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SUBSPACES OF THE FREE TOPOLOGICAL VECTOR SPACE ON THE UNIT INTERVAL

SAAK S. GABRIYELYAN and SIDNEY A. MORRIS✉

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Dedicated to the memory of Matatyahu Rubin

Abstract

For a Tychonoff space X, let $\mathbb{V}(X)$ be the free topological vector space over X, $A(X)$ the free abelian topological group over X and $\mathbb{I}$ the unit interval with its usual topology. It is proved here that if X is a subspace of $\mathbb{I}$, then the following are equivalent: $\mathbb{V}(X)$ can be embedded in $\mathbb{V}(\mathbb{I})$ as a topological vector subspace; $A(X)$ can be embedded in $A(\mathbb{I})$ as a topological subgroup; X is locally compact.

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1. Introduction

Free topological vector spaces were introduced in [2]. If X is a Tychonoff space, then $\mathbb{V}(X)$ is said to be the *free topological vector space* on X if X is a subspace of $\mathbb{V}(X)$ and every continuous mapping φ of X into any topological vector space E can be extended uniquely to a continuous linear mapping Φ of $\mathbb{V}(X)$ into E. It has been shown that $\mathbb{V}(X)$ exists and is unique up to isomorphism of topological vector spaces, and that X is a Hamel basis for $\mathbb{V}(X)$.

For over half a century, free topological groups and free abelian topological groups have been investigated. The following question turns out to be nontrivial: If Y is a subspace of X, under what circumstances can the free (free abelian) topological group $F(Y)$ (respectively, $A(Y)$) be embedded as a topological group in $F(X)$ (respectively, $A(X)$). Note that this question is quite different from asking whether the subgroup of $F(X)$ (respectively, $A(X)$) generated by the given space Y is the free topological group $F(Y)$ (respectively, the free abelian topological group $A(Y)$). In this paper, we examine the analogous question for free topological vector spaces and at the same time obtain a new result for free abelian topological groups.

As special cases of our results, we obtain the main results of [3] and [4].

Theorem 1.1. *Let $\mathbb{R}$ denote the set of all real numbers with the Euclidean topology:*

(i) $A(\mathbb{R})$ *embeds into* $A(\mathbb{I})$ *as a topological group* [4]*;*
(ii) $\mathbb{V}(\mathbb{R})$ *embeds into* $\mathbb{V}(\mathbb{I})$ *as a topological vector space* [3].

Our approach is to obtain a very useful description of locally compact subspaces of $\mathbb{I}$.

2. Results

We use the following notation. Set $\mathbb{N} := \{1, 2, \ldots\}$. For a subset A of a vector space E and a natural number $n \in \mathbb{N}$, $\mathrm{sp}_n(A)$ denotes the subset of E defined by

$$\mathrm{sp}_n(A) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \in [-n, n], x_i \in A, \forall i = 1, \ldots, n\},$$

and $\mathrm{sp}(A) := \bigcup_{n\in\mathbb{N}} \mathrm{sp}_n(A)$ is the span of A in E.

The *disjoint union* of a nonempty family $\{X_i\}_{i\in I}$ of topological spaces is the coproduct in the category of topological spaces and continuous functions and is denoted by $\bigsqcup_{i\in I} X_i$.

By an open interval in $\mathbb{I}$, we mean an interval of the form $[0, a), (a, b)$ or $(b, 1]$ for $a, b \in \mathbb{I}$.

Proposition 2.1. *Let X be a locally compact subspace of $\mathbb{I}$. Then there is a countable family $\{I_n : n \in N\}$, $N \subseteq \mathbb{N}$, of pairwise disjoint open intervals in $\mathbb{I}$, such that for every $n \in N$ there exists a countable family of increasing closed intervals $\{[l_{i,n}, r_{i,n}] : i \in M_n\}$ satisfying the following conditions:*

(i) $l_{i,n}, r_{i,n} \in X$*, for every $n \in N$ and each $i \in M_n$;*
(ii) $[l_{i,n}, r_{i,n}] \cap X$ *is a compact subset of X, for every $n \in N$ and each $i \in M_n$;*
(iii) *X is homeomorphic to the disjoint union*

$$X = \bigsqcup_{n\in N} (I_n \cap X) = \bigsqcup_{n\in N} \Big(\bigcup_{i\in M_n} [l_{i,n}, r_{i,n}] \cap X \Big).$$

Proof. We prove the proposition in four steps.

