1} x_i \right\| < 1 + \varepsilon, \quad \left\| \sum_{i=k+1}^J x_i \right\| < 1 + \varepsilon. \tag{1.2}$$

Indeed, for $k \in \mathbb{N}$ and $\gamma \in \Gamma_k \setminus \Gamma_{k-1}$, we have from (a)

$$|x_j(\gamma)| \leq \begin{cases} \varepsilon_j & j < k, \\ 1 & j = k, j \in \mathbb{N}. \\ 0 & j > k, \end{cases}$$

Further, x_k is bounded by ε_k on $\Gamma \setminus \bigcup_{k=1}^\infty \Gamma_k$ by (a). These observations verify (1.2).

For each $k \in \mathbb{N}$, we set

$$\varphi_{n_k}^1 = y_{n_k}^1|_X \quad \text{and} \quad \varphi_{n_k}^2 = y_{n_k}^2|_X.$$

For a fixed index $k \in \mathbb{N}$ and arbitrary $J > k$, we need to estimate

$$|\varphi_{n_k}(u_J)| = \left| \varphi_{n_k} \left(\sum_{i=1}^{k-1} x_i \right) + \varphi_{n_k}(x_k) + \varphi_{n_k} \left(\sum_{i=k+1}^J x_i \right) \right|. \tag{1.3}$$

Condition (b) and (1.2) ensure that

$$\left| \varphi_{n_k} \left(\sum_{i=1}^{k-1} x_i \right) \right| \leq \varepsilon \cdot \left\| \sum_{i=1}^{k-1} x_i \right\| < \varepsilon(1 + \varepsilon). \tag{1.4}$$

From (b) we also have

$$|\varphi_{n_k}(x_k)| > c - \varepsilon. \tag{1.5}$$

Finally, (a) and (c) give

$$\begin{aligned} \left| \varphi_{n_k} \left(\sum_{i=k+1}^J x_i \right) \right| &= \left| \left(\varphi_{n_k}^1 + \varphi_{n_k}^2 \right) \left(\sum_{i=k+1}^J x_i \right) \right| \\ &= \left| y_{n_k}^2 \left(\sum_{i=k+1}^J x_i \right) \right| \leq \varepsilon_k \cdot \left\| \sum_{i=k+1}^J x_i \right\| \\ &< \varepsilon_k (1 + \varepsilon). \end{aligned} \tag{1.6}$$

Using (1.4)–(1.6) in (1.3), we get

$$\begin{aligned} |\varphi_{n_k}(u_J)| &\geq c - \varepsilon - \varepsilon(1 + \varepsilon) - \varepsilon_k(1 + \varepsilon) \\ &\geq c - \varepsilon(3 + 2\varepsilon) \geq c - 5\varepsilon. \end{aligned} \tag{1.7}$$

It follows from (1.7) that, for $z_J = (1 + \varepsilon)^{-1}u_J$, we have $z_J \in B_X$ by (1.2) and

$$|\varphi_{n_k}(z_J)| > (1 + \varepsilon)^{-1}(c - 5\varepsilon), \quad k \in \mathbb{N}, J > k.$$

Let $z^{**} \in B_{X^{**}}$ be a weak* cluster point of (z_J) . Then

$$|\varphi_{n_k}(z^{**})| \geq (1 + \varepsilon)^{-1}(c - 5\varepsilon), \quad k \in \mathbb{N}.$$

It follows that each weak* cluster point of (φ_{n_k}) has norm at least $(1 + \varepsilon)^{-1}(c - 5\varepsilon)$. This completes the proof, as given $\eta > 0$, we can choose ε at the outset such that

$$(1 + \varepsilon)^{-1}(c - 5\varepsilon) > c - \eta. \quad \square$$

PROOF OF THEOREM 1.1. Let X be a subspace of $c_0(\Gamma)$ and (x_n^*) be a sequence in X^* bounded by a constant M . Suppose that $0 < c < ca(x_n^*)$. We extract subsequences (a_n) and (b_n) from (x_n^*) such that

$$c < \|a_n - b_n\|, \quad n \in \mathbb{N}.$$

We denote $\varphi_n = a_n - b_n$, $n \in \mathbb{N}$. We extend a_n to $A_n \in \ell_1(\Gamma)$ and φ_n to $z_n \in \ell_1(\Gamma)$ with preservation of the norm and set $B_n = A_n - z_n$. Then B_n is an extension of b_n (not necessarily preserving the norm). By passing to a subsequence if necessary, we can assume that (A_n) converges pointwise (and hence weak* in $\ell_1(\Gamma)$) to some $A \in \ell_1(\Gamma)$ and (B_n) converges pointwise to some $B \in \ell_1(\Gamma)$. (This is possible due to the fact that any sequence in $\ell_1(\Gamma)$ can be viewed as a sequence in $\ell_1(\Gamma')$ for a countable $\Gamma' \subset \Gamma$.) Then (z_n) weak* converges to $A - B$. Set $y_n = z_n - A + B$ for $n \in \mathbb{N}$. Then (y_n) weak* converges to 0 and $\|y_n|_X\| > c - \|(A - B)|_X\|$ for each $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrary. By Lemma 1.4, there is a subsequence (y_{n_k}) such that each weak* cluster point of $(y_{n_k}|_X)$ in X^{***} has norm at least

$$c - \|(A - B)|_X\| - \varepsilon.$$

Let a be a weak* cluster point of (a_{n_k}) in X^{***} . Let (a_τ) be a subnet of (a_{n_k}) weak* converging to a . Let b be a weak* cluster point of the net (b_τ) . Then a and b are weak* cluster points of (x_n^*) in X^{***} .

