BULL. AUSTRAL. MATH. SOC. 60G42, 60G46
VOL. 45 (1992) [43-52]

ON THE DUALITY OF SOME MARTINGALE SPACES

N.L. BASSILY AND A.M. ABDEL-FATTAH

Fefferman has proved that the dual space of the martingale Hardy space H_1 is the BMO_1-space. Garsia went further and proved that the dual of H_p is the so-called martingale K_p-space, where p and q are two conjugate numbers and $1 \leqslant p < 2$.

The martingale Hardy spaces H_Φ with general Young function Φ, were investigated by Bassily and Mogyoródi. In this paper we show that the dual of the martingale Hardy space H_Φ is the martingale Hardy space H_Ψ where (Φ, Ψ) is a pair of conjugate Young functions such that both Φ and Ψ have finite power. Moreover, two other remarkable dualities are presented.

1. BASIC NOTATIONS AND DEFINITIONS

Let $X \in L^1(\Omega, A, P)$ be a random variable defined on the probability space (Ω, A, P) and consider the regular martingale

$$X_n = E(X \mid F_n), \qquad n \geqslant 0,$$

where $\{F_n\}$, $n \geqslant 0$, is an increasing sequence of σ-fields of events such that

$$F_\infty = \sigma\left(\bigcup_{n=0}^{\infty} F_n\right) = A.$$

We suppose that $X_0 = 0$ almost surely. We denote by $d_0 = 0,\ d_1,\ d_2, \ldots$ the difference sequence corresponding to the martingale (X_n, F_n).

The K_p-spaces were investigated by Garsia (see [2]).

In [3] we generalised this notion. Consider a pair (Φ, Ψ) of conjugate Young functions and let

$$\mu_X^{(\Phi)} = \{\gamma : \gamma \in L^\Phi,\ E(|X - X_{n-1}| \mid F_n) \leqslant E(\gamma \mid F_n) \text{ almost surely } \forall n \geqslant 1\},$$

We say that $X \in K_\Phi$ if the set $\mu_X^{(\Phi)}$ is not empty. In this case we define

$$\|X\|_{K_\Phi} = \inf_{\gamma \in \mu_X^{(\Phi)}} \|\gamma\|_\Phi\,,$$

Received 14 January 1991

https://doi.org/10.1017/S0004972700036996 Published online by Cambridge University Press

where $\|.\|_\Phi$ denotes the Luxemburg norm in the Orlicz-space L^Φ. For the definition of the Young functions, Orlicz-spaces and Luxemburg norms we refer to [4] and [5]. It is easily proved that $\left(K_\Phi, \|.\|_{K_\Phi}\right)$ is a Banach space (see [3]). The space K_∞ is the well-known BMO_1-space.

We say that the random variable X belongs to the Hardy space H_Φ if the quadratic variation

$$S = S(X) = \left(\sum_{i=1}^{\infty} d_i^2\right)^{1/2} \in L^\Phi.$$

It is easy to show that H_Φ with the norm $\|X\|_{H_\Phi} = \|S\|_\Phi$ is a Banach space (see [3]).

A Young function Φ is said to be of moderated growth if its power

$$p = \sup_{x>0} (x\phi(x))/(\Phi(x))$$

is finite. Here $\phi(x)$ stands for the right-hand side derivative of Φ.

2. Auxiliary results

Lemma 1. *If the Young function Φ has a finite power, then $H_\Phi \subset K_\Phi$.*

Proof: In fact, the Burkholder-Davis-Gundy inequality (see [6, Theorem 15.1]) guarantees that $X \in H_\Psi$ implies that

$$X^* = \sup_{n\geqslant 1} |X_n| \in L^\Phi,$$

where $X_n = E(X \mid F_n)$, $n \geqslant 1$.

From this for all $n \geqslant 1$ we have almost surely.

$$E(|X - X_{n-1}| \mid F_n) \leqslant E(2X^* \mid F_n).$$

Consequently, $X \in K_\Phi$ with $\|X\|_{K_\Phi} \leqslant 2\|X^*\|_\Phi$.