Step 1. Let $x \in X$. Choose $\epsilon > 0$ and an open neighbourhood U of x of the form $U = (x - \varepsilon, x + \varepsilon) \cap X$ which has compact closure in X. Then $[x - \varepsilon/2, x + \varepsilon/2] \cap X$ is a compact subset of X. So, for every $x \in X$, there is a compact neighbourhood of x in X of the form $[a(x), b(x)] \cap X$, such that:

(i) $a(x) < x < b(x)$ if $0 < x < 1$;
(ii) $0 = a(x) < b(x)$ if $x = 0$; and
(iii) $a(x) < b(x) = 1$ if $x = 1$.

Step 2. For $x, y \in X$, set $x \sim y$ if the set $[\min\{x, y\}, \max\{x, y\}] \cap X$ is compact in X. It is easy to see that $\sim$ is an equivalence relation on X. For $x \in X$, we denote by $\mathbf{x}$ the equivalence class of x and set

$$a(\mathbf{x}) := \inf\{y : y \in \mathbf{x}\} \quad \text{and} \quad b(\mathbf{x}) := \sup\{y : y \in \mathbf{x}\}.$$

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Note that, by Step 1, $a(\mathbf{x}) \in X$ if and only if $a(\mathbf{x}) \in \mathbf{x}$, and $b(\mathbf{x}) \in X$ if and only if $b(\mathbf{x}) \in \mathbf{x}$. Then one of the following cases holds:

(1) $a(\mathbf{x}) \in X$ and $a(\mathbf{x}) = 0$. Set $c(\mathbf{x}) := a(\mathbf{x}) = 0$.
(2) $a(\mathbf{x}) \in X$, $a(\mathbf{x}) > 0$ and $[0, a(\mathbf{x})) \cap X = \emptyset$. Set $c(\mathbf{x}) := a(\mathbf{x})/2$.
(3) $a(\mathbf{x}) \in X$, $a(\mathbf{x}) > 0$ and $[0, a(\mathbf{x})) \cap X \neq \emptyset$. Set $a^-(\mathbf{x}) := \sup\{a : a \in [0, a(\mathbf{x})) \cap X\}$ and note that $a^-(\mathbf{x}) < a(\mathbf{x})$ (otherwise, by Step 1, one can find $a < a(\mathbf{x})$ such that $[a, a(\mathbf{x})] \cap X$ is compact, and hence $a \sim x$ which contradicts the choice of $a(\mathbf{x})$). Set $c(\mathbf{x}) := (2a(\mathbf{x}) + a^-(\mathbf{x}))/3$.
(4) $a(\mathbf{x}) \notin X$. Set $a^-(\mathbf{x}) := a(\mathbf{x})$ and $c(\mathbf{x}) := (2a(\mathbf{x}) + a^-(\mathbf{x}))/3 = a(\mathbf{x})$.

In particular, $c(\mathbf{x}) \in X$ if and only if $c(\mathbf{x}) = a(\mathbf{x}) = 0$.

Analogously, one of the following cases holds:

(1)′ $b(\mathbf{x}) \in X$ and $b(\mathbf{x}) = 1$. Set $d(\mathbf{x}) := b(\mathbf{x}) = 1$.
(2)′ $b(\mathbf{x}) \in X$, $b(\mathbf{x}) < 1$ and $(b(\mathbf{x}), 1] \cap X = \emptyset$. Set $d(\mathbf{x}) := (1 + b(\mathbf{x}))/2$.
(3)′ $b(\mathbf{x}) \in X$, $b(\mathbf{x}) < 1$ and $(b(\mathbf{x}), 1] \cap X \neq \emptyset$. Set $b^+(\mathbf{x}) := \inf\{b : b \in (b(\mathbf{x}), 1] \cap X\}$ and note that $b^+(\mathbf{x}) > b(\mathbf{x})$ (otherwise, by Step 1, one can find $b > b(\mathbf{x})$ such that $[b(\mathbf{x}), b] \cap X$ is compact, and hence $b \sim x$ which contradicts the choice of $b(\mathbf{x})$). Set $d(\mathbf{x}) := (2b(\mathbf{x}) + b^+(\mathbf{x}))/3$.
(4)′ $b(\mathbf{x}) \notin X$. Set $b^+(\mathbf{x}) := b(\mathbf{x})$ and $d(\mathbf{x}) := (2b(\mathbf{x}) + b^+(\mathbf{x}))/3 = b(\mathbf{x})$.