Obviously $a|_X = A|_X$ and $b|_X = B|_X$ and, $a - b - (a - b)|_X = a - b - (A - B)|_X$ is a weak* cluster point of $(y_{n_k}|_X)$ in X^{***} . Thus

$$\|a - b - (a - b)|_X\| \geq c - \|(A - B)|_X\| - \varepsilon.$$

Let $F \in (\ell_\infty(\Gamma))^* = c_0(\Gamma)^{***}$ be an extension of $a - b$ preserving the norm. Then

$$\begin{aligned} \|a - b\| &= \|F\| = \|F|_{c_0(\Gamma)}\| + \|F - F|_{c_0(\Gamma)}\| \geq \|F|_X\| + \|(F - F|_{c_0(\Gamma)})|_{X^{**}}\| \\ &= \|(A - B)|_X\| + \|a - b - (a - b)|_X\| \\ &\geq \|(A - B)|_X\| + c - \|(A - B)|_X\| - \varepsilon \\ &= c - \varepsilon. \end{aligned}$$

(For a Banach space Y and $G \in Y^{***}$, we denote by $G|_Y$ the respective element of Y^* canonically embedded into Y^{***} .) It follows that $\delta(x_k^*) \geq c - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\delta(x_k^*) \geq c$. Hence $ca(x_k^*) \leq \delta(x_k^*)$ and the proof is complete. \square

2. Quantitative Schur property and quantitative Dunford–Pettis property

It is well known that the Schur property is closely related to the Dunford–Pettis property. Recall that a Banach space X is said to have the *Dunford–Pettis property* if for any Banach space Y every weakly compact operator $T : X \rightarrow Y$ is completely continuous. Let us further recall that T is *weakly compact* if the image by T of the unit ball of X is relatively weakly compact in Y , and that T is *completely continuous* if it maps weakly convergent sequences to norm convergent ones, or, equivalently, if it maps weakly Cauchy sequences to norm Cauchy (hence norm convergent) ones. Obviously, any Banach space with the Schur property has the Dunford–Pettis property. Further, any Banach space whose dual has the Schur property enjoys the Dunford–Pettis property as well.

Quantitative variants of the Dunford–Pettis property were studied in [8] where two strengthenings of the Dunford–Pettis property were introduced (the *direct quantitative Dunford–Pettis property* and *dual quantitative Dunford–Pettis property*; see [8, Definition 5.6]). Section 6 of [8] shows several relations between the Schur property and the two variants of the quantitative Dunford–Pettis properties. In this section we focus on the relationship of the quantitative Schur property and quantitative versions of the Dunford–Pettis property.

The unexplained notation and notions in this section are taken from [8]. More specifically, the quantities $ca_{\rho^*}(\cdot)$ and $ca_{\rho}(\cdot)$ measure how far the given sequence is from being Cauchy in the Mackey topology of X^* or the restriction to X of the Mackey topology of X^{**} , respectively. The quantity $\widetilde{\delta}(\cdot)$ is defined by taking the infimum of $\delta(\cdot)$ over all subsequences. Similarly for $\widetilde{ca}(\cdot)$, $\widetilde{ca}_{\rho^*}(\cdot)$ and $\widetilde{ca}_{\rho}(\cdot)$. These quantities are defined and described in detail in [8, Section 2.3]. Further, $\widehat{d}(\cdot, \cdot)$ is the nonsymmetrised Hausdorff distance, $\chi(\cdot)$ denotes the Hausdorff measure of norm noncompactness, $\omega(\cdot)$ and $wk_X(\cdot)$ are measures of weak noncompactness; see [8, Section 2.5]. Applying a measure of (weak) noncompactness to an operator means applying it to the image of the unit ball (see [8, Section 2.6]). Finally, the quantity $cc(\cdot)$

measures how far the given operator is from being completely continuous; that is, if $T : X \rightarrow Y$ is an operator, then

$$cc(T) = \sup\{ca(Tx_k) : (x_k) \text{ is a weakly Cauchy sequence in } B_X\},$$

see [8, Section 2.4].

It is obvious that a Banach space X with the Schur property possesses also the direct quantitative Dunford–Pettis property (see [8, Proposition 6.2]). If we assume that X has a C -Schur property, we get the following result.

THEOREM 2.1. *Let X be a Banach space with the C -Schur property where $C > 0$.*

- (i) *Then $ca_\rho(x_n) \leq C\delta(x_n)$ for any bounded sequence (x_n) in X . In particular, X has both the direct and the dual quantitative Dunford–Pettis properties.*
- (ii) *The space X satisfies the following stronger version of the dual quantitative Dunford–Pettis property: if $A \subset X$ is a bounded set, then*

$$wk_X(A) \leq \omega(A) = \chi(A) \leq 2C wk_X(A). \tag{2.1}$$

PROOF. The inequality in assertion (i) follows from $ca_\rho(x_n) \leq ca(x_n)$ for any bounded sequence (x_n) in X which is an immediate consequence of the definitions. Thus X satisfies condition (iv) of [8, Theorem 5.5], that is, X possesses the dual quantitative Dunford–Pettis property. Further, from [8, Proposition 6.2] we know that X has the direct quantitative Dunford–Pettis property.

