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IDEMPOTENT ENDOMORPHISMS OF AN INDEPENDENCE ALGEBRA OF FINITE RANK*

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The result of Ballantine [1] to the effect that a singular matrix A is a product of k idempotent matrices if and only if the rank of $I-A$ does not exceed k times the nullity of A is generalized to endomorphisms of a class of independence algebras.

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1. Introduction

In 1966 Howie [8] showed that every singular selfmap of the set $[n]=\{1,2,\dots,n\}$ is expressible as a composition of idempotent selfmaps. An analogous result concerning the expressibility of every singular $n\times n$ matrix over a field as a product of idempotent matrices was proved by J. A. Erdos [3] in 1967.

In any semigroup S generated by its set E of idempotents there is for each element s a least k with the property that $s\in E^k$. Saito [12] gave a formula determining k for any singular selfmap of $[n]$—see also Iwahori [10] and Howie [9]—and a corresponding formula for singular matrices was given by Ballantine [1].

In effect, it was clear that there was a strong analogy between the properties of the endomorphism monoid of the finite set $[n]$ and those of the endomorphism monoid of an n-dimensional vector space, an analogy strong enough to prompt Fountain and Lewin [4, 5] to seek a common framework. The key lay in the idea of an *independence algebra of finite rank*, due to Narkiewicz [11] and Gould [6], of which both a set $[n]$ without structure and a finite dimensional vector space over a field are special cases. Fountain and Lewin [4] were able to show that every singular endomorphism of an independence algebra of finite rank is expressible as a product (that is to say, a composition) of idempotent endomorphisms. Both the Howie theorem and the Erdos theorem are special cases of this result.

Let V be an n-dimensional vector space over a field F, and let $\alpha\colon V\to V$ be an endomorphism (a linear transformation). Denote the image of α by $\operatorname{im}\alpha$ and let $\operatorname{fix}\alpha$ be the subspace $\{x\in V: x\alpha=x\}$. Let $d(\alpha)$ (*the defect* of α) be $n-\dim(\operatorname{im}\alpha)$, and let $s(\alpha)$ (the

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shift of α) be $n-\dim(\operatorname{fix}\alpha)$. Ballantine's result [1] can be regarded as saying that α is expressible as a product of k idempotents if and only if $s(\alpha)/d(\alpha) \leqq k$.

For a wide class of independence algebras A we can define $s(\alpha)$ and $d(\alpha)$ in an analogous manner. In Section 2 we show that half of Ballantine's result is then true. Precisely, we denote the set of singlular idempotent endomorphisms of A by E, and show that if a singular endomorphism α belongs to E^k, then $s(\alpha)/d(\alpha) \leqq k$.

The converse half of Ballantine's result is, however, known to be untrue in the case where A is simply a set $[n]$ without structure. Here we define

$$s(\alpha) = |\{x \in [n]: x\alpha \neq x\}|, \quad d(\alpha) = n - |\operatorname{im}\alpha|,$$

and it is clear that the largest possible value of $s(\alpha)/d(\alpha)$ (for a singular α) is n. On the other hand, it follows from Howie's result [9] that if n is odd then there exist elements α for which $\alpha \notin E^{3(n-3)/2}$.

It is natural therefore to seek to determine in an abstract fashion a class of independence algebras for which the full Ballantine property holds. We show that it holds for 'connected' independence algebras, a class of algebras that includes vector spaces over fields and a number of other less familiar types of algebra but does not include sets without structure.

2. Preliminaries

We follow the terminology of Fountain and Lewin [4]. We consider an algebra A (where $A \neq \varnothing$) with a collection (perhaps empty) of finitary operations and denote the smallest subalgebra of A containing a subset X of A by $\langle X \rangle$. In particular, the subalgebra $\langle \varnothing \rangle$ is the subalgebra generated by the set of constants (nullary operations) of A. (By convention, if A has no constants then we allow $\varnothing$ as a subalgebra.) An *endomorphism* of A is a map $\alpha: A \to A$ which respects all the operations of A. The composition of two endomorphisms is again an endomorphism, and indeed the set of all endomorphisms of A is a monoid, denoted by End A. An endomorphism which is also a bijection is called an *automorphism*, and the set Aut A of automorphisms of A forms a group under composition. We denote the set End $A \setminus$Aut A of *singular* endomorphisms by Sing_A. We shall consistently denote the set of idempotents in Sing_A by E.

