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THE EULER-MACLAURIN SUM FORMULA FOR A CLOSED DERIVATION

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Abstract

An operator form of the Euler-Maclaurin sum formula is obtained, expressing the sum of the Euler-Maclaurin infinite series in a closed derivation, whose spectrum is compact, not equal to $\{0\}$, and does not have 0 as a clusterpoint, as the difference between a summation operator and an antiderivation which is the local inverse of the derivation.

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1. Introduction

We prove that to any closed derivation D on a complex Banach algebra $\mathfrak{A}$ there corresponds, subject to certain conditions on the spectrum of D, an Euler-Maclaurin formula

$$(1)\qquad S_\omega x = \frac{1}{\omega}Kx + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k\omega^{2k-1}}{(2k)!}D^{2k-1}x$$

on a closed vector subspace $\mathfrak{J}$ of $\mathfrak{A}$; that is, D determines an antiderivation K and a summation operator S_ω such that (1) holds for $x \in \mathfrak{J}$ and all sufficiently small $|\omega|$, $\omega \in \mathbf{C}$. Here B_k is the kth Bernoulli number. Moreover $K \mid \mathfrak{J}$ is the inverse of $D \mid \mathfrak{J}$. The subspace $\mathfrak{J}$ does not contain the identity of $\mathfrak{A}$.

The classes of operators involved here can be described as follows. By a *derivation on* $\mathfrak{A}$ we shall mean a closed linear operator $D: \mathfrak{D} \to \mathfrak{A}$ whose domain $\mathfrak{D}$ is a subalgebra of $\mathfrak{A}$, satisfying the identity

$$D(xy) = Dx \cdot y + x \cdot Dy \quad \text{for all } x, y \in \mathfrak{D}. \tag{2}$$

The usual definitions of antiderivation and summation operator must be stretched a little in order that the assertion in the first paragraph be true, because of the possible failure of $\mathfrak{J}$ to be a subalgebra of $\mathfrak{A}$. Rather than adopt the stretched definitions here, we shall instead adhere to the usual ones and make clear the necessary modification in the statement of the main result, which is Theorem 3.2. Therefore, if $\mathfrak{M}$ denotes a subalgebra of $\mathfrak{A}$, by an *antiderivation on* $\mathfrak{M}$ we shall mean a bounded linear operator $K: \mathfrak{M} \to \mathfrak{M}$ satisfying

$$Kx \cdot Ky = K(Kx \cdot y + x \cdot Ky) \quad \text{for all } x, y \in \mathfrak{M}; \tag{3}$$

and by a *summation operator on* $\mathfrak{M}$ we shall mean a bounded linear operator S: $\mathfrak{M} \to \mathfrak{M}$ satisfying

$$Sx \cdot Sy = S(Sx \cdot y + x \cdot Sy - xy) \quad \text{for all } x, y \in \mathfrak{M}. \tag{4}$$

Identities (3) and (4) are the particular cases of the *Baxter identity*

$$Tx \cdot Ty = T(Tx \cdot y + x \cdot Ty - \theta xy) \tag{5}$$

where the parameter θ is 0 and 1. The structural consequences of these identities have been discussed by several authors, including G. Baxter [2], F. V. Atkinson [1], J. F. C. Kingman [5], G.-C. Rota [9], [10], N.-H. Bong [3] and the author [6], [7]. In particular, the way in which K and S reproduce formally the properties of integration and summation respectively is pointed out in [6]: it is best seen on the subalgebras of $\mathfrak{A}$ generated by e and Ke, or e and Se, where e is the identity of $\mathfrak{A}$. Since as we remarked $e \notin \mathfrak{J}$, the formulae in [7] for the resolvents of K and S are not available in the present situation.

We also use the notion of an *averaging operator*; see Lemma 2.2.