(ii) First we notice that (2.1) is indeed a stronger version of the dual quantitative Dunford–Pettis property. Indeed, using [8, diagram (3.1) and formula (2.6)] one can deduce from (2.1) the validity of condition (i) of [8, Theorem 5.5]. For the proof of (2.1), let A be a bounded set in X . If (x_k) in X is a bounded sequence, by taking consecutively infima in (1.1) over all subsequences, we obtain

$$\widetilde{ca}(x_k) \leq C\widetilde{\delta}(x_k). \tag{2.2}$$

By [9, Theorem 1],

$$\widetilde{\delta}(x_k) \leq 2\widehat{d}(\text{clust}_{X^{**}}(x_k), X) \tag{2.3}$$

for any bounded sequence (x_k) in an arbitrary Banach space, and (2.3) and (2.2) together yield

$$\widetilde{ca}(x_k) \leq 2C\widehat{d}(\text{clust}_{X^{**}}(x_k), X). \tag{2.4}$$

Since obviously (see [8, inequalities (2.2)])

$$\chi(A) \leq \sup\{\widetilde{ca}(x_k) : (x_k) \text{ is a sequence in } A\},$$

(2.4) yields

$$\chi(A) \leq 2C wk_X(A). \tag{2.5}$$

Since X has the C -Schur property, it has the Schur property, and thus any weakly compact subset of X is norm compact. Hence

$$\chi(A) = \omega(A). \tag{2.6}$$

A consecutive use of [8, inequality (2.4)], (2.6) and (2.5) gives

$$\text{wk}_X(A) \leq \omega(A) = \chi(A) \leq 2C \text{wk}_X(A),$$

which is the inequality (2.1). \square

If the dual X^* of a Banach space X possesses the Schur property, then we have by [8, Theorem 6.3] that X has the dual quantitative Dunford–Pettis property and, moreover, for any Banach space Y and an operator $T : X \rightarrow Y$ the following inequalities hold:

$$\text{wk}_Y(T) \leq \omega(T) \leq \chi(T) \leq \text{cc}(T) \leq 2\omega(T^*) = 2\chi(T^*) \leq 4\chi(T).$$

Thus the quantities $\chi(T)$, $\text{cc}(T)$, $\chi(T^*)$ and $\omega(T^*)$ are equivalent in this case. However, the quantities $\omega(T)$ and $\text{wk}_Y(T)$ need not be equivalent to the others, that is, X need not have the direct quantitative Dunford–Pettis property (see [8, Example 10.1]). However, if we assume that X^* has a quantitative version of the Schur property, then, for an operator T with domain X , the compactness (both norm and weak) of T and its adjoint are quantitatively equivalent to the complete continuity of T .

THEOREM 2.2. *Let X be a Banach space such that X^* has the C -Schur property for some $C \geq 0$. If Y is a Banach space and $T : X \rightarrow Y$ is a bounded linear operator, then*

$$\begin{aligned} \text{wk}_Y(T) \leq \omega(T) \leq \chi(T) \leq \text{cc}(T) \\ \leq 2\omega(T^*) = 2\chi(T^*) \leq 4C \text{wk}_{X^*}(T^*) \leq 8C \text{wk}_Y(T). \end{aligned} \quad (2.7)$$

In particular, X has both the direct and the dual quantitative Dunford–Pettis properties.

PROOF. The first five inequalities are contained in [8, Theorem 6.3(i)]. By Theorem 2.1 we get the sixth inequality. The last inequality follows from [8, Equation (2.8)]. Further, X^* has both the direct and dual quantitative Dunford–Pettis property by Theorem 2.1(i). Hence X itself possesses both the direct and dual quantitative Dunford–Pettis property by [8, Theorem 5.7]. \square

If we combine the previous theorem with Theorem 1.1, we immediately get the following corollary.

COROLLARY 2.3. *Let X be a subspace of $c_0(\Gamma)$. Then X has both the direct and dual quantitative Dunford–Pettis properties. Moreover, inequalities (2.7) are satisfied with $C = 1$.*

In the case where $X = c_0(\Gamma)$, [8, Theorem 8.2] yields even stronger inequalities (with $C = 1/2$). This is proved by a different method.

We continue with a characterisation of spaces whose dual has the quantitative Schur property. It is well known that the dual space X^* of a Banach space X has the Schur property if and only if X has the Dunford–Pettis property and contains no copy of ℓ_1 (see [4, Theorem 3]). The following theorem quantifies this assertion.

THEOREM 2.4. *Let X be a Banach space. Then X^* has the quantitative Schur property if and only if X has the direct quantitative Dunford–Pettis property and contains no copy of ℓ_1 .*

PROOF. Suppose that X^* has the quantitative Schur property. Then X contains no copy of ℓ_1 . Indeed, if X contains an isomorphic copy of ℓ_1 , by [12, Proposition 3.3] the dual space X^* contains an isomorphic copy of $C(\{0, 1\}^{\mathbb{N}})^*$, hence also an isomorphic copy of $C([0, 1])^*$. The space $C([0, 1])^*$ fails the Schur property as it contains a copy of $L^1(0, 1)$. Thus X^* fails the Schur property as well. Further, X has the direct quantitative Dunford–Pettis property by Theorem 2.1.

For the proof of the converse implication we need the following consequence of Rosenthal’s ℓ_1 -theorem.