A subset X of A is called *independent* if $x \notin \langle X \setminus \{x\} \rangle$ for every x in X. A *basis* of A is defined as a subset X which is independent and is such that $\langle X \rangle = A$. The algebra A is called an *independence algebra* if it has the properties:

(I1) for every independent subset X of A and every $u \notin \langle X \rangle$, the set $X \cup \{u\}$ is independent;

(I2) for every basis X of A and for every map α: $X \to A$ there is an endomorphism $\bar{\alpha}$ of A such that $\bar{\alpha}|_X = \alpha$.

The independence algebra A is called *strong* if:

(I3) for every pair X, Y of independent subsets, $\langle X\rangle \cap \langle Y\rangle = \langle \varnothing \rangle$ implies that $X \cup Y$ is independent.

Many of the standard techniques of linear algebra can be adapted to this more general class of algebras. It is convenient to list a number of properties that will be of use later in the article. Let A be a strong independence algebra with a finite basis.

(I4) Every subalgebra B of A has a finite basis, and all bases of B have the same number of elements; this number is called rank B, the *rank* of the subalgebra B.

(I5) Every set of independent elements in a subalgebra B can be extended to form a basis of B. If rank $B=r$, then every set of r independent elements of B is a basis, and so is every set of r elements generating B.

(I6) If $X=\{x_1,x_2,\dots,x_k\}\subseteq A$ then there is a subset Y of X such that Y is a basis of $\langle X\rangle$.

(I7) If B, C are subalgebras of A and $B \vee C$ is the smallest subalgebra of A containing B and C, then

$$\operatorname{rank}(B \vee C)=\operatorname{rank} B+\operatorname{rank} C-\operatorname{rank}(B \cap C).$$

3. Shift and defect

Let $\alpha \in \operatorname{End} A$, where A is a strong independence algebra of finite rank n. Then both $\operatorname{im}\alpha$ and $\operatorname{fix}\alpha$ $(=\{x\in A: x\alpha=x\})$ are subalgebras of A. We define $s(\alpha)$, the *shift* of α, to be $n-\operatorname{rank}(\operatorname{fix}\alpha)$, and $d(\alpha)$, the *defect* of α, to be $n-\operatorname{rank}(\operatorname{im}\alpha)$. We begin by establishing some elementary properties of shift and defect which will be of assistance in proving the main theorem of this section.

If ε belongs to the set E of singular idempotents of $\operatorname{End} A$, then $\operatorname{im}\varepsilon=\operatorname{fix}\varepsilon$, and so certainly

$$d(\varepsilon)=s(\varepsilon). \tag{1}$$

In general, for α in $\operatorname{End} A$ we have $\operatorname{fix}\alpha\subseteq\operatorname{im}\alpha$, and so

$$d(\alpha)\leqq s(\alpha). \tag{2}$$

If $\alpha,\beta\in\operatorname{End} A$ then it is clear that $\operatorname{im}(\alpha\beta)\subseteq\operatorname{im}\beta$; hence

$$d(\alpha\beta)\geqq d(\beta). \tag{3}$$

If $\operatorname{im}\alpha=\langle x_1,x_2,\dots,x_r\rangle$ then $\operatorname{im}(\alpha\beta)$ is generated by $\{x_1\beta,x_2\beta,\dots,x_r\beta\}$ and so has rank at most r. Thus

$$d(\alpha\beta)\geqq d(\alpha). \tag{4}$$

It is clear that $\operatorname{fix}\alpha\cap\operatorname{fix}\beta\subseteq\operatorname{fix}(\alpha\beta)$. Hence by (I7) we have

$$\operatorname{rank}(\operatorname{fix}(\alpha\beta))\geqq\operatorname{rank}(\operatorname{fix}\alpha\cap\operatorname{fix}\beta)$$

$$=\operatorname{rank}(\operatorname{fix}\alpha)+\operatorname{rank}(\operatorname{fix}\beta)-\operatorname{rank}(\operatorname{fix}\alpha\vee\operatorname{fix}\beta)$$

$$\geqq\operatorname{rank}(\operatorname{fix}\alpha)+\operatorname{rank}(\operatorname{fix}\beta)-n,$$

and from this it follows that

$$s(\alpha\beta)\leqq s(\alpha)+s(\beta). \tag{5}$$

We can now easily establish:

Theorem 1. *Let A be a strong independence algebra and let E be the set of singular idempotents in* End A. *If $\alpha\in E^k$ then $s(\alpha)/d(\alpha)\leqq k$.*

Proof. Suppose that $\alpha=\varepsilon_1\varepsilon_2\ldots\varepsilon_k$, where $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_k\in E$. Then

$$s(\alpha)\leqq s(\varepsilon_1)+s(\varepsilon_2)+\cdots+s(\varepsilon_k)\quad\text{by (5)}$$

$$=d(\varepsilon_1)+d(\varepsilon_2)+\cdots+d(\varepsilon_k)\quad\text{by (1)}$$

$$\leqq kd(\alpha)\quad\text{by (3) and (4)},$$

and the proof is complete.

4. Connected algebras

An independence algebra A of finite rank is called *connected* if it is strong and if for any two independent elements x, y in A there exists z in A such that

$$\langle x,y\rangle=\langle x,z\rangle=\langle y,z\rangle. \tag{6}$$

A vector space V over a field F certainly has this property—we simply take $z=x+y$. By contrast, the set $[n]$ (with no algebraic structure) does not have the property, for in this case $\langle x,y\rangle=\{x,y\}$, and the only z for which $\langle x,y\rangle=\langle x,z\rangle$ is the element y itself.

Another example, of a connected independence algebra, attributed to Narkiewicz [11], is quoted by Grätzer ([7, Exercise 5.26]). Let $(R,+,\cdot)$ be a division ring, let $(A,+)$ be a left module over R, and let A_0 be a submodule of A with the property that for all a in A_0 and all $r\neq 0$ in R there exists b in A_0 such that $a=rb$. Let T be the set of all n-ary operations f on A (with $n\geqq 0$) of the form

$$f(x_0, x_1, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} \lambda_i x_i + a, \tag{7}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \in R$ and $a \in A_0$. When $n=0$, (7) is to be interpreted as specifying a constant (0-ary operation) a. Then (A, T) is a connected independence algebra in which $\langle \emptyset \rangle = A_0$. The verifications are routine, and the element z required by the condition (6) is again $x+y$.

A third example, which we owe to Dr John Fountain, shows that a connected independence algebra need not have constants. Let $A = \{a, b, c\}$, and let $\circ$ be a binary relation specified by the table

$\circ$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

This is not a semigroup: $(a \circ b) \circ c = c$, $a \circ (b \circ c) = a$. It is, however, not hard to check that it is a strong independence algebra, in which $\langle x \rangle = \{x\}$ for all x, and $\langle x, y \rangle = \langle A \rangle = A$ for all $x \neq y$, and in which the non-empty independent sets are $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$. Every permutation of $\{a, b, c\}$ is an automorphism. Singular endomorphisms are scarcer: if, for example, $a\alpha = b\alpha = t$, (where $t \in A$), then

$$c\alpha = (a \circ b)\alpha = (a\alpha) \circ (b\alpha) = t \circ t = t$$

also. Hence the only singular endomorphisms are the constant maps

$$\begin{pmatrix} a & b & c \\ a & a & a \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ b & b & b \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ c & c & c \end{pmatrix}.$$

The algebra is, moreover, connected. Given $x \neq y$ (which implies that $\{x, y\}$ is independent), we take z as the unique element of $A \setminus \{x, y\}$ and immediately observe that $\langle x, y \rangle = \langle x, z \rangle = \langle y, z \rangle = A$.

We shall require the following technical lemma:

Lemma 1. *Let A be a connected independence algebra of finite rank, and let $\{y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s\}$ be independent, with $r \geqq s$. Let $\alpha \in \operatorname{End} A$ and let $f < r$ be such that $y_i\alpha = y_i$ for $1 \leqq i \leqq f$. Suppose that*

$$C = \langle y_1\alpha, \ldots, y_r\alpha, z_1\alpha, \ldots, z_s\alpha \rangle = \langle y_1, \ldots, y_f, y_{f+1}\alpha, \ldots, y_r\alpha, z_1\alpha, \ldots, z_s\alpha \rangle$$

has ranks $s + p \leqq r$. Then there exist $y'_{f+1}, \ldots, y'_r$ in A such that:

1. *$\{y_1, \ldots, y_f, y'_{f+1}, \ldots, y'_r, z_1, \ldots, z_s\}$ is independent;*

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2. $\langle y_1,\ldots,y_f,y'_{f+1}\alpha,\ldots,y'_r\alpha\rangle = C$.

