6

Free space solutions of the Dirac equation

In this chapter we display the plane wave solutions of the Dirac equation. We show that a Dirac particle has intrinsic spin $\hbar/2$, and we shall see how the Dirac equation predicts the existence of antiparticles.

6.1 A Dirac particle at rest

In Chapter 5 we showed that the Dirac equation for a particle in free space is equivalent to the coupled two-component equations

$$\begin{aligned} \mathrm{i}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\mathrm{L}} - m\psi_{\mathrm{R}} &= 0, \\ \mathrm{i}\sigma^{\mu}\partial_{\mu}\psi_{\mathrm{R}} - m\psi_{\mathrm{L}} &= 0. \end{aligned} \tag{6.1}$$

These equations have plane wave solutions of the form

$$\psi_{\mathrm{L}} = u_{\mathrm{L}}\mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)}, \ \psi_{\mathrm{R}} = u_{\mathrm{R}}e^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)}, \tag{6.2}$$

where u_{L} and u_{R} are two-component spinors. Since solutions of the Dirac equation also satisfy the Klein–Gordon equation (3.19), we must have

$$E^2 = p^2 + m^2. \tag{6.3}$$

It is simplest to find the solution in a frame K' in which the particle is at rest, and then obtain the solution in a frame in which the particle is moving with velocity v by making a Lorentz boost. Using primes to denote quantities in the frame K', the momentum $\mathbf{p}' = 0$, so that equations (6.1) and (6.3) become

$$\mathrm{i}\partial_0'\psi_{\mathrm{L}}' = m\psi_{\mathrm{R}}', \qquad \mathrm{i}\partial_0'\psi_{\mathrm{R}} = m\psi_{\mathrm{L}}',$$

and

$$E'^2 = m^2, \quad E' = \pm m. \tag{6.4}$$

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The solutions with positive energy $E' = m$ are

$$\psi'_L = u e^{-imt'}, \quad \psi'_R = u e^{-imt'}, \tag{6.5}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is an arbitrary two-component spinor and we are adopting the standard convention of quantum mechanics that the time dependence of an energy eigenstate is given by the phase factor e^{-iEt}.

In the rest frame K', the left-handed and right-handed positive energy spinors are identical. As a consequence this solution is invariant under space inversion (see Section 5.3). It is said to have *positive parity.*

6.2 The intrinsic spin of a Dirac particle

The intrinsic spin operator **S** of a particle with mass is defined to be its angular momentum operator in a frame in which it is at rest. The component of **S** along the z-direction is given by

$$S_z = i\hbar \lim_{\phi \to 0} [R_z(\phi) - 1] / \phi,$$

where $R_z(\phi)$ is the operator that rotates the state of the particle through an angle ϕ about Oz (cf. Section 4.7). A rotation of the state through an angle ϕ, is equivalent to rotating the axes through an angle $-\phi$, and then $\psi_L \to M\psi_L$, $\psi_R \to N\psi_R$ where, from (5.22),

$$M = N = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}.$$

Hence

$$S_z = i\hbar \lim_{\phi \to 0} \frac{1}{\phi} \begin{pmatrix} e^{-i\phi/2} - 1 & 0 \\ 0 & e^{i\phi/2} - 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \sigma_z.$$

In the state with $u_1 = 1, u_2 = 0$,

$$S_z \psi'_L = (\hbar/2)\psi'_L$$

and

$$S_z \psi'_R = (\hbar/2)\psi'_R.$$

Acting on the Dirac wave function, we have

$$S_z \begin{pmatrix} \psi'_{\mathrm{L}} \\ \psi'_{\mathrm{R}} \end{pmatrix} = (\hbar/2) \begin{pmatrix} \psi'_{\mathrm{L}} \\ \psi'_{\mathrm{R}} \end{pmatrix}. \tag{6.6}$$

Similarly, in the state with $u_1 = 0, u_2 = 1$,

$$S_z \begin{pmatrix} \psi'_{\mathrm{L}} \\ \psi'_{\mathrm{R}} \end{pmatrix} = -(\hbar/2) \begin{pmatrix} \psi'_{\mathrm{L}} \\ \psi'_{\mathrm{R}} \end{pmatrix}. \tag{6.7}$$