LEMMA 2.5. *Let X be a Banach space not containing an isomorphic copy of ℓ_1 . Then any bounded sequence (x_n^*) in X^* satisfies $\text{ca}(x_n^*) \leq 3 \text{ca}_{\rho^*}(x_n^*)$.*

PROOF. If (x_n^*) is norm Cauchy, then the inequality is obvious. So, suppose that $\text{ca}(x_n^*) > 0$ and fix any $c \in (0, \text{ca}(x_n^*))$. Then there is a sequence of natural numbers $l_n < m_n < l_{n+1}$, $n \in \mathbb{N}$, and a sequence (x_n) in B_X such that $|(x_{l_n}^* - x_{m_n}^*)(x_n)| > c$ for every $n \in \mathbb{N}$. By Rosenthal’s ℓ_1 -theorem, there is a weakly Cauchy subsequence of (x_n) . Let us assume, without loss of generality, that $l_n = 2n - 1$ and $m_n = 2n$ for every $n \in \mathbb{N}$ and that (x_n) is weakly Cauchy. Since, for every $k \in \mathbb{N}$, the singleton $\{x_k\}$ is a weakly compact set in B_X , there is some $n_k > k$ such that $|(x_{2n_k-1}^* - x_{2n_k}^*)(x_k)| < \text{ca}_{\rho^*}(x_n^*) + (1/k)$. Using this estimate and the fact that $\{(x_{n_k} - x_k)/2 : k \in \mathbb{N}\}$ is a relatively weakly compact subset of B_X , we can write

$$\begin{aligned} c &\leq \limsup |(x_{2n_k-1}^* - x_{2n_k}^*)(x_{n_k})| \\ &\leq 2 \limsup |(x_{2n_k-1}^* - x_{2n_k}^*)(2^{-1}(x_{n_k} - x_k))| + \limsup |(x_{2n_k-1}^* - x_{2n_k}^*)(x_k)| \\ &\leq 2 \text{ca}_{\rho^*}(x_n^*) + \limsup \left(\text{ca}_{\rho^*}(x_n^*) + \frac{1}{k} \right) = 3 \text{ca}_{\rho^*}(x_n^*). \end{aligned}$$

This completes the proof of the lemma. \square

For the converse implication in Theorem 2.4, suppose now that X has the direct Dunford–Pettis property. Then there exists $C > 0$ such that

$$\text{ca}_{\rho^*}(x_n^*) \leq C\delta(x_n^*)$$

for any bounded sequence (x_n^*) in X^* (see [8, Theorem 5.4(iv)]). By Lemma 2.5,

$$\text{ca}(x_n^*) \leq 3 \text{ca}_{\rho^*}(x_n^*) \leq 3C\delta(x_n^*)$$

for any bounded sequence (x_n^*) in X^* . Hence X^* has the $3C$ -Schur property. \square

3. Subspaces of $C(K)$ for K scattered

It is natural to ask whether Theorem 1.1 holds for a larger class of spaces in place of $c_0(\Gamma)$. The first attempt is to consider isomorphic ℓ_1 preduals, that is, spaces whose dual is isomorphic to ℓ_1 . But this has no chance due to the old result of Bourgain and Delbaen [1] later improved by Haydon [7]. In fact, Freeman *et al.* recently proved in [5] that any Banach space with separable dual can be embedded into a space whose dual is isomorphic to ℓ_1 .

The second attempt is to consider isometric ℓ_1 preduals, that is, spaces whose dual is isometric to ℓ_1 (or, more generally, $\ell_1(\Gamma)$). We focus on the case $C(K)$, K scattered. We can substitute $C(K)$ for $c_0(\Gamma)$ if and only if K has finite Cantor–Bendixson rank. Of course, the constant 1 should be replaced by a constant depending on the height of K . The positive part of this result is contained in Theorem 3.1 below which essentially follows from the Bessaga–Pełczyński classification of $C(K)$ for K countable. In Example 3.2 we show that the constant really depends on the height. This example can be viewed as an approximation of the example constructed in [13] which is recalled as a part of Example 3.3 below.

THEOREM 3.1. *For $n \in \mathbb{N}$, denote the Banach–Mazur distance of c_0 and $C[0, \omega^n]$ by C_n . Let K be a compact space satisfying $K^{(n+1)} = \emptyset$ for some n and let X be a Banach space isometric to a subspace of $C(K)$. Then X^* has the C_{n+1} -Schur property.*

PROOF. Let n , K and X satisfy the assumptions. Firstly, we will show that without loss of generality we may assume that X is separable. Indeed, let (x_k^*) be any bounded sequence in X^* . Denote by Z the closed linear span of this sequence. Then Z is separable. Let D be a countable norm-dense subset of Z . It is now easy to find a separable subspace $Y \subset X$ such that $\|x^*|_Y\| = \|x^*\|$ for each $x^* \in D$. Then the mapping $x^* \mapsto x^*|_Y$ is an isometric injection of Z into Y^* . Therefore the quantities $\text{ca}(x_k^*)$ and $\delta(x_k^*)$ are the same when computed in X^* , Z or Y^* . Therefore, if we know that Y^* has the C_{n+1} -Schur property, we deduce that $\text{ca}(x_k^*) \leq C_{n+1}\delta(x_k^*)$. Since (x_k^*) was arbitrary, this proves the C_{n+1} -Schur property of X^* .