Proof. Since $\{y_1,\ldots,y_f\}$ is an independent subset of $\langle y_1\alpha,\ldots,y_r\alpha\rangle$, we can find a subset Y of $\{y_{f+1}\alpha,\ldots,y_r\alpha\}$ such that $\{y_1,\ldots,y_f\}\cup Y$ is a basis for $\langle y_1\alpha,\ldots,y_r\alpha\rangle$. Relabelling if necessary, we write this basis as

$$\{y_1,\ldots,y_f,y_{f+1}\alpha,\ldots,y_l\alpha\}.$$

We can now extend this set to obtain a basis

$$\{y_1,\ldots,y_f,y_{f+1}\alpha,\ldots,y_l\alpha,z_1\alpha,\ldots,z_m\alpha\}$$

for C, where the elements $z_i\alpha$ have been relabelled if necessary, and where $l+m=s+p$.

For $i=1,\ldots,m$ the set $\{z_i,y_{l+i}\}$ is independent. Hence, since A is connected, there exists y'_{l+i} such that

$$\langle z_i,y_{l+i}\rangle = \langle z_i,y'_{l+i}\rangle = \langle y_{l+i},y'_{l+i}\rangle.$$

Let

$$B=\langle y_1,\ldots,y_l,y_{l+m+1},\ldots,y_r,z_{m+1},\ldots,z_s\rangle.$$

Then

$$\begin{aligned}&\langle y_1,\ldots,y_l,y'_{l+1},y'_{l+2},\ldots,y'_{l+m},y_{l+m+1},\ldots,y_r,z_1,\ldots,z_s\rangle\\ &=B\vee\langle y'_{l+1},z_1\rangle\vee\langle y'_{l+2},z_2\rangle\vee\cdots\vee\langle y'_{l+m},z_m\rangle\\ &=B\vee\langle y_{l+1},z_1\rangle\vee\langle y_{l+2},z_2\rangle\vee\cdots\vee\langle y_{l+m},z_m\rangle\\ &=\langle y_1,\ldots,y_r,z_1,\ldots,z_s\rangle,\end{aligned}$$

of rank $r+s$, and so the set

$$\{y_1,\ldots,y_l,y'_{l+1},\ldots,y'_{l+m},y_{l+m+1},\ldots,y_r,z_1,\ldots,z_s\}$$

must be independent.

Next, we show that the set

$$\begin{aligned}D&=\{y_1\alpha,\ldots,y_l\alpha,y'_{l+1}\alpha,\ldots,y'_{l+m}\alpha\}\\ &=\{y_1,\ldots,y_f,y_{f+1}\alpha,\ldots,y_l\alpha,y'_{l+1}\alpha,\ldots,y'_{l+m}\alpha\}\end{aligned}$$

is independent. Since $z_i\in\langle y_{l+i},y'_{l+i}\rangle$ for $i=1,2,\ldots,m$, it follows that $z_i\alpha\in\langle y_{l+i}\alpha,y'_{l+i}\alpha\rangle$. Now the elements

$$y_1,\ldots,y_f,y_{f+1}\alpha,\ldots,y_l\alpha$$

were chosen so as to generate $\langle y_1\alpha,\ldots,y_r\alpha\rangle$; hence both $y_{l+i}\alpha$ and $y'_{l+i}\alpha$ are in $\langle D\rangle$, and it follows that $z_i\alpha\in\langle D\rangle$. Since $\langle D\rangle$ contains the independent set

$$\{y_1,\ldots,y_f,y_{f+1}\alpha,\ldots,y_l\alpha,z_1\alpha,\ldots,z_m\alpha\},$$

it must have rank at least $l+m$. But $|D|=l+m$, and so the rank is exactly $l+m=s+p$. Thus D is independent. Finally we conclude that $\langle D\rangle$, being a subalgebra of C of rank $s+p$, is equal to C.

