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PERIODIC POINTS AND CHAOS FOR EXPANDING SELF-MAPS OF THE INTERVAL

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It is shown that expanding self-maps of the interval with a finite number of turning points must have periodic points whose periods are not integral power of 2 and therefore are chaotic.

Introduction

Let I be the unit interval $[0, 1]$ of the real line. A continuous map f from I to itself is piecewise monotonic if I can be subdivided into finite number of subintervals $I_1, I_2, \ldots, I_l$ on which f is either strictly increasing or strictly decreasing. Each such maximal interval on which f is monotonic is called a lap of f, and $l = l(f)$ is the lap number of f. The separating points $c_1, c_2, \ldots, c_{l-1}$ at which f has a local minimum or maximum are called the turning points of f. The limit $S(f) = \lim_{n\to\infty} l(f^n)^{1/n}$ is a real number in the interval $[1, l(f)]$ called the growth number of f. A piecewise monotonic map f from I to itself is expanding if there exists a constant $\lambda > 1$ such that $|f(x)-f(y)| \geq \lambda|x-y|$ whenever both x and y belong to the same lap. Call λ an expansion constant for f.

In recent years there has been considerable interest in the dynamical

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properties of difference equations defined by self-maps of the unit interval. The complicated asymptotic behaviour which often arises has been emphasized by the use of the term "chaotic" to characterize certain dynamical properties of a large class of such equations [1], [3], [5], [6], [8], [9]. This complexity can be dealt with statistically if the transformation admits an invariant measure [4] especially one which is absolutely continuous with respect to Lebesgue measure. Thus there has been much interest in proving the existence of such measures [7].

However, the connection between these two ideas has not yet been clarified. It is known that transformations with a periodic point whose period is not an integral power of 2 must exhibit chaotic behaviour [1], [3], [11]. On the other hand, if there exists some natural number m such that the map f^m is expanding then f admits an absolutely continuous invariant measure [10]. In [2], Byers has shown that expanding maps with a unique turning point must have a periodic point of period $2^n \cdot 3$ and therefore are chaotic. In this paper we generalize the result and show that if there exists some natural number m such that the map f^m is expanding then f admits a periodic point whose period is not an integral power of 2 and therefore is chaotic.

Discussion

LEMMA 1. *If $f : I \to I$ is a continuous expanding map with expansion constant λ then, for any natural number n, f^n is an expanding map with expansion constant λ^n.*

Proof. Let $c_1, c_2, \ldots, c_{l-1}$ be the turning points of f. Set

$$E = \bigcup_{j=0}^{n-1} \left(\bigcup_{i=1}^{l-1} f^{-j}(c_i) \right) .$$

We shall first show that E is exactly the set of all turning points of f^n. For any $x_0 \in E$, there must exist $1 \le i_0 \le l-1$, $0 \le j_0 \le n-1$, such that $f^{j_0}(x_0) = c_{i_0}$. Hence, for any natural number $j \ge j_0 + 1$, f^j

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has a local extreme value at x_0 , in particular, so does f^n . Thus x_0 is a turning point of f^n . Now let us assume that x^* is a turning point of f^n . If $x^* \notin E$, then, for any j which satisfies the condition that $0 \le j \le n-1$, $f^j(x^*)$ is not a turning point of f ; that is f is monotonic at $f^j(x^*)$, in particular at $f^{n-1}(x^*)$. Hence f^n is monotonic at x^* and this contradicts the assumption that x^* is a turning point of f^n . Therefore $x^* \in E$.

Finally we shall show that f^n is an expanding map with expansion constant λ^n . Let us take an arbitrary lap $I_k^{(n)}$ of f^n . For any x and y belonging to $I_k^{(n)}$, we may assume that $x < y$ without loss of generality. Then for each fixed $j = 0, 1, \ldots, n-1$, $f^j(x)$ and $f^j(y)$ belong to the same lap of f . Otherwise from the continuity of f there must exist x_0 which belongs to (x, y) and c_{i_0} such that

$f^j(x_0) = c_{i_0}$, that is, $x_0 \in E$ and this contradicts the assumption

that $x, y \in I_k^{(n)}$. Therefore

$$|f^n(x)-f^n(y)| \ge \lambda|f^{n-1}(x)-f^{n-1}(y)| \ge \ldots \ge \lambda^n|x-y| .$$

That is $|f^n(x)-f^n(y)| \ge \lambda^n|x-y|$. □

LEMMA 2 [10]. *Suppose the continuous map $f : I \to I$ is piecewise monotonic. If $S(f) > 1$, then f admits a periodic point whose period is not an integral power of 2 .*

THEOREM. *If $f : I \to I$ is a continuous expanding map with a finite number of turning points, then f admits a periodic point whose period is not an integral power of 2 and therefore is chaotic.*

Proof. For an arbitrary natural number n , we may take any lap $I_k^{(n)} = \left[c_{k-1}^{(n)}, c_k^{(n)}\right]$ of f^n . Let λ be an expansion constant of f , then we have $\left|f^n\left(c_k^{(n)}\right)-f^n\left(c_{k-1}^{(n)}\right)\right| \ge \lambda^n\left|c_k^{(n)}-c_{k-1}^{(n)}\right|$ by Lemma 1. Hence

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$\left|c_k^{(n)}-c_{k-1}^{(n)}\right| \le \lambda^{-n}$ since $\left|f^n\left(c_k^{(n)}\right)-f^n\left(c_{k-1}^{(n)}\right)\right| \le 1$, that is, the length of $I_k^{(n)}$ is equal to or less than λ^{-n} . Thus the lap number $l(f^n)$ of f^n is equal to or larger than $1/\lambda^{-n} = \lambda^n$, that is $l(f^n)^{1/n} \ge \lambda$; hence $S(f) \ge \lambda > 1$. By Lemma 2, f admits a periodic point whose period is not an integral power of 2 and therefore is chaotic. □

COROLLARY. *Suppose the continuous map $f : I \to I$ is piecewise monotonic. If there exists a natural number m such that f^m is expanding, then f admits a periodic point whose period is not an integral power of 2 and therefore is chaotic.*

Proof. Let $g = f^m$. Then g satisfies the conditions of the theorem; hence g admits a periodic point whose period is not an integral power of 2 . So f possesses a periodic point of period not equal to 2^n for any natural number n and therefore is chaotic. □

We give the following example to illustrate the corollary.

EXAMPLE. $f : I \to I$ is defined in the following way:

$$f(x) = \begin{cases} -3x/2 + 1 , & x \in [0, 2/3] , \\ 3x/4 - 1/2 , & x \in [2/3, 1] . \end{cases}$$

Hence

$$f^2(x) = \begin{cases} -9x/8 + 1/4 , & x \in [0, 2/9] , \\ 9x/4 - 1/2 , & x \in [2/9, 2/3] , \\ -9x/8 + 7/4 , & x \in [2/3, 1] . \end{cases}$$

It is obvious that f is not expanding, but f^2 is an expanding map.

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