So in the rest of the proof we will suppose that X is separable. Let \tilde{X} be the closed algebra generated by X and the constant function 1. Then \tilde{X} is canonically isometric to $C(L)$, where L is a quotient of K . (This is a well-known consequence of the Stone–Weierstrass theorem: define on K an equivalence \sim by $k \sim l$ if and only if $x(k) = x(l)$ for $x \in \tilde{X}$ (equivalently for $x \in X$). Then $L = K/\sim$ is a compact space and \tilde{X} is isometric to $C(L)$.) Since X is separable, \tilde{X} is separable as well, hence L is metrisable. Further, since $K^{(n+1)} = \emptyset$, we get also $L^{(n+1)} = \emptyset$. (Indeed, let q be the quotient mapping of K onto L . It is easy to check that $L' \subset q(K')$ and by induction we get $L^{(k)} \subset q(K^{(k)})$ for $k \in \mathbb{N}$.)

Therefore, without loss of generality K is countable. It follows that K is homeomorphic to $[0, \alpha]$ for an ordinal $\alpha < \omega^{n+1}$. Since $C[0, \alpha]$ is isometric to a subspace of $C[0, \beta]$ for $\alpha < \beta$, X is isometric to a subspace of $C[0, \omega^{n+1}]$. This space is isomorphic to c_0 by the Bessaga–Pełczyński classification of $C(K)$ for K countable. Let

$d > C_{n+1}$ be arbitrary. It follows that there is an onto isomorphism $T : C[0, \omega^{n+1}] \rightarrow c_0$ with $\|T\| \cdot \|T^{-1}\| < d$. Then $T(X)$ is an isometric subspace of c_0 , so $T(X)^*$ has the 1-Schur property by Theorem 1.1. Further, $S = (T|_X)^*$ is an isomorphism of X^* onto $T(X)^*$ with $\|S\| \cdot \|S^{-1}\| < d$. Let (x_n^*) be a bounded sequence in X^* . Then

$$\text{ca}(x_n^*) \leq \|S^{-1}\| \text{ca}(Sx_n^*) = \|S^{-1}\| \delta(Sx_n^*) \leq \|S^{-1}\| \cdot \|S\| \delta(x_n^*) \leq d\delta(x_n^*).$$

Hence X^* has the d -Schur property. Since $d > C_{n+1}$ was arbitrary, X^* has the C_{n+1} -Schur property. □

THEOREM 3.2. *For each $n \in \mathbb{N}$ there exists a Banach space X_n with the following properties.*

- (i) X_n is isomorphic to c_0 .
- (ii) X_n is isometric to a subspace of $C[0, \omega^{n+1}]$.
- (iii) There are sequences (e_k) in X_n and (e_k^*) in X_n^* with the following properties:
 - (a) $\|e_k\|_n = 1$ for each $k \in \mathbb{N}$;
 - (b) the sequence (e_k) converges weakly to zero in X_n ;
 - (c) for any $x^{**} \in X_n^{**}$ with $\|x^{**}\| \leq 1$, we have $\limsup |x^{**}(e_k^*)| \leq (2/n)$;
 - (d) $x_k^*(x_k) = 1$ for each $k \in \mathbb{N}$.

In particular, X_n does not have the direct quantitative Dunford–Pettis property in the sense of [8, Theorem 5.4(iii)] with constant $C < n/2$ and X_n^ does not have the C -Schur property for $C < n/16$.*

PROOF. For $x = (x_k) \in c_0$ and $p \in \mathbb{N}$ set

$$A_p(x) = \sup\{|x_p + x_{i_1} + \dots + x_{i_p}| : p < i_1 < i_2 < \dots < i_p\}.$$

It is easy to check that

$$\max(|x_p|, \frac{1}{2} \max\{|x_k| : k > p\}) \leq A_p(x) \leq (p + 1)\|x\|_\infty$$

for any $x \in c_0$. Next let us fix some $n \in \mathbb{N}$. For $x \in c_0$ we set

$$\|x\|_n = \max\{A_p(x) : 1 \leq p \leq n\}.$$

It follows that $\|\cdot\|_n$ is an equivalent norm on c_0 . Set $X_n = (c_0, \|\cdot\|_n)$. Then (i) is obviously fulfilled.

Let us show (ii). For $k \in \mathbb{N}$, let Λ_k be the subset of $[0, \omega]^k$ formed by nondecreasing sequences equipped with the lexicographic order. Then it is easy to check that Λ_k is order-isomorphic to the ordinal interval $[0, \omega^k]$. Further, set $\Lambda = \bigcup_{k=1}^{n+1} \Lambda_k$. Define an order on Λ such that the shorter sequences are smaller and Λ_k is ordered lexicographically. The set Λ equipped with this order is order-isomorphic to the ordinal interval $[0, \omega^{n+1}]$. Define a mapping $\varphi : X_n \rightarrow \mathbb{R}^\Lambda$ by the formula

$$\varphi(x)(i_1, \dots, i_k) = \begin{cases} x(i_1) + \dots + x(i_l) & \text{if } 1 \leq i_1 \leq n \text{ and } l \leq k \text{ is maximal such that} \\ & l - 1 \leq i_1 < i_2 < \dots < i_l, \\ 0 & \text{otherwise,} \end{cases}$$

where we use the convention that $x(\omega) = 0$. Then φ is a well-defined isometry of X_n into $\ell_\infty(\Lambda)$ and, moreover, $\varphi(X_n) \subset C(\Lambda)$ (where Λ is considered with the order topology).

