BORDER REDUCTION IN EXISTENCE PROBLEMS OF HARMONIC FORMS

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Dedicated to Professor KIYOSHI NOSHIRO on his 60th birthday

1. Given an arbitrary Riemannian n-space V let σ be a harmonic field in the complement $V - V_0$ of a regular region V_0. The problem of constructing in V a harmonic field ρ with the property $\|\rho - \sigma\|^2_{V-\bar{V}_0} = \int_{V-\bar{V}_0} (\rho - \sigma) \wedge *(\rho - \sigma) < \infty$ was given a complete solution in [2]. The corresponding problem for harmonic forms σ, ρ remains open in the general case. In the special case of locally flat spaces the construction can be carried out by replacing $\|\cdot\|$ by the point norm [3].

In the present paper we shall introduce another device that appears promising of generalization: We shall show that the border ∂V_0 can be reduced, in a sense to be specified, to an $(n-1)$-sphere. This will enable us to perform all calculations in V_0 in terms of one coordinate system.

The present paper is thus methodological in nature. However, we believe the border reduction also has some interest in its own right.

For any form $\varphi = \varphi_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ the point norm $|\varphi|$ defined by the invariant function

$$|\varphi|^2 = \varphi_{i_1 \ldots i_p} \varphi^{i_1 \cdots i_p} \tag{1}$$

reduces to $(\sum \varphi^2_{i_1 \cdots i_p})^{1/2}$ in the present case of a Euclidean metric.

A form is said to be defined in a set $E \subset V$ if it is defined in an open set containing E. The same is true of properties such as harmonicity that involve differentials.

2. *A* Riemannian space V is *locally flat* if we can find an admissible coordinate system at each point such that the metric tensor takes the form

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$g_{ij}=\delta_{ij}$ throughout the parametric neighborhood. In this case the Laplace-Beltrami operator $\Delta=d\delta+\delta d$ applied to a p-form $\varphi=\varphi_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ gives

$$\Delta\varphi=(\Delta\varphi_{i_1\ldots i_p})\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

with

$$\Delta\varphi_{i_1\ldots i_p}=-\left(\frac{\partial^2}{(\partial x^1)^2}+\cdots+\frac{\partial^2}{(\partial x^n)^2}\right)\varphi_{i_1\ldots i_p}.$$

Therefore a *harmonic form* $\varphi=\varphi_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ is nothing but a form whose coefficients $\varphi_{i_1\ldots i_p}$ are harmonic functions in the Euclidean coordinate system.

A Riemannian space V is called *hyperbolic* if V carries no nonconstant positive superharmonic functions, i.e., functions v of class C^2 with $\Delta v\leq 0$ in V. Take a regular region B in V. Then V is hyperbolic if and only if there exists a continuous function $u_{\partial B}$ in V such that (a) $u_{\partial B}$ is harmonic in $V-\overline{B}$, (b) $u_{\partial B}|B\equiv 1$, (c) $0<u_{\partial B}<1$ in $V-\overline{B}$, and (d) if u is continuous in V and harmonic in $V-\overline{B}$ with $u=0$ on $\overline{B}$, then $|u|\leq u_{\partial B}$ implies $u\equiv 0$.

3. Let V_0 be a regular region [4] of V and let σ be a harmonic p-form in the closure of $V_\sigma=V-\overline{V}_0$. Take an exhaustion [4] of V, say $\{V_n\}_0^\infty$ and let φ be a p-form on ∂V_n with tangential component $t\varphi$ and normal component $n\varphi$ on ∂V_n. For a p-form ψ we write

$$\varphi=\psi \text{ on } \partial V_n \tag{2}$$

if $t\varphi=t\psi$ and $n\varphi=n\psi$. This is equivalent to

$$\varphi_{i_1\ldots i_p}=\psi_{i_1\ldots i_p} \text{ on } \partial V_n \tag{3}$$

for every admissible coordinate system at each point of ∂V_n. This in turn is equivalent to

$$|\varphi-\psi|^2=0 \text{ on } \partial V_n \tag{4}$$

and implies

$$|\varphi|^2=|\psi|^2 \text{ on } \partial V_n. \tag{5}$$

Let φ be given in a vicinity of ∂V_0. By Duff-Spencer [1] there exists a unique harmonic p-form $L_n\varphi$ on $\overline{V}_n-V_0$ such that $L_n\varphi=\varphi$ on ∂V_0 and $L_n\varphi=0$ on ∂V_n.