In particular, $d(\mathbf{x}) \in X$ if and only if $d(\mathbf{x}) = b(\mathbf{x}) = 1$.

Let $I(\mathbf{x})$ be the open interval in $\mathbb{I}$ with endpoints $c(\mathbf{x})$ and $d(\mathbf{x})$ such that, if $c(\mathbf{x}) = 0$ or $d(\mathbf{x}) = 1$, then $I(\mathbf{x})$ contains $c(\mathbf{x})$ or $d(\mathbf{x})$, respectively. By construction, the length of $I(\mathbf{x})$ is positive.

Step 3. We claim that if $x \nsim y$, then $I(\mathbf{x}) \cap I(\mathbf{y}) = \emptyset$. Indeed, assume that $x < y$ and note that $d(\mathbf{x}) < 1$ by (1)′ and $c(\mathbf{y}) > 0$ by (1). So $d(\mathbf{x}) \notin I(\mathbf{x}) \cup X$ and $c(\mathbf{y}) \notin I(\mathbf{y}) \cup X$. Therefore, to prove the claim it is sufficient to show that $d(\mathbf{x}) \leq c(\mathbf{y})$.

First, we note that $b(\mathbf{x}) \leq a(\mathbf{y})$. Indeed, if $b(\mathbf{x}) > a(\mathbf{y})$, there is a $z \in \mathbf{y}$ such that $b(\mathbf{x}) > z \geq a(\mathbf{y})$. Therefore, $z \sim x$. Hence, $x \sim y$, a contradiction.

Next we show that

$$b^+(\mathbf{x}) \leq a(\mathbf{y}) \quad \text{and} \quad b(\mathbf{x}) \leq a^-(\mathbf{y}). \tag{2.1}$$

Indeed, if $b(\mathbf{x}) \notin X$, then $b^+(\mathbf{x}) = b(\mathbf{x}) \leq a(\mathbf{y})$ by the above. Assume that $b(\mathbf{x}) \in X$, so only (3)′ holds for $b(\mathbf{x})$ since $x < y$. Now, if $a(\mathbf{y}) \in X$, then $b(\mathbf{x}) < a(\mathbf{y})$ (otherwise, $b(\mathbf{x}) = a(\mathbf{y}) \in X$ and hence $x \sim y$, a contradiction), and if $a(\mathbf{y}) \notin X$, then also $b(\mathbf{x}) < a(\mathbf{y})$. Thus, $b^+(\mathbf{x}) \leq a(\mathbf{y})$ by the definition of $b^+(\mathbf{x})$. Analogously one can prove that $b(\mathbf{x}) \leq a^-(\mathbf{y})$.

Since $y \in X$, for $d(\mathbf{x})$ only one of the cases (3)′ and (4)′ can hold. Analogously, since $x \in X$, for $c(\mathbf{y})$ only one of the cases (3) and (4) can hold. In all these cases, by (2.1),

$$c(\mathbf{y}) = \tfrac{1}{3}(2a(\mathbf{y}) + a^-(\mathbf{y})) \geq \tfrac{1}{3}(2b^+(\mathbf{x}) + b(\mathbf{x})) \geq \tfrac{1}{3}(2b(\mathbf{x}) + b^+(\mathbf{x})) = d(\mathbf{x}).$$

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Step 4. Since the length of $I(\mathbf{x})$ is positive for every $x \in X$, Step 3 implies that there is only a countable family of equivalence classes. Let $\{\mathbf{x}_n\}_{n\in N}$ be an enumeration of all equivalence classes. For every $n \in N$, set $I_n := I(\mathbf{x}_n)$, and consider the following cases:

(a) If $a(\mathbf{x}_n), b(\mathbf{x}_n) \in X$, set $M_n := \{1\}$, $l_{1,n} := a(\mathbf{x}_n)$ and $r_{1,n} := b(\mathbf{x}_n)$.

(b) If $a(\mathbf{x}_n) \in X$ and $b(\mathbf{x}_n) \notin X$, choose arbitrarily a strictly increasing sequence $\{b_i(\mathbf{x}_n)\}_{i\in\mathbb{N}} \subseteq \mathbf{x}_n$ converging to $b(\mathbf{x}_n)$. Set $M_n := \mathbb{N}$ and, for every $i \in M_n$, put $l_{i,n} := a(\mathbf{x}_n)$ and $r_{i,n} := b_i(\mathbf{x}_n)$.