If we now define $y'_i=y_i$ for $i=f+1,\ldots,l$ and $i=l+m+1,\ldots,r$, we have a set $\{y'_{f+1},\ldots,y'_r\}$ with the required properties. We are now ready to prove:

Theorem 2. *Let A be a connected independence algebra of finite rank n and let $\alpha\in\mathrm{Sing}_A$. Denote the set of singular idempotents in* End A *by E. Then $\alpha\in E^k$ if and only if $s(\alpha)\leqq k\,d(\alpha)$.*

Proof. In view of Theorem 1, we need only consider the converse half. We prove the result by induction on k. Certainly if $k=1$, so that $s(\alpha)\leqq d(\alpha)$, we deduce using (2) that $\mathrm{fix}\,\alpha=\mathrm{im}\,\alpha$; hence $(x\alpha)\alpha=x\alpha$ for all x in A, and so $\alpha\in E$.

Suppose now that $k\geqq 2$ and that

$$(k-1)d(\alpha)<s(\alpha)\leqq k\,d(\alpha). \tag{8}$$

We write $d(\alpha)=d$, $\mathrm{rank}(\mathrm{fix}\,\alpha)=f$ (so that $s(\alpha)=n-f$), $b=(k-2)d(\geqq 0)$, $a=n-f-(k-1)d$. The condition (8) is equivalent to

$$0<a\leqq d.$$

Choose a basis $\{y_1,\ldots,y_f\}$ for fix α, noting that from $a=n-f-(k-1)d$ we have

$$f+a=n-(k-1)d\leqq n-d.$$

Thus we can extend to obtain an independent subset $\{y_1,\ldots,y_f,z_1,\ldots,z_a\}$ of $\mathrm{im}\,\alpha$, and then extend again to obtain a basis

$$\{y_1,\ldots,y_{f+(k-1)d},z_1,\ldots,z_a\}$$

of A. Notice that

$$\langle y_1,\ldots,y_f,y_{f+1}\alpha,\ldots,y_{f+(k-1)d}\alpha,z_1\alpha,\ldots,z_a\alpha\rangle=\mathrm{im}\,\alpha,$$

and so has rank $n-d$. Let us denote this set by C.

We now apply Lemma 1 to this set C, with $r=f+(k-1)d$, $s=a$. The conditions of the lemma are satisfied, since

$$a \leqq n-d \leqq f+(k-1)d.$$

We conclude that there exist elements $y'_{f+1},\ldots,y'_{f+(k-1)d}$ in A such that

$$\operatorname{im}\alpha=\langle y_1,\ldots,y_f,y'_{f+1}\alpha,\ldots,y'_{f+(k-1)d}\alpha\rangle$$

and

$$\{y_1,\ldots,y_f,y'_{f+1},\ldots,y'_{f+(k-1)d},z_1,\ldots,z_a\}$$

is independent. Since $a+f+(k-1)d=n$, this set must be a basis of A. Now write

$$x_i=y_i \quad \text{for} \quad i=1,\ldots,f,$$

$$x_{f+i}=z_i \quad \text{for} \quad i=1,\ldots,a,$$

$$x_{f+a+i}=y'_{f+i} \quad \text{for} \quad i=1,\ldots,(k-1)d,$$

and obtain a basis $\{x_1,\ldots,x_n\}$ for A such that $\operatorname{fix}\alpha=\langle x_1,\ldots,x_f\rangle$, $\langle x_1,\ldots,x_{f+a}\rangle\subseteq\operatorname{im}\alpha$ and

$$\operatorname{im}\alpha=\langle x_1,\ldots,x_f,x_{f+a+1}\alpha,\ldots,x_n\alpha\rangle.$$

For $i=1,\ldots,a$ there is a term T_i such that

$$x_{f+i}\alpha=T_i(x_1,\ldots,x_f,x_{f+a+1}\alpha,\ldots,x_n\alpha).$$

Now define $\varepsilon\in\operatorname{End}A$ by

$$x_j\varepsilon=x_j \quad \text{if } j=1,\ldots,f \text{ or if } j=f+a+1,\ldots,n,$$

$$x_{f+i}\varepsilon=T_i(x_1,\ldots,x_f,x_{f+a+1},\ldots,x_n) \text{ for } i=1,\ldots,a.$$

Since

$$T_i(x_1,\ldots,x_f,x_{f+a+1},\ldots,x_n)\in\langle x_1,\ldots,x_f,x_{f+a+1},\ldots,x_n\rangle$$

foreach i, we easily see that ε is idempotent.
Next, define β in $\operatorname{End}A$ by

$$x_j\beta=\begin{cases}x_j & \text{for } j=1,\ldots,f+\alpha\\ x_j\alpha & \text{for } j=f+a+1,\ldots,n.\end{cases}$$

Then $\operatorname{im}\beta\subseteq\operatorname{im}\alpha$ and so $d(\beta)\geqq d$. Thus

$$s(\beta)\leqq n-f-a=(k-1)d\leqq(k-1)d(\beta),$$

and so $\beta\in E^{k-1}$ by the induction hypothesis.