Indeed, the inequality $\|x\|_n \leq \|\varphi(x)\|_\infty$ is obvious. To see the converse, we fix l and i_1, \dots, i_l such that $1 \leq i_1 \leq n$ and $l - 1 \leq i_1 < i_2 < \dots < i_l$. First suppose that $i_l < \omega$. If $l - 1 = i_1$, then

$$|x(i_1) + \dots + x(i_l)| \leq A_{l-1}(x) \leq \|x\|_n.$$

If $l - 1 < i_1$, then

$$|x(i_1) + \dots + x(i_l)| = \lim_{m \rightarrow \infty} \left| x(i_1) + \dots + x(i_l) + \sum_{j=1}^{i_1-l+1} x(i_l + k + j) \right| \leq A_{i_1}(x) \leq \|x\|_n.$$

If $i_l = \omega$, then $x(i_l) = 0$, hence

$$|x(i_1) + \dots + x(i_l)| = |x(i_1) + \dots + x(i_{l-1})| \leq \|x\|_n$$

by the previous case. This completes the proof that φ is an isometry.

Finally, let us show that $\varphi(x)$ is a continuous function for each $x \in X_n$. Fix $x \in X_n$ and $\mathbf{i} = (i_1, \dots, i_k) \in \Lambda$. If $i_k < \omega$, then \mathbf{i} is an isolated point of Λ , so $\varphi(x)$ is continuous at this point. So suppose that $i_k = \omega$. If $i_1 > n$, then

$$\{(j_1, \dots, j_k) : j_1 > n\} = \{\mathbf{j} \in \Lambda_k : \mathbf{j} > (n, \omega, \dots, \omega)\}$$

is an open set containing \mathbf{i} on which $\varphi(x)$ is zero. Hence $\varphi(x)$ is continuous at \mathbf{i} . Similarly, if $i_1 = 0$, then

$$\{(j_1, \dots, j_k) : j_1 = 0\} = \{\mathbf{j} \in \Lambda_k : \mathbf{j} < (1, 1, \dots, 1)\}$$

is an open set containing \mathbf{i} on which $\varphi(x)$ is zero. Next suppose that $1 \leq i_1 \leq n$. Let m be the smallest index such that $i_m = \omega$. Necessarily $m \geq 2$. For $r \in \mathbb{N}$, $r \leq i_{m-1}$, let

$$V_r = \{(i_1, \dots, i_{m-1}, j_m, j_{m+1}, \dots, j_k) : j_m > r\} = ((i_1, \dots, i_{m-1}, r, \omega, \dots, \omega), \mathbf{i}].$$

These sets form a neighbourhood basis of \mathbf{i} in Λ . Let l be maximal such that $l - 1 \leq i_1 < i_2 < \dots < i_l$. Necessarily $l \leq m$. If $l < m - 1$, then $\varphi(x)$ is constant on $V_{i_{m-1}}$. If $l = m - 1$, then necessarily $l - 1 = i_1$ and hence $\varphi(x)$ is again constant on $V_{i_{m-1}}$. If $l = m$, then $\varphi(x)(\mathbf{i}) = x(i_1) + \dots + x(i_{m-1})$. For $\mathbf{j} \in V_r$,

$$|\varphi(x)(\mathbf{i}) - \varphi(x)(\mathbf{j})| \leq (k - l + 1) \sup_{j \geq r} |x(j)|.$$

Since $x \in c_0$, this shows that $\varphi(x)$ is continuous at \mathbf{i} and completes the proof that $\varphi(x) \in C(\Lambda)$.

Finally, let us prove (iii). Let (e_k) be the canonical basis of X_n . It follows from the definition that $\|e_k\|_n = 1$ for each k , hence (a) holds. Moreover, e_k converges weakly to zero, as in c_0 , so (b) holds as well. Let (e_k^*) be the sequence of biorthogonal functionals to (e_k) . Then clearly (d) holds.

Let us show (c). Fix any $x^{**} \in X_n^{**}$ satisfying $\|x^{**}\| \leq 1$. Suppose for contradiction that $\limsup |x^{**}(e_k^*)| > 2/n$. Then there is $\eta > 2/n$ such that $|x^{**}(e_k^*)| > \eta$ for infinitely

many k . Without loss of generality we may suppose that $x^{**}(e_k^*) > \eta$ for infinitely many k (otherwise we would replace x^{**} by $-x^{**}$). Therefore we can find indices $n < i_1 < i_2 < \dots < i_n$ such that $x^{**}(e_{i_j}^*) > \eta$ for $j = 1, \dots, n$. By Goldstine’s theorem there is $x \in B_{X_n}$ with $e_{i_j}^*(x) > \eta$ for $j = 1, \dots, n$. Then

$$1 \geq \|x\|_n \geq A_n(x) \geq |x_n + x_{i_1} + \dots + x_{i_n}| \geq n\eta - |x_n| \geq n\eta - A_n(x) \geq n\eta - 1,$$

hence $\eta \leq 2/n$, a contradiction.