Thus in the rest frame of the particle there are two independent states which are eigenstates of S_z with eigenvalues $\pm(\hbar/2)$. The operator S_z on a Dirac wave function is represented by the matrix

$$\mathbf{\Sigma}_z = (\hbar/2) \begin{pmatrix} \sigma_z & \mathbf{0} \\ \mathbf{0} & \sigma_z \end{pmatrix}. \tag{6.8}$$

More generally, $\mathbf{S}$ is represented by

$$\mathbf{\Sigma} = (\hbar/2) \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}. \tag{6.9}$$

Also, every Dirac wave function is an eigenstate of the square of the spin operator,

$$\mathbf{\Sigma}^2 = (3/4)\hbar^2 \mathbf{I},$$

with eigenvalue $(3/4)\hbar^2 = (1/2)((1/2)+1)\hbar^2$. Recalling that the square J^2 of the angular momentum for a state with angular momentum j is $j(j+1)\hbar^2$; it is appropriate to say that a Dirac particle has intrinsic spin $\hbar/2$.

6.3 Plane waves and helicity

We now transform to a frame K in which the frame K', and the particle, are moving with velocity v. For simplicity we take $\mathbf{v} = (0, 0, v)$, along the z-axis with $v > 0$, and consider the state with $u_1 = 1, u_2 = 0$.

Transformations between K and K' are then given by (5.23), along with (5.24). Using (5.19) and (5.20),

$$\psi_{\mathrm{L}} = M^{-1}\psi'_{\mathrm{L}} = \begin{pmatrix} \mathrm{e}^{-\theta/2} & 0 \\ 0 & \mathrm{e}^{\theta/2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}mt'} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{e}^{-\mathrm{i}mt'}\mathrm{e}^{-\theta/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\psi_{\mathrm{R}} = N^{-1}\psi'_{\mathrm{R}} = \begin{pmatrix} \mathrm{e}^{\theta/2} & 0 \\ 0 & \mathrm{e}^{-\theta/2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}mt'} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{e}^{-\mathrm{i}mt'}\mathrm{e}^{\theta/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Finally, substituting $t' = t\cosh\theta - z\sinh\theta$ (and noting that $m\cosh\theta = \gamma m = E$, $m\sinh\theta = \gamma m v = p$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ we have

$$\psi_{\rm L} = {\rm e}^{{\rm i}(pz-Et)}\begin{pmatrix} {\rm e}^{-\theta/2} \\ 0 \end{pmatrix}, \qquad \psi_{\rm R} = {\rm e}^{{\rm i}(pz-Et)}\begin{pmatrix} {\rm e}^{\theta/2} \\ 0 \end{pmatrix}. \tag{6.10}$$

The *helicity* operator is useful in classifying plane wave states. It is defined by

$$\text{helicity} = \frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|}. \tag{6.11}$$

The expectation value of this operator in a given state is a measure of the alignment of a particle's intrinsic spin with its direction of motion in that state. For $\mathbf{p} = (0, 0, p)$, $p > 0$, the helicity operator $\boldsymbol{\Sigma}\cdot\mathbf{p}/|\mathbf{p}| = \Sigma_z$. Thus the state (6.10) is an eigenstate of the helicity operator with positive helicity 1/2, which we can write as a Dirac spinor

$$\psi_+ = \frac{1}{\sqrt{2}}{\rm e}^{{\rm i}(pz-Et)}\begin{pmatrix} {\rm e}^{-\theta/2} \\ 0 \\ {\rm e}^{\theta/2} \\ 0 \end{pmatrix}, \quad p > 0. \tag{6.12}$$

We have inserted the normalisation factor $1/\sqrt{2}$ to conform with the standard normalisation of the Lorentz scalar $\bar\psi\psi$:

$$\bar\psi\psi = \psi^\dagger\gamma^0\psi = \psi_{\rm L}^\dagger\psi_{\rm R} + \psi_{\rm R}^\dagger\psi_{\rm L} = 1.$$