In general let ψ be harmonic in a Euclidean neighborhood U. Then since the $\psi_{i_1\ldots i_p}$ are harmonic in U, $\psi_{i_1\ldots i_p}\psi^{i_1\cdots i_p}=\psi^2_{i_1\cdots i_p}$ (a single term, not summation) is subharmonic in U, and so is $|\psi|^2=\sum\psi_{i_1\ldots i_p}\psi^{i_1\cdots i_p}=\sum\psi^2_{i_1\cdots i_p}$.

In particular $|L_n\varphi|^2$ is subharmonic in $V_n-\overline{V}_0$, with $|L_n\varphi|^2=|\varphi|^2$ on ∂V_0 and $|L_n\varphi|^2=0$ on ∂V_n. Set $M=\max_{\partial V_0}|\varphi|^2$ and let $u_{\partial V_0}$ be the harmonic measure of ∂V_0. Then

$$0\leq|L_n\varphi|^2\leq Mu_{\partial V_0}$$

and $(L_n\varphi)_{i_1\ldots i_p}$ is bounded harmonic in each Euclidean neighborhood. Therefore, there exists a continuous form $\tilde{\varphi}$ in $V-V_0$, harmonic in $V-\overline{V}_0$, such that a subsequence of $\{L_n\varphi\}$ converges to $\tilde{\varphi}$, i.e., each coefficient of $L_n\varphi$ converges to that of $\tilde{\varphi}$ in each neighborhood. Clearly

$$0\leq|\tilde{\varphi}|^2\leq Mu_{\partial V_0}$$

and $\tilde{\varphi}=\varphi$ on ∂V_0.

Take another such form $\tilde{\varphi}'$. Then of course $0\leq|\tilde{\varphi}'|^2\leq Mu_{\partial V_0}$ and $\tilde{\varphi}'=\varphi$ on ∂V_0. Thus

$$|\tilde{\varphi}-\tilde{\varphi}'|^2\leq(|\tilde{\varphi}|+|\tilde{\varphi}'|)^2\leq2|\tilde{\varphi}|^2+2|\tilde{\varphi}'|^2\leq4\,Mu_{\partial V_0}.$$

On the other hand, $\tilde{\varphi}-\tilde{\varphi}'=0$ and $|\tilde{\varphi}-\tilde{\varphi}'|^2=0$ on ∂V_0. Thus by the subharmonicity of $|\tilde{\varphi}-\tilde{\varphi}'|^2$ and the "vanishing property" of $u_{\partial V_0}$ at the ideal boundary, $\tilde{\varphi}=\tilde{\varphi}'$. We conclude that

$$L\varphi=\lim_n L_n\varphi \quad (\text{i.e., } |L\varphi-L_n\varphi|^2\to0)$$

exists and satisfies

$$(\alpha)\qquad L(a\varphi_1+b\varphi_2)=aL\varphi_1+bL\varphi_2,$$

$$(\beta)\qquad L\varphi=\varphi \text{ on } \partial V_0,$$

$$(\gamma)\qquad |L\varphi|^2\leq(\max_{\partial V_0}|\varphi|^2)u_{\partial V_0} \text{ on } V-V_0,$$

$$(\delta)\qquad L^2=L.$$

4. We shall now prove our contention that V_0 may be assumed to be a ball as small as we wish. To this end take an (open) ball B such that $\overline{B}\subset V_0\subset\overline{V}_0\subset V_1$.