The following assertion gives a sufficient condition which ensures that the martingale Hardy space H_Φ and the martingale space K_Φ coincide and the corresponding norms are equivalent. ∎

Theorem 1. *Suppose that Φ and its conjugate Ψ have finite powers p and q respectively. Then, the spaces H_Φ and K_Φ coincide. More precisely, there exist positive constants $c_\Phi^{(1)}$ and $C_\Phi^{(1)}$ depending only on Φ such that*

$$c_\Phi^{(1)} \|X\|_{K_\Phi} \leqslant \|X\|_{H_\Phi} \leqslant C_\Phi^{(1)} \|X\|_{K_\Phi}.$$

PROOF: Suppose that $X \in K_\Phi$. Let $X_n = E(X \mid F_n)$, $n \geqslant 1$ be the corresponding regular martingale and let us define

$$X_n^* = \max_{1 \leqslant \ell \leqslant n} |X_\ell|, \qquad n \geqslant 1.$$

This random variable with arbitrary constants $\beta > \alpha > 0$ satisfies the inequality

$$(\beta - \alpha) P(X_n^* \geqslant \beta) \leqslant E\big(\gamma \chi_{(X_n^* \geqslant \alpha)}\big),$$

where $\gamma \in \mu_X^{(\Phi)}$ is arbitrary and $\chi_{(B)}$ stands for the indicator of B.

For arbitrary $A > 0$ define

$$X_n^{**} = \min(X_n^*, a).$$

Then $X_n^{**} \in L_\infty$ and for arbitrary $\lambda > 0$ we have

$$\chi_{(X_n^{**} \geqslant \lambda)} = \begin{cases} 0 & \text{if } \lambda > a \\ \chi_{(X_n^* \geqslant \lambda)}, & \text{if } \lambda \leqslant a. \end{cases}$$

Consequently, since $\beta > \alpha > 0$, it follows that

$$(\beta - \alpha) P(X_n^{**} \geqslant \beta) \leqslant E\big(\gamma \chi_{(X_n^{**} \geqslant \alpha)}\big).$$

Choose $\beta = c\alpha$ where $c > 1$ is a constant, and integrate the above inequality with respect to the measure $d\phi(\alpha)$ and using Fubini's theorem we get

$$(c-1) E\left(\Psi\left(\phi\left(\frac{X_n^{**}}{c}\right)\right)\right) \leqslant E(\gamma \phi(X_n^{**})).$$

Since Φ has finite power, then for any $c > 1$ there exists a constant $A = A(c) > 0$ such that

$$\phi(cx) \leqslant A\phi(x), \qquad x \geqslant 0.$$

From the preceding inequality we get

$$(c-1) E\left(\Psi\left(\phi\left(\frac{X_n^{**}}{c}\right)\right)\right) \leqslant AE\left(\gamma \phi\left(\frac{X_n^{**}}{c}\right)\right).$$

Applying Young's inequality and rearranging, we have

$$(\rho - 1) E\left(\Psi\left(\phi\left(\frac{X_n^{**}}{c}\right)\right)\right) \leqslant E(\Phi(\gamma/b)),$$

https://doi.org/10.1017/S0004972700036996 Published online by Cambridge University Press

where $b = (c-1)/(A\rho)$ and $\rho > 1$ is arbitrary.

Let $A \uparrow +\infty$, $X_n^{**} \uparrow X_n^*$ and by the monotone convergence theroem we have

$$(\rho - 1)E\left(\Psi\left(\phi\left(\frac{X_n^*}{c}\right)\right)\right) \leqslant E(\Phi(\gamma/b)).$$

Applying the so obtained inequality to the new martingale

$$\left(\frac{X_k}{\|\gamma\|_\Phi} b,\ F_k\right), \qquad k = 1, 2, \ldots$$

we get

$$(\rho - 1)E\left(\Psi\left(\phi\left(\frac{X_n^*}{\rho\frac{c}{c-1}A\,\|\gamma\|_\Phi}\right)\right)\right) \leqslant 1.$$

