On an Asymptotic Expansion of the Hypergeometric Function.

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In a previous paper* the author has employed the expansion

$$F(\alpha, \beta, \gamma, z) = \sum_{r=1}^{s} T_r + P_s, \quad \text{.....................(1)}$$

where

$$T_r = \frac{\alpha(\alpha+1)\dots(\alpha+r-2)\beta(\beta+1)\dots(\beta+r-2)}{(r-1)!\,\gamma(\gamma+1)\dots(\gamma+r-2)} z^{r-1},$$

$$P_s = \frac{1}{B(\beta+s, \gamma-\beta)} T_{s+1} \int_0^1 s(1-t)^{s-1} I dt,$$

and †
$$I = \int_0^1 \zeta^{\beta+s-1}(1-\zeta)^{\gamma-\beta-1}(1-\zeta t z)^{-\alpha-s} d\zeta,$$

to establish the theorem that, if $-\pi/2 < amp\,\gamma < \pi/2$, the function P_s/T_{s+1} remains finite as $\gamma \rightarrow \infty$. This theorem is valid provided that z is not real and $\geqq 1$.

It will now be shown that the theorem is true for a more extended range of values of $amp\,\gamma$.

In the integral I put $\zeta = 1 - e^{-\lambda}$‡; then

$$I = \int_0^\infty (1-e^{-\lambda})^{\beta+s-1} e^{-\lambda(\gamma-\beta)} \{1 - tz(1-e^{-\lambda})\}^{-\alpha-s} d\lambda, \quad \text{.........(2)}$$

the path of integration being the real axis from 0 to ∞. This path may now be deformed into that straight line from the origin to infinity which makes an acute angle $-\phi$ with the positive real axis, and the integral is still convergent provided that, on the path of integration, $1 - tz(1-e^{-\lambda}) \neq 0$, and provided that

$$-\pi/2 < amp\,(\lambda\gamma) < \pi/2.$$

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† $R(\gamma)$ and s are taken so large that $R(\gamma-\beta) > 0$ and $R(\beta+s) > 0$.

‡ *Cf.* G. N. Watson, *Trans. Camb. Phil. Soc.*, Vol. 22, 1918, p. 299.

Since $amp\,\lambda = -\phi$ the latter inequality may be written

$$-\pi/2 + \phi < \text{amp}\,\gamma < \pi/2 + \phi,$$

and, by reversing the transformation $\zeta = 1 - e^{-\lambda}$, it can be made clear that the first condition may be replaced by the proviso that $1 - tz\zeta$ must not vanish at any point on the contour in the ζ-plane which corresponds to the path of integration in the λ-plane.

To determine this contour put $\lambda = \mu(1 - it)$, where μ is real and $t = tan\phi$; then

$$\zeta = \xi + i\eta = 1 - e^{-\lambda} = 1 - e^{-\mu + i\mu t},$$

so that

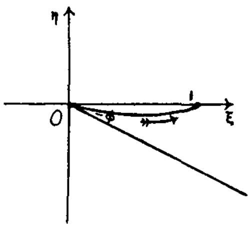
$$\xi = 1 - e^{-\mu}\cos\mu t, \quad \eta = -e^{-\mu}\sin\mu t.$$

Hence, if (r, θ) are the polar coordinates of the point (ξ, η) referred to axes parallel to the ξ and η axes and passing through the point $(1, 0)$ in the ζ-plane

$$r = e^{-\mu}, \ \tan\theta = \tan\mu t,$$

and the contour is an equiangular spiral (see Fig.) whose equation may be written

$$r = e^{-(\theta + \pi)\cot\phi},$$



where $\theta = -\pi$ and $r = 1$ when $\mu = 0$, $\lambda = 0$, $\zeta = 0$; at this point $\dfrac{d\zeta}{d\lambda} = 1$, so that the contour makes an angle $-\phi$ with the ξ-axis, and the path is described from $\zeta = 0$ in the direction indicated by the arrow.

It can easily be shown that the entire contour of integration lies between the lines $amp\zeta = \pm\phi$. Now $1 - tz\zeta$ must not vanish for any point ζ on this contour. But, since $0 \leqq t \leqq 1$, the values of z which satisfy $z = 1/(t\zeta)$ will lie in the region to the right of the η-axis which is bounded by the lines $amp\zeta = \pm\phi$. Also the

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hypergeometric expansion is valid within the unit circle; hence the expansion (1) is valid for

$$-\pi/2+\phi<\text{amp}\,\gamma<\pi/2+\phi$$

at all points external to a region B which is bounded by the lines $amp\zeta=\pm\phi$ and the circle $|\zeta|=1$.