(c) If $a(\mathbf{x}_n) \notin X$ and $b(\mathbf{x}_n) \in X$, choose arbitrarily a strictly decreasing sequence $\{a_i(\mathbf{x}_n)\}_{i\in\mathbb{N}} \subseteq \mathbf{x}_n$ converging to $a(\mathbf{x}_n)$. Set $M_n := \mathbb{N}$ and, for every $i \in M_n$, put $l_{i,n} := a_i(\mathbf{x}_n)$ and $r_{i,n} := b(\mathbf{x}_n)$.

(d) if $a(\mathbf{x}_n) \notin X$ and $b(\mathbf{x}_n) \notin X$, choose arbitrarily a strictly decreasing sequence $\{a_i(\mathbf{x}_n)\}_{i\in\mathbb{N}} \subseteq \mathbf{x}_n$ converging to $a(\mathbf{x}_n)$ and a strictly increasing sequence $\{b_i(\mathbf{x}_n)\}_{i\in\mathbb{N}} \subseteq \mathbf{x}_n$ converging to $b(\mathbf{x}_n)$. Set $M_n := \mathbb{N}$ and, for every $i \in M_n$, put $l_{i,n} := a_i(\mathbf{x}_n)$ and $r_{i,n} := b_i(\mathbf{x}_n)$.

By Step 3 and (a)–(d), we see that (i)–(iii) are satisfied. □

LEMMA 2.2. *Let $\{X_i\}_{i\in\mathbb{N}}$ and $\{Y_i\}_{i\in\mathbb{N}}$ be families of Tychonoff spaces:*

(i) *if $\mathbb{V}(X_i)$ embeds into $\mathbb{V}(Y_i)$ as a topological vector subspace for every $i \in \mathbb{N}$, then $\mathbb{V}(\bigsqcup_{i\in\mathbb{N}} X_i)$ embeds into $\mathbb{V}(\bigsqcup_{i\in\mathbb{N}} Y_i)$ as a topological vector subspace;*

(ii) *if $A(X_i)$ embeds into $A(Y_i)$ as a topological subgroup for every $i \in \mathbb{N}$, then $A(\bigsqcup_{i\in\mathbb{N}} X_i)$ embeds into $A(\bigsqcup_{i\in\mathbb{N}} Y_i)$ as a topological subgroup.*

PROOF. We prove only (i) as (ii) can be proved similarly. Set $X := \bigsqcup_{i\in\mathbb{N}} X_i$ and $Y := \bigsqcup_{i\in\mathbb{N}} Y_i$ and note that $\mathbb{V}(X)$ and $\mathbb{V}(Y)$ are canonically topologically isomorphic to the direct sums

$$\Big(\bigoplus_{i\in\mathbb{N}} \mathbb{V}(X_i), \mathcal{T}_b\Big) \quad \text{and} \quad \Big(\bigoplus_{i\in\mathbb{N}} \mathbb{V}(Y_i), \mathcal{T}_b\Big),$$

respectively, where $\mathcal{T}_b$ denotes the box topology on the direct sums. (See [2, Proposition 2.8].) For every $i \in \mathbb{N}$, let $p_i : \mathbb{V}(X_i) \to \mathbb{V}(Y_i)$ be an embedding of topological vector spaces. Denote by $p : \mathbb{V}(X) \to \mathbb{V}(Y)$ the map defined by

$$p((u_i)) := (p_i(u_i)), \quad (u_i) \in \mathbb{V}(X).$$

We claim that p is an embedding of topological vector spaces. Clearly, p is continuous. To prove that p is relatively open, for every $i \in \mathbb{N}$, take arbitrarily an open neighbourhood U_i of zero in $\mathbb{V}(X_i)$ and choose an open neighbourhood of zero in $\mathbb{V}(Y_i)$ such that

$$p_i(U_i) = V_i \cap p(\mathbb{V}(X_i)).$$

Now to prove the claim, it is sufficient to show that

$$p\Big(\prod_i U_i \cap \mathbb{V}(X)\Big) = \prod_i V_i \cap p(\mathbb{V}(X)).$$

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The inclusion '⊆' is clear. Conversely, let $(v_i) \in \prod_i V_i \cap p(\mathbb{V}(X))$. Take $(u_i) \in \mathbb{V}(X)$ such that $p((u_i)) = (p_i(u_i)) = (v_i)$, and if $v_i = 0$ then also $u_i = 0$. Then $p_i(u_i) = v_i \in V_i \cap p_i(\mathbb{V}(X_i))$ and hence $u_i \in U_i$ for every $i \in \mathbb{N}$. Noting that all but finitely many of the v_i are zero, $(u_i) \in \prod_i U_i \cap \mathbb{V}(X)$. Thus, $(v_i) \in p(\prod_i U_i \cap \mathbb{V}(X))$. □

We shall also use the following proposition.