Since q, the power of Ψ is finite it follows that with $\rho = q$

$$\|X_n^*\|_\Phi \leqslant q\frac{c}{c-1}A\,\|X\|_{K_\Phi}\,.$$

REMARK. Especially, with $\Phi(x) = x^p/p$, $p > 1$, we have $\phi(x) = x^{p-1}$ and $\Psi(x) = x^q/q$, $q > 1$ where $1/p + 1/q = 1$. Thus, if $K \in K_\Phi = K_p$ we have

$$\|X_n^*\|_p \leqslant q\frac{c^p}{c-1}\,\|X\|_{K_p}\,.$$

This is the inequality obtained by Garsia ([**2**, Theorem III.5.2]). The constant $c > 1$ is used to optimise the coefficient on the right hand side in the preceding inequaity. The minimal value of $(c^p)/(c-1)$ is obtained when $c = p/(p-1)$. Thus we get

$$\|X_n^*\|_p \leqslant pq^p\,\|X\|_{K_p} \leqslant pqe\,\|X\|_{K_p}\,.$$

Now, let us denote $X^* = \sup_{n \geqslant 1} |X_n|$, then by the monotone convergence theorem we have

$$\|X^*\|_\Phi \leqslant q\frac{c}{c-1}A\,\|X\|_{K_\Phi}\,.$$

We deduce that $X^* \in L^\Phi$. By the above mentioned Burkholder-Davis-Gundy inequality it follows that $X \in H_\Phi$ and with some $C'_\Phi > 0$ we have

$$c'_\Phi\,\|X\|_{H_\Phi} \leqslant \|X^*\|_\Phi \leqslant q\frac{c}{c-1}A\,\|X\|_{K_\Phi}\,.$$

This proves the right hand side of our inequality.

Conversely, suppose that $X \in H_\Phi$, then using Lemma 1, with some constant $c''_\Phi > 0$, we have

$$\|X\|_{K_\Phi} \leqslant 2\,\|X^*\|_\Phi \leqslant 2c''_\Phi\,\|X\|_{H_\Phi}\,.$$

This proves the left hand side of our inequality. ∎

Lemma 2. *Let (Φ, Ψ) be a pair of conjugate Young functions and suppose that both Φ and Ψ have finite power p and q respectively. Then for every $X \in H_\Phi$ there exist positive constants $c_\Phi^{(2)}$ and $C_\Phi^{(2)}$ depending only on Φ such that the following two sided inequality holds:*

$$c_\Phi^{(2)} \sup_{n\geqslant 0} \|X - X_n\|_\Phi \leqslant \|X\|_{H_\Phi} \leqslant C_\Phi^{(2)} \sup_{n\geqslant 0} \|X - X\|_\Phi .$$

Here $X_n = E(X \mid F_n)$, $n \geqslant 0$.

Proof: Denote $X^* = \sup_{n\geqslant 0} |X_n|$. Since Φ has finite power, then by the Burkholder-Davis-Gundy inequality we have

$$c'_\Phi \|X\|_{H_\Phi} \leqslant \|X^*\|_\Phi \leqslant C''_\Phi \|X\|_{H_\Phi}, \tag{1}$$

where c'_Φ and c''_Φ are positive constants depending only on Φ. Since Ψ has a finite power q, then using Doob's maximal inequality (see [7]) we have

$$\sup_{n\geqslant 0} \|X_n\|_\Phi \leqslant \|X^*\|_\Phi \leqslant q \sup_{n\geqslant 0} \|X_n\| . \tag{2}$$

Remarking that $X_0 = 0$ almost surely and that $\|X_n\|_\Phi \uparrow \|X\|_\Phi$ by using Jensen's inequality and by [4, Appendix (Proposition A-3-4)], (2) implies that

$$\frac{1}{2} \sup_{n\geqslant 0} \|X - X_n\|_\Phi \leqslant \sup_{n\geqslant 0} \|X_n\|_\Phi \leqslant \|X^*\|_\Phi \leqslant q \|X\|_\Phi \leqslant q \sup_{n\geqslant 0} \|X - X_n\|_\Phi .$$

holds. Thus, using (1) our inequality is proved with

$$c_\Phi^{(2)} = 1/2c''_\Phi \quad \text{and} \quad C_\Phi^{(2)} = q/c'_\Phi .$$

∎

Ishak and Mogyoródi (see [8, 9] proved the following result:

Theorem 2. *Let Φ be a Young function with finite power p and Ψ denotes its conjugate Young function, not necessarily with finite power. If $X \in H_\Phi$ and $Y \in K_\Phi$ then the following Fefferman-Garsia type inequality holds*

$$|E(X_n Y_n)| \leqslant c_\Phi^{(3)} \|X_n\|_{H_\Phi} \|Y_n\|_{K_\Psi} ,$$

where $c_\Phi^{(3)}$ is a constant depending only on Φ. Further, the limit $\lim_{n\to+\infty} E(X_n Y_n)$ exists and we have

$$\left| \lim_{n\to+\infty} E(X_n Y_n) \right| \leqslant c_\Phi^{(3)} \|X\|_{H_\Phi} \|Y\|_{K_\Psi} .$$

Here $X_n = E(X \mid F_n)$ and $Y_n = E(Y \mid F_n)$, $n \geqslant 0$.