Similarly, taking $u_1 = 0, u_2 = 1$, we can construct an eigenstate of negative helicity $-1/2$:

$$\psi_- = \frac{1}{\sqrt{2}}{\rm e}^{{\rm i}(pz-Et)}\begin{pmatrix} 0 \\ {\rm e}^{\theta/2} \\ 0 \\ {\rm e}^{-\theta/2} \end{pmatrix}, \ p > 0. \tag{6.13}$$

All plane waves with positive energy can be generated by applying rotations to the states we have found. The helicity of a state is unchanged by a rotation, since it is defined by a scalar product. The evident generalisations of (6.12) and (6.13) to a wave with wave vector $\mathbf{p}$ are

$$\psi_+ = {\rm e}^{{\rm i}(\mathbf{p}\cdot\mathbf{r}-Et)}u_+(\mathbf{p}) \tag{6.14}$$

where

$$u_+(\mathbf{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} {\rm e}^{-\theta/2}\,|+\rangle \\ {\rm e}^{\theta/2}\,|+\rangle \end{pmatrix},$$

https://doi.org/10.1017/9781009401685.008 Published online by Cambridge University Press

and

$$\psi_- = e^{i(\mathbf{p}\cdot\mathbf{r}-Et)}u_-(\mathbf{p}) \tag{6.15}$$

where

$$u_-(\mathbf{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\theta/2}\,|-\rangle \\ e^{-\theta/2}\,|-\rangle \end{pmatrix}.$$

The Pauli spin states $|\pm\rangle$ are here the eigenstates of the operators $\boldsymbol{\sigma}\cdot\mathbf{p}/|\mathbf{p}|$ with eigenvalues ± 1 (Problem 6.6). A general state of positive energy can be constructed as a superposition of plane waves.

6.4 Negative energy solutions

In the frame K' in which the particle is at rest, there are also negative energy solutions of (6.4) with $E' = -m$:

$$\psi'_L = v e^{imt'}, \quad \psi'_R = -v e^{imt'}. \tag{6.16}$$

In this case the left-handed and right-handed spinors v differ in sign. Thus the negative energy solution changes sign under space inversion (see Section 5.3). It is said to have *negative parity*.

The same Lorentz boost we used above in Section 6.3 gives solutions ψ_+ and ψ_- with positive and negative helicity, respectively, which we can write as Dirac spinors

$$\psi_+ = \frac{1}{\sqrt{2}}e^{i(-pz+Et)}\begin{pmatrix} 0 \\ e^{\theta/2} \\ 0 \\ -e^{-\theta/2} \end{pmatrix}, \quad \psi_- = \frac{1}{\sqrt{2}}e^{i(-pz+Et)}\begin{pmatrix} -e^{-\theta/2} \\ 0 \\ e^{\theta/2} \\ 0 \end{pmatrix}, \quad p > 0. \tag{6.17}$$

These solutions generalise to

$$\psi_+ = e^{i(-\mathbf{p}\cdot\mathbf{r}+Et)}v_+(\mathbf{p}) \tag{6.18}$$

where

$$v_+(\mathbf{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\theta/2}\,|-\rangle \\ -e^{-\theta/2}\,|-\rangle \end{pmatrix},$$

and

$$\psi_- = e^{i(-\mathbf{p}\cdot\mathbf{r}+Et)}v_-(\mathbf{p}) \tag{6.19}$$

where

$$v_-(\mathbf{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{\theta/2}\,|+\rangle \\ e^{-\theta/2}\,|+\rangle \end{pmatrix}.$$

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$|+\rangle$ and $|-\rangle$ remain eigenstates of $\boldsymbol{\sigma} \cdot \mathbf{p}/|\mathbf{p}|$ as defined below (6.15). Note that the Lorentz invariant $\bar{\psi}\psi$ acquires a minus sign; in the case of the negative energy solutions,

$$\bar{\psi}\psi = \psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{R}} + \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}} = -1.$$