Proposition 2.3 [2]. *Let $X = \bigcup_{n\in\mathbb{N}} C_n$ be a k_ω-space and let Y be a subset of $\mathbb{V}(X)$ such that Y is a vector space basis for the subspace,* sp(Y)*, that it generates. Assume that $K_1, K_2, \ldots$ is a sequence of compact subsets of Y such that $Y = \bigcup_{n\in\mathbb{N}} K_n$ is a k_ω-decomposition of Y inducing the same topology on Y that Y inherits as a subset of $\mathbb{V}(X)$. If for every $n \in \mathbb{N}$ there is a natural number m such that* $\mathrm{sp}(Y) \cap \mathrm{sp}_n(C_n) \subseteq \mathrm{sp}_m(K_m)$*, then* sp$(Y)$ *is $\mathbb{V}(Y)$, and both* sp(Y) *and Y are closed subsets of $\mathbb{V}(X)$.*

Now we prove the main result of the paper.

Theorem 2.4. *For a subspace X of $\mathbb{I}$ the following assertions are equivalent:*

(i) *$\mathbb{V}(X)$ embeds into $\mathbb{V}(\mathbb{I})$ as a topological vector space;*
(ii) *$A(X)$ embeds into $A(\mathbb{I})$ as a topological group;*
(iii) *X is locally compact.*

Proof. (i) ⇒ (iii) Since $A(X)$ is a closed subgroup of $\mathbb{V}(X)$ by [2, Proposition 5.1], $A(X)$ is a subgroup of $\mathbb{V}(\mathbb{I})$. As X is metrisable, the group $A(X)$ is complete by [6]. Since $\mathbb{V}(\mathbb{I})$ is a k_ω-space by [2, Theorem 3.1], we see that $A(X)$ and hence also X are closed subspaces of $\mathbb{V}(\mathbb{I})$. So X is a k_ω-space. Being also metrisable, X is locally compact by [1, Exercise 3.4.E(c)].

(ii) ⇒ (iii) As X is metrisable, the group $A(X)$ is complete by [6]. Since $A(\mathbb{I})$ is a k_ω-space by [5], we see that $A(X)$ and hence also X are closed subspaces of $A(\mathbb{I})$. So X is a k_ω-space. Being also metrisable, X is locally compact by [1, Exercise 3.4.E(c)].

(iii) ⇒ (i), (ii) By [2, (the proof of) Theorem 4.2], if $\{K_n\}_{n\in\mathbb{N}}$ is a sequence of disjoint compact subsets of $\mathbb{R}$, then $\mathbb{V}(\bigsqcup_{n\in\mathbb{N}} K_n)$ embeds onto a closed vector subspace of $\mathbb{V}(\mathbb{I})$, and $A(\bigsqcup_{n\in\mathbb{N}} K_n)$ embeds onto a closed subgroup of $A(\mathbb{I})$. Now Proposition 2.1 and Lemma 2.2 imply that to prove the theorem it is sufficient to show the following: if the subspace X of $\mathbb{I}$ has the form

$$\bigcup_{i\in\mathbb{N}} [u_i, v_i] \cap X,$$

where $u_i, v_i \in X$, $[u_i, v_i] \cap X$ is a compact subspace of X, $u_i < v_i$, $u_{i+1} \le u_i$ and $v_i \le v_{i+1}$ for every $i \in \mathbb{N}$, then $\mathbb{V}(X)$ and $A(X)$ embed into $\mathbb{V}(\mathbb{I})$ and $A(\mathbb{I})$, respectively. Below we consider only the most difficult and general case when $u_{i+1} < u_i$ and $v_i < v_{i+1}$ for every $i \in \mathbb{N}$.