Now, combining the results of Theorems 1 and 2, we have

https://doi.org/10.1017/S0004972700036996 Published online by Cambridge University Press

THEOREM 3. *Let (Φ, Ψ) be a pair of conjugate Young functions and suppose that both have finite power. If $X \in H_\Phi$ and $Y \in Y_\Psi$ then*

$$|E(XY)| \leqslant C \, \|X\|_{H_\Phi} \, \|Y\|_{H_\Psi} \, ,$$

where $E(XY) = \lim_{n\to+\infty} E(X_n Y_n)$ and C is a constant depending on Φ and Ψ such that $C = c_\Phi^{(3)} c_\Psi^{(1)}$.

PROOF: Using the result of Theorem 1, we have $Y \in K_\Psi$ and

$$\|Y\|_{K_\Psi} \leqslant 1/c_\Psi^{(1)} \, \|Y\|_{H_\Psi} \, .$$

And using the result of Theorem 2 we have $\lim_{n\to+\infty} E(X_n Y_n) = E(XY)$ and

$$|E(XY)| \leqslant C \, \|X\|_{H_\Phi} \, \|Y\|_{H_\Psi} \, , \text{ where } C = c_\Phi^{(3)} / c_\Psi^{(1)}.$$

Let $(T_0, \|.\|_0)$, $(T_1, \|.\|_1)$, ... be a sequence of Banach spaces, and let us define the following Banach spaces

$$T^{(1)} = \left\{ x = (x_0, x_1, \ldots) \in (T_0 \times T_1 \times \ldots) : \|x\|^{(1)} = \sum_{n=0}^{\infty} \|x_n\|_n < +\infty \right\},$$

$$T^{(\infty)} = \left\{ x = (x_1, x_1, \ldots) \in (T_0 \times T_1 \times, \ldots) : \|x\|^{(\infty)} = \sup_{n\geqslant 0} \|x_n\|_n < +\infty \right\},$$

and

$$T_0^{(\infty)} = \left\{ x \in T^{(\infty)} : \lim_{n\to+\infty} \|x_n\|_n = 0, \ \|x\|^{(\infty)} = \sup_{n\geqslant 0} \|x_n\|_n \right\}.$$

∎

Now, we formulate the following lemma without proof (see [10]).

LEMMA 3. *Let B_n be the dual space of T_n, $n = 0, 1, 2, \ldots$. Then, the dual space of $\left(T_0^{(\infty)}, \|.\|^{(\infty)}\right)$ is isomorphic to $\left(B^{(1)}, \|.\|^{(1)}\right)$ and isomorphism can be given by the formula*

$$B^{(1)} \ni y \mapsto f_y = \sum_{n=0}^{\infty} \langle \cdot, y_n \rangle$$

with $\|f_y\| = \sum_{n=0}^{\infty} \|y_n\|_n = \|y\|^{(1)}$.

3. Main result

Theorem 4. *Let (Φ, Ψ) be a pair of conjugate Young functions and suppose that both of them have finite power. Then, the dual space of the martingale Hardy space H_Φ is the martingale Hardy space H_Ψ.*

Proof: If $Y \in H_\Psi$ is fixed and X varies on H_Φ then $\lim_{n\to+\infty} E(X_nY_n)$ is a continuous and linear functional on H_Φ with norm $\leqslant C\,\|Y\|_{H_\Psi}$. Conversely, suppose f is a continuous and linear functional on $\left(H_\Phi, \|.\|_{H_\Phi}\right)$. Then by Lemma 2, f is also continuous with respect to the norm $\sup_{n\geqslant 0}\|X - X_n\|_\Phi$. Consider the Banach space $T_0^{(\infty)}(\Phi)$ defined by the formula

$$T_0^{(\infty)}(\Phi) = \{\lambda = (\lambda_0, \lambda_1, \ldots),\ \lambda_n \in L^\Phi,\ n \geqslant 0,\ \lim_{n\to+\infty}\|\lambda_n\| = 0\}$$

furnished with the norm

$$\|\lambda\|_{T_0^{(\infty)}(\Phi)} = \sup_{n\geqslant 0}\|\lambda_n\|_\Phi\,.$$