Negative energy solutions of the Dirac equation appear at first sight to be an embarrassment. In quantum theory a particle can make transitions between states. Hence all Dirac states would seem to be unstable to a transition to lower energy. Dirac's solution to the difficulty was to assume that nearly all negative energy states are occupied, so that the Pauli exclusion principle forbids transitions to them. An unoccupied negative energy state, or *hole*, will behave as a positive energy *antiparticle*, of the same mass but opposite momentum, spin, and electric charge. Left unfilled, the negative energy state ψ_+ of (6.17) corresponds to an antiparticle of positive energy E and positive momentum p, and positive helicity, since the spin of the hole is also opposite to that of the negative energy state.

A particle falling into an empty negative energy state will be seen as the simultaneous annihilation of a particle–antiparticle pair with the emission of electromagnetic energy $\geq 2mc^2$. Conversely, the excitation of a particle from a negative energy state to a positive energy state will be seen as pair production. The existence of the positron, the antiparticle of the electron, was established experimentally in 1932, and the observation of pair production soon followed.

The uniform background sea of occupied negative energy states, with its associated infinite electric charge, is assumed to be unobservable. In any case, it is clearly quite arbitrary whether, say, the electron is regarded as the particle and the positron as antiparticle, or vice versa. Evidently our starting interpretation of the Dirac equation as a single particle equation is not tenable. We are led, inevitably, to a quantum field theory in which particles and antiparticles appear as the quanta of the field, in somewhat the same way as photons appear as the quanta of the electromagnetic field. We shall take up this theme in Chapter 8.

6.5 The energy and momentum of the Dirac field

The Lagrangian density of the Dirac field is given by (5.31), which we display in more detail:

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\psi \\ &= \mathrm{i}\psi_a^{*}\partial_0\psi_a + \bar{\psi}_b\left(\mathrm{i}\gamma_{ba}^{i}\partial_i - m\delta_{ba}\right)\psi_a.\end{aligned} \tag{6.20}$$

As in Section 5.1 we may treat the fields ψ_a and $\psi_a{}^*$ as independent, and take the energy–momentum tensor to be

$$T_{\nu}^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi_a)}\partial_{\nu}\psi_a - \mathcal{L}\delta_{\nu}^{\mu} \tag{6.21}$$

($\mathcal{L}$ does not depend on $\partial_{\mu}\psi_a{}^*$).

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In particular, the energy density is

$$\begin{aligned} T_0^0 &= \mathrm{i}\psi_a^* \partial_0 \psi_a - \mathcal{L} \\ &= \bar{\psi}(-\mathrm{i}\gamma^i \partial_i + m)\psi \end{aligned} \tag{6.22}$$

and the momentum density is

$$T_i^0 = \mathrm{i}\psi_a^* \partial_i \psi_a = \mathrm{i}\psi^\dagger \partial_i \psi. \tag{6.23}$$

The general solution of the free space Dirac equation is a superposition of all possible plane waves, which we will write

$$\psi = \frac{1}{\sqrt{V}} \sum_{\mathbf{p},\varepsilon} \sqrt{\frac{m}{E_\mathbf{p}}} \left(b_{\mathbf{p}\varepsilon} u_\varepsilon(\mathbf{p}) \mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} - E_p t)} + d_{\mathbf{p}\varepsilon}^* v_\varepsilon(\mathbf{p}) \mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r} + E_p t)} \right). \tag{6.24}$$

ε is the helicity index, $\pm$, and $b_{\mathbf{p}\varepsilon}$ and $d_{\mathbf{p}\varepsilon}$ are arbitrary complex numbers. The factors $\sqrt{(m/E_p)}$ take the place of the factors $1/\sqrt{2\omega_k}$ we inserted in the boson field expansions of Chapter 3 and Chapter 4.