For a form φ on $\overline{V}_1$ there exist, again by Duff-Spencer [1], unique harmonic forms $K\varphi$ and $K_0\varphi$ in V_1 and $V_1-\overline{B}$ respectively such that

$$(6)\qquad K\varphi=K_0\varphi=\varphi \text{ on } \partial V_1$$

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and

(7) $$K_0\varphi = 0 \text{ on } \partial B.$$

The operator $K\varphi$ will be used later, and for the present we only consider $K_0\varphi$.

We wish *to find a continuous form* $\bar{\sigma}$ *on* $V - B$, *harmonic on* $V - \overline{B}$, *such that*

(8) $$\bar{\sigma} = 0 \textit{ on } \partial B$$

and

(9) $$L(\bar{\sigma} - \sigma) = \bar{\sigma} - \sigma \textit{ on } V_\sigma = V - \overline{V}_0,$$

i.e., $\bar{\sigma}$ *behaves like* σ *near the ideal boundary of* V.

We have only to find a form φ on ∂V_1 and a form ϕ on ∂V_0 such that

(10) $$\begin{cases} \varphi - \sigma = L(\phi - \sigma) & \text{on } \partial V_1, \\ \phi = K_0\varphi & \text{on } \partial V_0. \end{cases}$$

In fact, then we put

$$\bar{\sigma} = \begin{cases} K_0\varphi \text{ in } \overline{V}_1 - B, \\ L\phi + \sigma - L\sigma \text{ in } \overline{V}_\sigma. \end{cases}$$

It is well defined since on ∂V_0, $K_0\varphi = \phi = L\phi = L\phi + \sigma - L\sigma$, and on ∂V_1, $K_0\varphi = \varphi = L\phi + \sigma - L\sigma$, and by Duff-Spencer [1], $K_0\varphi = L\phi - \sigma - L\sigma$ in $(\overline{V}_1 - B) \cap \overline{V}_\sigma$.

Now $\bar{\sigma}$ is harmonic in $V - \overline{B}$, $\bar{\sigma} = 0$ on ∂B, and $L(\bar{\sigma} - \sigma) = L^2\phi + L\sigma - L^2\sigma - L\sigma = L(\phi - \sigma) = \bar{\sigma} - \sigma$ in $\overline{V}_\sigma$.

Clearly solving (10) is equivalent to finding a form φ on ∂V_1 such that

(11) $$\varphi - \sigma = L(K_0\varphi - \sigma) \text{ on } \partial V_1.$$

This is rewritten in the form

(12) $$(I - LK_0)\varphi = \sigma_0 \text{ on } \partial V_1$$

with the identity operator I and $\sigma_0 = \sigma - L\sigma$. Since $(LK_0)^n\sigma_0$ is defined on ∂V_1, we are led to the Neumann series

(13) $$\varphi = \sum_{n=0}^{\infty} (LK_0)^n \sigma_0.$$

We have to show the convergence on ∂V_1.

5. Let F_0 be the family of continuous p-forms φ in $\overline{V}_1 - B$, harmonic in $V_1 - \overline{B}$, such that $\varphi = 0$ on ∂B. The q-lemma for 0-forms [4] has the following

counterpart:

There exists a constant q with $0<q<1$ *such that*

$$\max_{\partial V_0}|\varphi|\leq q\max_{\partial V_1}|\varphi| \tag{14}$$

for all $\varphi\in F_0$.

For the proof take the function ω continuous in $\overline{V}_1-B$ and harmonic in $V_1-\overline{B}$ such that $\omega=0$ on ∂B and $\omega=1$ on ∂V_1. Let

$$q=\sqrt{\max_{\partial V_0}\omega}\,.$$

By the maximum principle it is clear that $0<q<1$. This is the desired constant q. In fact, to see (14) we may clearly assume that $|\varphi|^2>0$ on ∂V_1, for otherwise $|\varphi|\equiv 0$. We may also assume that $\max_{\partial V_1}|\varphi|=1$ and have $|\varphi|^2\leq\omega$ in $\overline{V}_1-B$, and $|\varphi|^2\leq q^2$ on ∂V_0.