Then, the space $\left(H_\Phi, \sup_{n\geqslant 0}\|X - X_n\|_\Phi\right)$ which can be considered as the set of the sequences

$$\widetilde{X} = (X - X_0, X - X_1, X - X_2, \ldots), \qquad X \in H_\Phi$$

is a subspace of $T_0^{(\infty)}(\Phi)$ since X_n converges to X almost surely and in L^Φ-norm. The continuous and linear functional f given on $\left(H_\Phi, \sup_{n\geqslant 0}\|X - X_n\|_\Phi\right)$ can be extended to a linear functional $G(\lambda)$ on $T_0^{(\infty)}(\Phi)$ with the same norm as that of f. This can be done by means of the Hahn-Banach theorem.

Remarking that the dual space of L^Φ is L^Ψ and choosing $T_i = L^\Phi(\Omega, A, P)$, $i = 0, 1, 2, \ldots$, by Lemma 3 there exists a sequence $(\sigma_n)_{n=0}^\infty$ of random variables such that $\sigma_n \in L^\Psi$ with

$$\sum_{n=0}^{\infty}\|\sigma_n\|_\Psi \leqslant \|G\| = \|f\|\,.$$

We also have

$$G(\lambda) = \sum_{n=0}^{\infty} E(\lambda_n\sigma_n) \quad \text{for all } \lambda \in T_0^{(\infty)}(\Phi).$$

Consider now the special sequence

$$\widetilde{X} = (X - X_0, X - X_1, X - X_2, \ldots, S - X_n, \ldots)$$

Putting $\widetilde{X}_n = (X_n - X_0, X_n - X_1, \ldots, X_n - X_{n-1}, 0, 0, \ldots)$, we see that

$$\left\|\widetilde{X} - \widetilde{X}_n\right\|_{T_0^{(\infty)}(\Phi)} = \sup_{k\geqslant n}\|X - X_k\|_\Phi \to 0$$

https://doi.org/10.1017/S0004972700036996 Published online by Cambridge University Press

as $n \to +\infty$. Consequently,

$$G\left(\widetilde{X}\right) = \lim_{n \to +\infty} G\left(\widetilde{X}_n\right).$$

Now, easy calculations show that

$$\begin{aligned} G\left(\widetilde{X}_n\right) &= \sum_{i=0}^{n-1} E[(X_n - X_i)\sigma_i] = \sum_{i=0}^{n-1} E\{[E(X_n \mid F_n) - E(X_n \mid F_i)]\sigma_i\} \\ &= \sum_{i=0}^{n-1} E\{X_n[E(\sigma_i \mid F_n) - E(\sigma_i \mid F_i)]\} \\ &= E\left\{X_n \left[\sum_{i=0}^{n-1} (E(\sigma_i \mid F_n) - E(\sigma_i \mid F_i))\right]\right\}. \end{aligned}$$

Writing

$$\Delta_n = \sum_{i=0}^{n-1} [E(\sigma_i \mid F_n) - E(\sigma_i \mid F_i)],$$

we have

$$G\left(\widetilde{X}\right) = \lim_{n \to +\infty} G\left(\widetilde{X}_n\right) = \lim_{n \to +\infty} E(X_n \Delta_n).$$

It is easy to see that (Δ_n, F_n) is a martingale which satisfies

$$\begin{aligned} \|\Delta_n\|_\Psi &\leqslant \sum_{i=0}^{n-1} \|E(\sigma_i \mid F_n) - E(\sigma_i \mid F_i)\|_\Psi \leqslant 2 \sum_{i=0}^{n-1} \|\sigma_i\|_\Psi \\ &\leqslant 2 \sum_{i=0}^{\infty} \|\sigma_i\|_\Psi \leqslant 2 \|G\|. \end{aligned}$$