For every $i \in \mathbb{N}$, define $c_i := v_i$ and $c_{1-i} := u_i$. For every $k \in \mathbb{Z}$, set $J_k := [c_k, c_{k+1}] \cap X$ and recall that $c_k \in X$ and J_k is a compact subset of X. Below we proceed as in the proof of [3, Theorem 3.5] and prove the implication (iii) ⇒ (i). Replacing $\mathbb{V}(X)$ and $\mathbb{V}(\mathbb{I})$ by $A(X)$ and $A(\mathbb{I})$, respectively, we prove (iii) ⇒ (ii).

Step 1. The basic construction. Take two sequences $\{a_k\}_{k\in\mathbb{Z}}, \{b_k\}_{k\in\mathbb{Z}} \subset \mathbb{I}$ such that

$$0 < a_0 < b_0 < a_1 < b_1 < a_{-1} < b_{-1} < a_2 < b_2 < a_{-2} < b_{-2} < \cdots < 1,$$

and set $I_k = [a_k, b_k]$ for every $k \in \mathbb{Z}$.

For $k = 0$, define the continuous injection $g_{0,0} : J_0 \to I_0$ by

$$g_{0,0}(x) := a_0 + (b_0 - a_0) \cdot \frac{x - c_0}{c_1 - c_0}.$$

For every $k \in \mathbb{Z} \setminus \{0\}$, set

$$S_k := 8(T_1 + \cdots + T_{|k|}) \quad \text{and} \quad A_k := \tfrac{1}{2} S_k, \quad \text{where } T_n := 1 + \cdots + n.$$

For every $k \in \mathbb{Z} \setminus \{0\}$ and $i \in \mathbb{N}$ such that $1 \le i \le A_k$, we define pairwise disjoint closed intervals by

$$I_{i,k} := \left[a_k + \frac{b_k - a_k}{S_k}(2i - 1), a_k + \frac{b_k - a_k}{S_k} 2i \right] \subset I_k,$$

and define the continuous injection $g_{i,k} : J_k \to I_{i,k}$ by

$$g_{i,k}(x) := a_k + \frac{b_k - a_k}{S_k}(2i - 1 + (x - c_k)).$$

For every $k \in \mathbb{Z}$, define the maps $H_k : J_k \to \mathbb{V}(\mathbb{I})$ by

$$H_k(x) := \begin{cases} g_{0,0}(x) & \text{if } k = 0, \\ g_{1,k}(x) + g_{2,k}(x) + \cdots + g_{A_k,k}(x) & \text{if } k \ne 0, \end{cases}$$

where '+' denotes the vector space addition in $\mathbb{V}(\mathbb{I})$.

Now we define the map $\chi : X \to \mathbb{V}(\mathbb{I})$ inductively as follows: if $x \in J_0$, set

$$\chi(x) := H_0(x);$$

if $k \in \mathbb{N}$ and $x \in J_k$, put

$$\chi(x) := H_k(x) - H_k(c_k) + \chi(c_k) = H_k(x) - \sum_{i=1}^{k} (H_i(c_i) - H_{i-1}(c_i)),$$

and if $-k \in \mathbb{N}$ and $x \in J_k$, set

$$\chi(x) := H_k(x) - H_k(c_{k+1}) + \chi(c_{k+1}) = H_k(x) - \sum_{i=k}^{-1} (H_i(c_{i+1}) - H_{i+1}(c_{i+1})).$$

Clearly, χ is well-defined and continuous. Since all the intervals $I_{i,k}$ are disjoint and the functions $g_{i,k}$ are injective, the map χ is one-to-one. For every $n \in \mathbb{N}$, set $Y_n := \chi([u_n, v_n] \cap X)$ and put $Y := \chi(X)$.

Step 2. For every $s \in \mathbb{N}$ there is $M(s) \in \mathbb{N}$ such that $\mathrm{sp}(Y) \cap \mathrm{sp}_s(\mathbb{I}) \subseteq \mathrm{sp}_{M(s)}(Y_{M(s)})$. Indeed, fix $t \in \mathrm{sp}(Y) \cap \mathrm{sp}_s(\mathbb{I})$. So there are distinct $x_1, \ldots, x_s \in \mathbb{I}$, distinct $y_1, \ldots, y_m \in Y$, nonzero real numbers $a_1, \ldots, a_m$ and nonzero numbers $\lambda_1, \ldots, \lambda_s \in [-s, s]$ such that

$$t = a_1 y_1 + \cdots + a_m y_m = \lambda_1 x_1 + \cdots + \lambda_s x_s.$$

By construction, there are $r \in \mathbb{N}$, integers $n_1 < \cdots < n_r$, natural numbers $q_1, \ldots, q_r$ with $q_1 + \cdots + q_r = m$, and pairwise distinct elements $z_{j,i} \in X$, where $1 \le j \le q_i$ for $1 \le i \le r$, such that:

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(1) $z_{1,i}, \dots, z_{q_i,i} \in \begin{cases} [c_0, c_1] \cap X & \text{if } n_i = 0, \\ (c_{n_i}, c_{n_i+1}] \cap X & \text{if } n_i > 0, \\ [c_{n_i}, c_{n_i+1}) \cap X & \text{if } n_i < 0, \end{cases}$

(2) for every $y \in \{y_1, \dots, y_m\}$ there is a unique pair (j, i) such that $y = \chi(z_{j,i})$.

So we can uniquely represent t in the form

$$t = \sum_{i=1}^{r} \sum_{j=1}^{q_i} a_{j,i}\chi(z_{j,i}) = \lambda_1 x_1 + \cdots + \lambda_s x_s. \tag{2.2}$$

Since all the intervals $I_{i,k}$ are disjoint and the functions $g_{i,k}$ are injective, the construction of the map χ and (2.2) imply the following: if $z_{j,i} \in (c_{n_i}, c_{n_i+1}) \cap X$, then $a_{j,i} \in \{\lambda_1, \dots, \lambda_s\}$ and $\chi(z_{j,i})$ has at least A_{n_i} distinct summands from the basis $\mathbb{I}$ of $\mathbb{V}(\mathbb{I})$, which do not appear in another summand in the middle sum of (2.2). Therefore

$$(q_1 - 1) + \cdots + (q_r - 1) \le s. \tag{2.3}$$

Assume that $n_r > 0$. Then $\chi(z_{q_r,r})$ has at least A_{n_r} distinct basic elements from $\mathbb{I}$ which do not appear in other summands in the middle sum of (2.2). Therefore, (2.2) implies that $A_{n_r} \le s$ and $|a_{q_r,r}| \le s$. If $n_1 < 0$, the same argument shows that $A_{n_1} \le s$ and $|a_{q_1,1}| \le s$. Since $A_k \ge 4|k|$, this implies in particular that $r \le s$, and therefore (2.3) yields

$$m = q_1 + \cdots + q_r \le 2s.$$

Now if $n_r > 0$, let $w \in \mathbb{N}$ be the least index such that $n_w \ge 0$. By the definition of χ and (2.2), for every i with $1 \le i \le n_r$ the coefficient of $H_i(c_i)$ in the sum $\sum_{j=1}^{q_r} a_{j,r}\chi(z_{j,r})$ is $-\sum_{j=1}^{q_r} a_{j,r}$, and hence

$$\Big|\sum_{j=1}^{q_r} a_{j,r}\Big| \le q_r \cdot s \le 2s \cdot s. \tag{2.4}$$

Therefore, if $w < r$ and $z_{q_{r-1},r-1} = c_{n_{r-1}+1}$, by (2.2), the coefficient $a_{q_{r-1},r-1}$ satisfies

$$|a_{q_{r-1},r-1}| \le s + q_r \cdot s = (q_r + 1) \cdot s \le 2s^2. \tag{2.5}$$

Analogously, assume that $w < r - 1$ and $z_{q_{r-2},r-2} = c_{n_{r-2}+1}$. Then the coefficient of $H_{n_{r-2}}(c_{n_{r-2}})$ in the middle sum of (2.2) is

$$a_{q_{r-2},r-2} - \sum_{j=1}^{q_{r-1}-1} a_{j,r-1} - a_{q_{r-1},r-1} - \sum_{j=1}^{q_r} a_{j,r}.$$

This and (2.2)–(2.5) imply

$$|a_{q_{r-2},r-2}| \le s + (q_{r-1} - 1) \cdot s + (q_r + 1) \cdot s + q_r \cdot s = s(2q_r + q_{r-1} + 1) < s \cdot 4s.$$

Continuing this process, $M_+ > 0$ such that $|a_{q_i,i}| \le M_+$ for every $i > w$. In a similar way, one can show that if $n_1 < 0$, then there is $M_- > 0$ such that $|a_{q_i,i}| \le M_-$ for every i such

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that $n_i > 0$. Now, if $n_w = 0$, the boundedness of $a_{j,i}$ corresponding to $i \neq w$ and (2.2) easily imply that there exists an M such that

$$|a_{j,i}| \leq M \quad \text{for } 1 \leq i \leq r \text{ and } 1 \leq j \leq q_i.$$

Then $M(s) := \max\{2s, M\}$ is as desired.