But if z is any interior point of the region A consisting of the entire complex plane bounded by a cross-cut along the positive real axis from $+1$ to $+\infty$, ϕ can be chosen so small $(<|\text{amp}\,z|)$ that z does not lie in B. Hence for any interior point of A a ϕ can be found such that the expansion defined by (1) and (2) is valid for $-\pi/2+\phi<amp\,\gamma<\pi/2+\phi$.

It remains to prove that the theorem is true under these conditions. Now, for points on the path of integration in the λ-plane

$$\begin{aligned}|1-e^{\lambda}| &= \surd(1-2e^{-\mu}\cos\mu t+e^{-2\mu})\\ &= \surd\{(1-e^{-\mu})^2+2e^{-\mu}(1-\cos\mu t)\}\\ &=(1-e^{-\mu})\sqrt{\left\{1+\left(\frac{\sin\frac{1}{2}\mu t}{\sinh\frac{1}{2}\mu}\right)^2\right\}}\\ &<C(1-e^{-\mu}),\end{aligned}$$

where C is a definite positive constant $(|\phi|<\pi/2)$. Also, let $\gamma=ge^{i\phi}$, so that $-\pi/2<amp\,g<\pi/2$, and note that $\lambda=\mu\sec\phi e^{-i\phi}$. Then, if $\beta=\sigma+i\tau$ and $g=l+im$,

$$R\{\lambda(\gamma-\beta)\}=\mu(l\sec\phi-\sigma-\tau\tan\phi).$$

Accordingly, for any point z within the region A,

$$|I|<K\int_0^\infty(1-e^{-\mu})^{\sigma+s-1}e^{-\mu(l\sec\phi-\sigma-\tau\tan\phi)}d\mu,$$

where K is a definite constant. Here put $\xi=1-e^{-\mu}$ and get

$$\begin{aligned}|I| &<K\int_0^1\xi^{\sigma+s-1}(1-\xi)^{l\sec\phi-\sigma-\tau\tan\phi-1}d\xi\\ &=KB(\sigma+s,\,l\sec\phi-\sigma-\tau\tan\phi):\end{aligned}$$

thus

$$\left|\frac{P_s}{T_{s+1}}\right|<K\frac{B(\sigma+s,\,l\sec\phi-\sigma-\tau\tan\phi)}{|B(\beta+s,\,\gamma-\beta)|}$$

$$=K\frac{\Gamma(\sigma+s)}{|\Gamma(\beta+s)|}\,\frac{\Gamma(l\sec\phi-\sigma-\tau\tan\phi)}{\Gamma(l\sec\phi-\tau\tan\phi+s)}\left|\frac{\Gamma(\gamma+s)}{\Gamma(\gamma-\beta)}\right|,$$

and when $\gamma\to\infty$, $l\to\infty$, and

$$\left|\frac{P_s}{T_{s+1}}\right| < K\frac{\Gamma(\sigma+s)}{|\,\Gamma(\beta+s)\,|}\,\frac{|\,\gamma^{\beta+s}\,|}{(l\sec\phi)^{\sigma+s}} = K\frac{\Gamma(\sigma+s)}{|\,\Gamma(\beta+s)\,|}\left(\frac{|\,\gamma\,|}{l\sec\phi}\right)^{\sigma+s}e^{-\chi\tau},$$

where $amp\,\gamma = \chi$. But $|\,\gamma\,|\,/l$ is finite; hence the theorem holds.

Similarly the theorem can be shown to hold for the region $-\pi/2-\phi < amp\,\gamma < \pi/2-\phi$; thus it holds for the entire region

$$-\pi/2-\phi < \text{amp}\,\gamma < \pi/2+\phi.$$

The Asymptotic Expansion of $P_n^m(z)$. It follows that, for any interior point z of the region in which the asymptotic expansion of $P_n^m(z)$ for n large is valid, a ϕ can be found such that the asymptotic expansion holds for $-\pi/2-\phi < amp\,n < \pi/2+\phi$. Hence, as $P_{-n-1}^m(z) = P_n^m(z)$, the function possesses an asymptotic expansion for any value of *amp n*.

By means of the formula

$$Q_n^m(z) = Q_{-n-1}^m(z) + \frac{\pi e^{m\pi i}\cos n\pi}{\sin(n-m)\pi}\,\frac{\Pi(n+m)}{\Pi(n-m)}\,P_{-n-1}^{-m}(z)$$

a similar result can be obtained for the function $Q_n^m(z)$.