This martingale (Δ_n, F_n) is L^Ψ-bounded. It follows that (Δ_n, F_n) is a regular martingale (see [11]) and there exists a random variable $\Delta \in L^\Psi$ such that $\Delta_n = E(\Delta \mid F_n)$. We also show that $\Delta \in K_\Psi = H_\Psi$. This follows from the Doob maximal inequality according to which $\Delta^* = \sup_{n \geqslant 0} |\Delta_n| \in L^\Psi$, since

$$\|\Delta^*\|_\Psi \leqslant \sup_{n \geqslant 0} \|\Delta_n\| \leqslant 2p \sum_{n=0}^{\infty} \|\sigma_n\|_\Psi <\to \infty.$$

This in fact implies that

$$E(|\Delta - \Delta_{n-1}| \mid F_n) \leqslant E(2\Delta^* \mid F_n) \quad \text{almost surely for all } n \geqslant 1,$$

and so $\Delta \in K_\Psi$ and $\|\Delta\|_{K_\Psi} \leqslant 2\|\Delta^*\|_\Psi$. Using the result of Theorem 1, it follows that $\Delta \in H_\Psi$ and

$$\|\Delta\|_{H_\Psi} \leqslant C_\Psi^{(1)} \|X\|_{K_\Psi} \leqslant 2C_\Psi^{(1)} \|\Delta^*\|_\Psi,$$

where $C_\Psi^{(1)}$ is a constant depending only on Ψ defined in Theorem 1. This proves our assertion. ∎

4. Some remarkable dualities

As a direct consequence of our main result proved in Section 3, we are now in a position to present the following remarkable dualities:

Theorem 5. *If (Φ, Ψ) is a pair of conjugate Young functions such that both Φ and Ψ have finite power then:*

(i) *The martingale space K_Φ is the dual space of the martingale K_Ψ-space.*

(ii) *The martingale Hardy space H_Φ is the dual space of the martingale K_Ψ-space.*

In the special case when $\Phi(x) = x^p/p$ and $\Psi(x) = x^q/q$, $1 < p < +\infty$ and $1 < q < +\infty$, it follows that the dual of the space H_p is the space K_q, where $1/p + 1/q = 1$, for all the values of p such that $1 < p < +\infty$. This can be considered as an extension of Garsia's result (see [2]).

References

[1] C. Fefferman, 'Characterizations of bounded mean oscillations', *Bull. Amer. Math. Soc.* **77** (1971), 587–588.

[2] A.M. Garsia, *Martingale Inequalities* (Benjamin Readings, Massachusetts, 1973).

[3] N.L. Bassily and J. Mogyoródi, 'On the K_Φ-spaces with general Young function Φ', *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **XXVII** (1985), 205–214.

[4] J. Neveu, *Discrete parameter martingales* (North-Holland, Amsterdam, 1975).

[5] M.A. Kranoselskii and Ya.B. Rutickii, *Convex functions and Orlicz spaces* (Noordhoff, Gröningen, 1961).

[6] D.L. Burkholder, B. Davis and R.F. Gundy, 'Integral inequalities for convex functions of operators on martingales', in *Proceedings 6th Berkeley symposium on mathematical statistics and probability*, pp. 223–240 (University of California Press, 1972).

[7] J. Mogyoródi and F. Móri, 'Necessary and sufficient condition for the maximal inequality of convex Young functions', *Acta Sci. Math.* **45** (1983), 325–332.

[8] S. Ishak and J. Mogyoródi, 'On the generalization of the Fefferman-Garsia inequality', Stochastic Differential Systems, in *Lecture notes in control and information sciences*, pp. 85–97 (Springer-Verlag, Berlin, 1981).

[9] S. Ishak and J. Mogyoródi, 'On the P_Φ-spaces and the generalization of Hertz' and Fefferman's inequalities I, II and III', *Studia Sci. Math. Hungar.* **17** (1982), 229–234. **18**, pp. 205–210 and **18** (1983), 211–219.

[10] F. Schipp, 'The dual space of the martingale VMO-space', in *Proceedings of the 3rd Pannonian symposium on mathematical statistics*, pp. 305–311 (Visagrád, Hungary, 1982).

[11] D.L. Burkholder, 'Distribution function inequalities for martingales', *Ann. Prob.* **1** (1973), 19–42.

Department of Mathematics
American University of Cairo
P.O. Box 2511
Cairo
Egypt

The Institute of Statistical Studies
Cairo University
Cairo
Egypt