It remains to prove the ‘in particular’ part. It follows from (c) that any w^* -cluster point of (e_k^*) in X_n^{***} has norm at most $2/n$. Now it immediately follows that the space X_n does not satisfy condition (iii) of [8, Theorem 5.4] for $C < n/2$. Further, if X_n^* has the C -Schur property, by Theorem 2.1(i), X_n^* satisfies condition (iv) of [8, Theorem 5.6] with constant C . It follows from the proof of the quoted theorem that X_n^* satisfies condition (iii) of that theorem with constant $8C$. Finally, by [8, Remark 5.8] the space X_n satisfies condition (iii) of [8, Theorem 5.4] with the same constant $8C$. By the above we get $8C \geq n/2$, hence $C \geq n/16$ and the proof is complete. \square

EXAMPLE 3.3. There are Banach spaces Y_1 and Y_2 with the following properties:

- (i) Both Y_1 and Y_2 are isometric to subspaces of $C[0, \omega^\omega]$.
- (ii) Y_1 fails the Dunford–Pettis property, so Y_1^* fails the Schur property.
- (iii) Y_2^* has the Schur property, so Y_2 has the dual quantitative Dunford–Pettis property. Y_2 fails the direct quantitative Dunford–Pettis property.

PROOF. The existence of Y_1 is a result of [13]. The above spaces X_n are in a sense approximations of Y_1 . For the sake of completeness we recall the definition of Y_1 .

The quantity $A_p(x)$ defined above makes sense for any $x \in \mathbb{R}^{\mathbb{N}}$. Further, it is finite if and only if x is bounded. The space $(Y_1, \|\cdot\|)$ is defined as

$$Y_1 = \{x \in \ell_\infty : A_p(x) \rightarrow 0\}, \quad \|x\| = \sup\{A_p(x) : p \in \mathbb{N}\}.$$

It is proved in [13] that Y_1 is isometric to a subspace of $C[0, \omega^\omega]$, the canonical basis (e_k) of Y_1 is unconditional, the orthogonal functionals (e_k^*) form also an unconditional basis of Y_1^* , e_k weakly converge to zero, e_k^* as well, while $e_k^*(e_k) = 1$. This proves the failure of the Dunford–Pettis property.

The space Y_2 can be taken to be the c_0 -sum of the spaces X_n , $n \in \mathbb{N}$. Then all the properties easily follow. \square

In view of the previous example the following question seems to be natural.

QUESTION 3.4. Let X be a Banach space isometric to a subspace of $C(K)$ with K scattered. Suppose that X has the Dunford–Pettis property. Does X^* have the Schur property? Does X have the dual quantitative Dunford–Pettis property?

A related topic is the study of spaces having the hereditary Dunford–Pettis property, that is, spaces all of whose subspaces enjoy the Dunford–Pettis property. Within $C(K)$ spaces, they are exactly those such that K has finite height (as explicitly formulated in [3, Theorem 1] as a consequence of [13]). Further, spaces with the Schur property enjoy the hereditary Dunford–Pettis property as well. The space constructed by Hagler in [6] has the hereditary Dunford–Pettis property by [3, Proposition 2]. It seems to us that the following questions are interesting.

QUESTION 3.5.

- (1) Is the space Y_2 from Example 3.3 hereditarily Dunford–Pettis?
- (2) Let $X \subset C(K)$ with K scattered be hereditarily Dunford–Pettis. Is X contained in $C(L)$ for some L with finite height?
- (3) Does X^* have the Schur property for any subspace of the space from [6]?

4. Subspaces of the space of compact operators

The space $K(\ell_2)$ of all compact operators on the Hilbert space ℓ_2 can be viewed as a noncommutative version of c_0 and its dual $N(\ell_2)$, the space of all nuclear operators on ℓ_2 equipped with the nuclear norm, can be viewed as a noncommutative version of ℓ_1 . The noncommutative versions share many properties of the commutative ones, but the Schur property and Dunford–Pettis property are essentially commutative.

Indeed, $N(\ell_2)$ does not have the Schur property and, moreover, $K(\ell_2)$ does not enjoy the Dunford–Pettis property. This is witnessed by the following easy example. Let (e_n) denote the standard basis in ℓ_2 . Consider the operators $T_n(x) = \langle x, e_1 \rangle e_n$, $x \in \ell_2$, and $S_n(x) = \langle x, e_n \rangle e_1$. These operators are rank-one operators, thus they are nuclear and hence compact. Moreover, both sequences converge weakly to 0 both in $K(\ell_2)$ and $N(\ell_2)$. The Schur property of $N(\ell_2)$ can be disproved by observing that $\|S_n\| = \|T_n\| = \|e_1\| \|e_n\| = 1$. Moreover, the failure of the Dunford–Pettis property of $K(\ell_2)$ follows by the fact that $\text{Tr}(S_n T_n) = 1$.

This easy observation was strengthened in [14], where the authors show that a subspace of $K(\ell_p)$, the space of compact operators on ℓ_p , enjoys the Dunford–Pettis property if and only if it is isomorphic to a subspace of c_0 (that is, only in the ‘commutative case’). Theorem 1.1 enables us to complement and strengthen their result to show that such a space automatically has a quantitative Dunford–Pettis property.

More precisely, we prove the following:

THEOREM 4.1. *Let X be a subspace of the space $K(\ell_p)$ of compact operators on ℓ_p where $1 < p < \infty$. Then the following assertions are equivalent:*

- (i) X has the Dunford–Pettis property.
- (ii) X^* has the Schur property.
- (iii) X is isomorphic to a subspace of c_0 . Moreover, in this case, there is for each $\varepsilon > 0$ an isomorphic embedding $T : X \rightarrow c_0$ such that $\|T\| \|T^{-1}\| < 4 + \varepsilon$.
- (iv) X^* has the 4-Schur property.
- (v) For each Banach space Y and each bounded linear operator $T : X \rightarrow Y$, inequalities (2.7) hold with $C = 4$.
- (vi) The space X has both the dual and the direct quantitative Dunford–Pettis properties.