6. Fix a neighborhood N of ∂V_1. Clearly $|\sigma_0|$ is bounded in N: $|\sigma_0|\leq M$, say. By (14),

$$\max_{\partial V_0}|K_0(LK_0)^{n-1}\sigma_0|\leq q\max_{\partial V_1}|K_0(LK_0)^{n-1}\sigma_0|=q\max_{\partial V_1}|(LK_0)^{n-1}\sigma_0|. \tag{15}$$

Clearly by (γ),

$$|(LK_0)^n\sigma_0|\leq\max_{\partial V_0}|K_0(LK_0)^{n-1}\sigma_0| \tag{16}$$

on $V_o=V-\overline{V}_0$ and of course on N. Thus by (15) and (16)

$$\max_N|(LK_0)^n\sigma_0|\leq q\max_N|(LK_0)^{n-1}\sigma_0| \tag{17}$$

and we conclude that

$$\max_N|(LK_0)^n\sigma_0|\leq q^nM. \tag{18}$$

This completes the proof of our claim that ∂V_0 can be reduced to an arbitrarily small sphere.

7. In [3] we proved the existence of a harmonic form ρ in a locally flat V with bounded $|\rho-\sigma|$ in $V-V_0$. Although it is not the main purpose of the present paper to discuss this existence, we close by showing that it follows on a hyperbolic locally flat V.

By the above reduction we may assume that V_0 and V_1 are concrete balls

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(and $\sigma = 0$ on ∂V_0). Instead of L and K_0 we take L and K, and by exactly the same procedure our problem is reduced to the proof of the uniform convergence of

$$\varphi = \sum_{n=0}^{\infty} (LK)^n \sigma_0.$$

8. Again we need a q-lemma of another kind. Let N be a neighborhood of ∂V_1 with compact closure such that $\overline{N} \cap \overline{V}_0 = \phi$.

Let $\mathfrak{F}$ be the family of continuous p-forms φ on $\overline{V}_\sigma = V - V_0$, harmonic in $V - \overline{V}_0$, such that $L\varphi = \varphi$ on $\overline{V}_\sigma$.

Lemma. *There exists a constant q such that $0 < q < 1$ and*

$$\max_N |\varphi| \leq q \max_{\partial V_0} |\varphi| \tag{18}$$

for any φ in $\mathfrak{F}$.

The important fact to be observed is that V_σ is connected and thus the harmonic measure $u_{\partial V_0}$ of ∂V_0 with respect to V_σ satisfies $0 < u_{\partial V_0} < 1$ in V_σ. The remainder of the proof is similar to that of (14).

9. The convergence is established as follows. Clearly $|LK\sigma_0|$ is bounded in N, $|LK\sigma_0| \leq M$, say. The norm $|K\varphi|$ attains its maximum on ∂V_1, and therefore

$$\max_{\partial V_0} |K(LK)^{n-1}\sigma_0| \leq \max_{\partial V_1} |K(LK)^{n-1}\sigma_0| \leq \max_N |(LK)^{n-1}\sigma_0|. \tag{19}$$

By (18),

$$\max_N |(LK)^n \sigma_0| \leq q \max_{\partial V_0} |LK(LK)^{n-1}\sigma_0| = q \max_{\partial V_0} |K(LK)^{n-1}\sigma_0|, \tag{20}$$

and by (19) and (20),

$$\max_N |(LK)^n \sigma_0| \leq q \max_N |(LK)^{n-1}\sigma_0|.$$

We conclude that

$$\max_N |(LK)^n \sigma_0| \leq q^{n-1} M.$$

This completes the proof.

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