We can express the total energy and total momentum of the Dirac field in terms of the wave amplitudes, by inserting the field expansion into T_0^0 and T_i^0, and integrating over the normalisation volume V. The results are

$$H = \sum_{\mathbf{p},\varepsilon} \left(b_{\mathbf{p}\varepsilon}^* b_{\mathbf{p}\varepsilon} - d_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon}^* \right) E_\mathbf{p}, \tag{6.25}$$

$$P = \sum_{\mathrm{p},\varepsilon} \left(b_{\mathbf{p}\varepsilon}^* b_{\mathbf{p}\varepsilon} - d_{\mathbf{p}\varepsilon} d_{\mathbf{p}\varepsilon}^* \right) \mathbf{p}. \tag{6.26}$$

$\varepsilon = \pm 1$ is the helicity index.

The (somewhat tedious) derivation of these results is left to the reader. Note that each plane wave is a solution of the Dirac equation (5.32), which implies

$$\begin{aligned} (\gamma^0 E_p - \gamma^i p^i) u_\varepsilon(\mathbf{p}) &= m u_\varepsilon(\mathbf{p}), \\ (\gamma^0 E_p - \gamma^i p^i) v_\varepsilon(\mathbf{p}) &= -m v_\varepsilon(\mathbf{p}). \end{aligned} \tag{6.27}$$

It is also necessary to use various orthogonality relations, which are set out in Problem 6.3.

For later convenience, we rewrite the Dirac field ψ (6.24) in terms of ψ_L and ψ_R. Using (6.14), (6.15), (6.18) and (6.19) gives

$$\begin{aligned} \psi_\mathrm{L} = \frac{1}{\sqrt{V}} \sum_\mathbf{p} \sqrt{\frac{m}{2E_p}} \Big[&\left(b_{\mathbf{p}+} \mathrm{e}^{-\theta/2} \left|+\right\rangle + b_{\mathbf{p}-} \mathrm{e}^{\theta/2} \left|-\right\rangle \right) \mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} - Et)} \\ &+ \left(d_{\mathbf{p}+}^* \mathrm{e}^{\theta/2} \left|-\right\rangle - d_{\mathbf{p}-}^* \mathrm{e}^{-\theta/2} \left|+\right\rangle \right) \mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r} + Et)} \Big] \end{aligned} \tag{6.28}$$

$$\begin{aligned} \psi_\mathrm{R} = \frac{1}{\sqrt{V}} \sum_\mathbf{p} \sqrt{\frac{m}{2E_p}} \Big[&\left(b_{\mathbf{p}+} \mathrm{e}^{\theta/2} \left|+\right\rangle + b_{\mathbf{p}-} \mathrm{e}^{-\theta/2} \left|-\right\rangle \right) \mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r} - Et)} \\ &+ \left(-d_{\mathbf{p}+}^* \mathrm{e}^{-\theta/2} \left|-\right\rangle + d_{\mathbf{p}-}^* \mathrm{e}^{\theta/2} \left|+\right\rangle \right) \mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r} + Et)} \Big] \end{aligned} \tag{6.29}$$

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6.6 Dirac and Majorana fields

The expansion (6.24) is the general solution of the free field Dirac equation. For every momentum $\mathbf{p}$ there are four independent complex coefficients: $b_{\mathbf{p}+}$, $b_{\mathbf{p}-}$, $d^*_{\mathbf{p}+}$ and $d^*_{\mathbf{p}-}$, which correspond to particles with helicities $+1/2$, $-1/2$ and antiparticles with helicities $+1/2$, $-1/2$, respectively.

It will be of interest, in Chapter 21, to consider solutions in which we impose the constraint that $d_{\mathbf{p}+} = b_{\mathbf{p}+}$, $d_{\mathbf{p}-} = b_{\mathbf{p}-}$, and hence $d^*_{\mathbf{p}+} = b^*_{\mathbf{p}+}$, $d^*_{\mathbf{p}-} = b^*_{\mathbf{p}-}$. These solutions are known as *Majorana fields*. On quantisation, we shall see that the Dirac fields create and annihilate particles, and antiparticles. For example, if ψ is an electron field it creates positrons and annihilates electrons, $\psi^\dagger$ creates electrons and annihilates positrons. With the Majorana constraint, particles and antiparticles are identical. Majorana fields are irrelevant for electrically charged particles, but it is possible that the electrically neutral neutrino fields have this property. It is still an open question whether neutrino fields are Dirac or Majorana.