PROOF. The implication (ii) \Rightarrow (i) is well known (see [4, Theorem 3]).

(i) \Rightarrow (iii). If $X \subset K(\ell_p)$ has the Dunford–Pettis property, it is embeddable into c_0 by [14, Theorem 1]. Moreover, the constant of embedding can be explicitly computed from [14, Lemmas 1 and 2]. Indeed, the embedding $T : X \rightarrow c_0$ is constructed as the composition $\psi \circ \phi_A$, where ϕ_A is provided by [14, Lemma 1] and ψ is provided by [14, Lemma 2]. The operator ψ satisfies $\|\psi\| \|\psi^{-1}\| \leq 4$ by [14, page 420]. Further, ϕ_A satisfies $\|\phi_A\| \|\phi_A^{-1}\| \leq 3$ (see the computation in [14, page 418]), but it can be easily modified to be an almost isometry. Indeed, if we replace in [14, formula (3), page 420] the number $\frac{1}{4}$ by $\varepsilon/2$, then we will obtain $\|\phi_A\| \|\phi_A^{-1}\| \leq (1 + \varepsilon)/(1 - \varepsilon)$. This completes the proof.

The implication (iii) \Rightarrow (iv) follows from Theorem 1.1. Indeed, let $T : X \rightarrow c_0$ be an embedding with $\|T\| = 1$ and $\|T^{-1}\| \leq 4 + \varepsilon$. Let (x_n^*) be a bounded sequence in X^* . Then $((T^*)^{-1}x_n^*)$ is a bounded sequence in $(T(X))^*$ satisfying $\delta((T^*)^{-1}x_n^*) \leq (4 + \varepsilon)\delta(x_n^*)$. By Theorem 1.1 we get $\text{ca}((T^*)^{-1}x_n^*) \leq (4 + \varepsilon)\delta(x_n^*)$, hence $\text{ca}(x_n^*) \leq (4 + \varepsilon)\delta(x_n^*)$ as well. Since $\varepsilon > 0$ is arbitrary, the proof is finished.

The implications (iv) \Rightarrow (v) and (v) \Rightarrow (vi) follow from Theorem 2.2.

Finally, the implications (vi) \Rightarrow (i) and (iv) \Rightarrow (ii) are trivial. \square

References

- [1] J. Bourgain and F. Delbaen, ‘A class of special \mathcal{L}_∞ spaces’, *Acta Math.* **145**(3–4) (1980), 155–176.
- [2] S. W. Brown, ‘Weak sequential convergence in the dual of an algebra of compact operators’, *J. Operator Theory* **33**(1) (1995), 33–42.
- [3] P. Cembranos, ‘The hereditary Dunford–Pettis property on $C(K, E)$ ’, *Illinois J. Math.* **31**(3) (1987), 365–373.
- [4] J. Diestel, ‘A survey of results related to the Dunford–Pettis property’, in: *Proceedings of the Conference on Integration, Topology, and Geometry in Linear Spaces (Univ. North Carolina, Chapel Hill, NC, 1979 (Providence, RI, 1980))*, Contemporary Mathematics, 2 (American Mathematical Society, Providence, RI) 15–60.
- [5] D. Freeman, E. Odell and T. Schlumprecht, ‘The universality of ℓ_1 as a dual space’, *Math. Ann.* **351**(1) (2011), 149–186.
- [6] J. Hagler, ‘A counterexample to several questions about Banach spaces’, *Studia Math.* **60**(3) (1977), 289–308.
- [7] R. Haydon, ‘Subspaces of the Bourgain–Delbaen space’, *Studia Math.* **139**(3) (2000), 275–293.
- [8] M. Kačena, O. F. Kalenda and J. Spurný, ‘Quantitative Dunford–Pettis property’, *Adv. Math.* **234** (2013), 488–527.
- [9] O. F. K. Kalenda, H. Pfitzner and J. Spurný, ‘On quantification of weak sequential completeness’, *J. Funct. Anal.* **260**(10) (2011), 2986–2996.
- [10] O. F. K. Kalenda and J. Spurný, ‘On a difference between quantitative weak sequential completeness and the quantitative Schur property’, *Proc. Amer. Math. Soc.* **140**(10) (2012), 3435–3444.
- [11] N. J. Kalton and D. Werner, ‘Property (M) , M -ideals, and almost isometric structure of Banach spaces’, *J. reine angew. Math.* **461** (1995), 137–178.
- [12] A. Pełczyński, ‘On Banach spaces containing $L_1(\mu)$ ’, *Studia Math.* **30** (1968), 231–246.
- [13] A. Pełczyński and W. Szlenk, ‘An example of a non-shrinking basis’, *Rev. Roumaine Math. Pures Appl.* **10** (1965), 961–966.
- [14] E. Saksman and H.-O. Tylli, ‘Structure of subspaces of the compact operators having the Dunford–Pettis property’, *Math. Z.* **232** (1999), 411–425.

ONDŘEJ F. K. KALENDA, Department of Mathematical Analysis,
Faculty of Mathematics and Physics, Charles University,
Sokolovská 83, 186 75, Praha 8, Czech Republic
e-mail: kalenda@karlin.mff.cuni.cz

JIŘÍ SPURNÝ, Department of Mathematical Analysis,
Faculty of Mathematics and Physics, Charles University,
Sokolovská 83, 186 75, Praha 8, Czech Republic
e-mail: spurny@karlin.mff.cuni.cz