Step 3. Next we show that if $a_1y_1 + \cdots + a_my_m = 0$, then $a_1 = \cdots = a_m = 0$. Indeed, we can represent 0 in the form (2.2). Now, as above, if $n_r > 0$, then $\chi(z_{q_r,r})$ has at least A_{n_r} distinct basic elements from $\mathbb{I}$ which do not appear in other summands in the middle sum of (2.2). So $a_{q_r,r} = 0$. Analogously, if $n_1 < 0$, $a_{q_1,1} = 0$. Therefore, $r = 1$ and $n_1 = 0$. In this case we also easily obtain $a_{j,1} = 0$ for $1 \leq j \leq q_1$. Thus, $a_1 = \cdots = a_m = 0$ as desired.

Step 4. We claim that Y is a closed subset of $\mathbb{V}(\mathbb{I})$. First, $Y \cap \mathrm{sp}_s(\mathbb{I}) = Y_s \cap \mathrm{sp}_s(\mathbb{I})$ for every $s \in \mathbb{N}$. Indeed, let

$$y := \chi(x) = \lambda_1x_1 + \cdots + \lambda_sx_s \in Y \cap \mathrm{sp}_s(\mathbb{I}). \tag{2.6}$$

If $x \in J_0$, then $y \in Y_1$ and we are done. Suppose that $x \in (c_k, c_{k+1}] \cap X$ for some $k > 0$, or $x \in [c_k, c_{k+1}) \cap X$ for some $k < 0$. If $y \notin Y_s$, then either $k \geq s$ or $k + 1 \leq -s$. In both cases y has at least $A_k \geq 4|k| > s$ distinct basic summands from $\mathbb{I}$ which contradicts (2.6). Hence, $y \in Y_s$. Thus, $Y \cap \mathrm{sp}_s(\mathbb{I}) \subseteq Y_s \cap \mathrm{sp}_s(\mathbb{I})$. The converse inclusion is clear.

Now fix a closed subset F of X. Then, for every $s \in \mathbb{N}$,

$$\begin{aligned}\chi(F) \cap \mathrm{sp}_s(\mathbb{I}) &= \chi(F) \cap (Y \cap \mathrm{sp}_s(\mathbb{I})) = \chi(F) \cap (Y_s \cap \mathrm{sp}_s(\mathbb{I})) \\ &= (\chi(F) \cap \chi([u_s, v_s] \cap X)) \cap \mathrm{sp}_s(\mathbb{I}) = \chi(F \cap [u_s, v_s]) \cap \mathrm{sp}_s(\mathbb{I}).\end{aligned}$$

Since, by the definition of u_s and v_s, the set $F \cap [u_s, v_s] = F \cap ([u_s, v_s] \cap X)$ is a compact subset of X and χ is continuous, we see that $\chi(F) \cap \mathrm{sp}_s(\mathbb{I})$ is a closed subset of $\mathrm{sp}_s(\mathbb{I})$. As $\mathbb{V}(\mathbb{I}) = \bigcup_{s\in\mathbb{N}} \mathrm{sp}_s(\mathbb{I})$ is a k_ω-space by [2, Theorem 3.1], it follows that $\chi(F)$ is closed in $\mathbb{V}(\mathbb{I})$. Therefore, χ is a closed map. Thus, χ is a homeomorphism of X onto Y.

Finally, by Steps 2–4, we can apply Proposition 2.3 to show that $\mathbb{V}(X)$ is linearly isomorphic to the closed linear subspace $\mathrm{sp}(Y)$ of $\mathbb{V}(\mathbb{I})$. □

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SAAK S. GABRIYELYAN, Department of Mathematics,
Ben-Gurion University of the Negev, Beer-Sheva PO 653, Israel
e-mail: saak@math.bgu.ac.il

SIDNEY A. MORRIS, Faculty of Science and Technology,
Federation University Australia, PO Box 663, Ballarat, Victoria 3353, Australia
and
Department of Mathematics and Statistics, La Trobe University, Melbourne,
Victoria 3086, Australia
e-mail: morris.sidney@gmail.com