6.7 The $E \gg m$ limit, neutrinos

The coefficients of the plane waves in the expansions (6.25) and (6.26) may be expressed as

$$\sqrt{(m/2E)}\mathrm{e}^{\pm\theta/2} = \{(1 \pm v/c)/2\}^{1/2}, \tag{6.30}$$

where v is the particle velocity (Problem 6.1). In the high energy limit, $E \gg m$, the velocity $v \to c$. The only significant terms in the field expansions which survive in this limit are

$$\psi_{\mathrm{L}} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \left(b_{\mathbf{p}-} \left|-\right\rangle \mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)} + d^*_{\mathbf{p}+} \left|-\right\rangle \mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r}+Et)} \right), \tag{6.31}$$

$$\psi_{\mathrm{R}} = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \left(b_{\mathbf{p}+} \left|+\right\rangle \mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{r}-Et)} + d^*_{\mathbf{p}-} \left|+\right\rangle \mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r}+Et)} \right). \tag{6.32}$$

In the limit, ψ_{L} and ψ_{R} are completely independent: ψ_{L} involves only negative helicity particles and positive helicity antiparticles; ψ_{R} involves only positive helicity particles and negative helicity antiparticles.

Since neutrinos are electrically neutral, they are accessible to experimental investigation only through the weak interaction and we shall see in Chapter 9 that in the weak interaction Nature only employs ψ_{L}. In practice neutrino energies are usually many orders of magnitude greater than their mass, so that only negative helicity neutrinos and positive helicity antineutrinos are readily observed. It has not so far been established that the ‘hard to see’ positive helicity neutrino is different from the ‘easy to see’ positive helicity antineutrino.

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Problems

6.1 With the normalisaion of ψ_+ determined by equation (6.14), show that

$$\psi_+^\dagger \psi_+ = \cosh\theta = E/m.$$

(Note that this is not the usual normalisation of particle quantum mechanics.)

Show that the probability of this positive helicity state being in the right-handed mode is

$$e^\theta/(2\cosh\theta) = (1 + v/c)/2$$

and the probability of its being in the left-handed mode is $(1 - v/c)/2$. What are the corresponding results for ψ_-?

6.2 Show that the negative energy positive helicity state of equation (6.18) has probability $(1 + v/c)/2$ of being in the left-handed mode.

6.3 Show that

$$\begin{aligned}
&u_\pm^\dagger(\mathbf{p})u_\pm(\mathbf{p}) = v_\pm^\dagger(\mathbf{p})v_\pm(\mathbf{p}) = E_p/m,\\
&u_\pm^\dagger(\mathbf{p})u_\mp(\mathbf{p}) = v_\pm^\dagger(\mathbf{p})v_\mp(\mathbf{p}) = 0,\\
&u_\pm^\dagger(\mathbf{p})v_\pm(-\mathbf{p}) = v_\pm^\dagger(-\mathbf{p})u_\pm(\mathbf{p}) = u_\pm^\dagger(\mathbf{p})v_\mp(-\mathbf{p}) = v_\mp^\dagger(-\mathbf{p})u_\pm(\mathbf{p}) = 0.
\end{aligned}$$

These results are useful in Problem 6.4.

6.4 Using the plane wave expansion (6.24) and the energy–momentum tensor components (6.22) and (6.23), show that the energy and momentum carried by the wave ψ are given by (6.25) and (6.26).

6.5 Consider a momentum $\mathbf{p}$ in the direction specified by the polar coordinates θ and ϕ.

$$\hat{\mathbf{p}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Show that

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

and the Pauli spin states

$$|+\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} -\sin(\theta/2)e^{-i\phi} \\ \cos(\theta/2) \end{pmatrix}$$

are the helicity eigenstates appearing in (6.14) and (6.15). An overall phase is undetermined.