Finally, observe that $\varepsilon\beta=\alpha$; for we have

$$x_j\varepsilon\beta=x_j\beta=x_j=x_j\alpha\ (j=1,\dots,f);$$

$$\begin{aligned}x_{f+i}\varepsilon\beta&=T_i(x_1,\dots,x_f,x_{f+a+1},\dots,x_n)\beta\\&=T_i(x_1,\dots,x_f,x_{f+a+1}\beta,\dots,x_n\beta)\\&=T_i(x_1,\dots,x_f,x_{f+a+1}\alpha,\dots,x_n\alpha)\\&=x_{f+i}\alpha\ (i=1,\dots,a);\end{aligned}$$

$$x_j\varepsilon\beta=x_j\beta=x_j\alpha\ (j=f+a+1,\dots,n).$$

Thus $\alpha\in E^k$ as required.

It is clear that for a singular element α of End A the maximum value of $s(\alpha)$ is n and the minimum value of $d(\alpha)$ is 1. In the case of vector spaces both bounds are attainable, but in a more general connected independence algebra A this may fail to be the case. However, provided the algebra A has a non-empty set of constants we can attain the bounds. For the following argument we are indebted to Dr John Fountain. Let A contain at least one constant c. If $\{x_1,x_2,\dots,x_n\}$ is a basis of A, define α in End A by the rule that

$$x_i\alpha=x_{i+1},(i=1,2,\dots,n-1),\quad x_n\alpha=c.$$

Then $\operatorname{im}\alpha=\langle x_2,\dots,x_n\rangle$, and so $d(\alpha)=1$. Also $x_i\alpha^n=c$ for $i=1,2,\dots,n$. Let $z\in\operatorname{fix}\alpha$, where z is given in terms of the basis by means of some term t: $z=t(x_1,x_2,\dots,x_n)$. Then

$$\begin{aligned}z=z\alpha=z\alpha^n&=t(x_1\alpha^n,x_2\alpha^n,\dots,x_n\alpha^n)\\&=t(c,c,\dots,c)\in\langle\varnothing\rangle.\end{aligned}$$

Thus $\operatorname{fix}\alpha=\langle\varnothing\rangle$, of rank 0, and so $s(\alpha)=n$.

If A, with basis $\{x_1,x_2,\dots,x_n\}$, has no constants, then we can consider a slightly different endomorphism α given by

$$x_i\alpha=x_{i+1},(i=1,2,\dots,n-1),\ x_n\alpha=x_n.$$

Then again $d(\alpha)=1$, and $x_i\alpha^{n-1}=x_n$ for $i=1,2,\dots,n$. If $z=t(x_1,x_2,\dots,x_n)\in\operatorname{fix}\alpha$, then

$$z=z\alpha^{n-1}=t(x_n,x_n,\dots,x_n)\in\langle x_n\rangle;$$

hence $\operatorname{fix} \alpha = \langle x_n \rangle$ has rank 1. It follows that $s(\alpha) = n-1$.

Accordingly we have the following generalization of a result proved by Dawlings [2], in the linear algebra context:

Corollary 1. *Let A be a connected independence algebra with finite rank n, let Sing_A be the semigroup of all singular endomorphisms of A, and let E be the set of indempotents of Sing_A. If A contains at least one constant, then $\Delta(\mathrm{Sing}_A) = n$. If A contains no constants then $\Delta(\mathrm{Sing}_A) \geqq n-1$.*

For an example of an algebra A with no constants in which $\Delta(\mathrm{Sing}_A) = n-1$ we need look no further than the earlier Fountain example quoted earlier, in which $n=2$ and $\Delta(\mathrm{Sing}_A) = 1$.

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