Further notation: For any closed vector subspace $\mathfrak{X}$ of $\mathfrak{A}$, $\mathfrak{B}(\mathfrak{X})$ denotes the Banach algebra of all bounded linear operators on $\mathfrak{X}$ into $\mathfrak{X}$, with identity I; for $T \in \mathfrak{B}(\mathfrak{X})$, $\mathrm{Sp}(T)$ and $\nu(T)$ denote the spectrum and spectral radius of T, and $\mathrm{Res}(T) = \mathbf{C}\backslash\mathrm{Sp}(T)$; $R(\lambda, T)$ denotes the resolvent $(\lambda I - T)^{-1} \in \mathfrak{B}(\mathfrak{X})$, with domain $\mathrm{Res}(T)$. For closed operator T with bounded spectrum, $\nu(T) = \sup\{|\lambda|: \lambda \in \mathrm{Sp}(T)\}$.

$\mathfrak{A}$ need not have an identity element, though it is of interest to assume its existence at one or two places, such as 2.3. If D is unbounded then ∞ is a singularity of $R(\lambda, D)$. The conditions we assume on $\mathrm{Sp}(D)$ are: either

$$\mathrm{Sp}(D) = \{0\} \cup \sigma \quad \textit{where } \sigma \textit{ is nonempty and compact, and } 0 \notin \sigma; \tag{*}$$

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or

$[*]_0$ $\mathrm{Sp}(D) = \sigma$, *nonempty compact*, *with* $0 \notin \sigma$.

The case where D is a inner derivation on $\mathfrak{A}$, and therefore is in $\mathfrak{B}(\mathfrak{A})$, has been discussed in Miller [8], and is recalled in Section 5 below; Theorem 3.2 extends the results in [8] to a larger class of inner derivations and to larger domains for the formula (1). Unfortunately condition [*] seems to inhibit the application of the theorem to the differentiation operator on algebras of functions.

If $e \in \mathfrak{D}$ then $De = 0$ so D cannot have an inverse. A first step to establishing an Euler-Maclaurin formula is therefore to find a restriction of D which is one-one; that is, we need somehow to excise 0 from the spectrum of D, and remove e from the domain. The excision of 0 in case [*] is readily done using the functional calculus. Once a local inverse K of D has been found it is easy to see that K must behave like an antiderivation. The results in case $[*]_0$ come by simplification of these for [*].

2. D and K on the subspace $\mathfrak{J}$

Assume for this section that derivation D is given, and [*] holds. There exist positive numbers δ, ρ such that σ lies in the annulus $\{\lambda\colon \delta < |\lambda| < \rho\}$. Let γ and Γ be the positively oriented circles $|\lambda| = \delta$ and $|\lambda| = \rho$ respectively, and write

$$J_0 = \frac{1}{2\pi i}\int_\gamma R(\lambda, D)\,d\lambda, \qquad J = \frac{1}{2\pi i}\int_{\Gamma-\gamma} R(\lambda, D)\,d\lambda; \tag{6}$$

these are the residue idempotents of D for the spectral sets $\{0\}$, σ respectively. J_0 is of minor interest, but J is important for the theory. Now consider the defining equation of the sequence of Bernoulli numbers (see for example [11], page 127), which we write as

$$\frac{e^{\omega\lambda}}{e^{\omega\lambda} - 1} = \frac{1}{\omega\lambda} + \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{B_k\omega^{2k-1}}{(2k)!}\lambda^{2k-1}. \tag{7}$$

The series here converges if $|\omega\lambda| < 2\pi$; therefore if ρ is chosen to satisfy $\nu(D) < \rho < 2\pi|\omega|^{-1}$ the series can be integrated term by term around $\Gamma - \gamma$. The choice is possible if

$$|\omega| < \frac{2\pi}{\nu(D)}, \tag{8}$$

which we assume is the case. Multiplying (7) by $R(\lambda, D)$ and integrating with respect to λ gives

$$S = \frac{1}{\omega}K + \frac{1}{2}J + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{B_k\omega^{2k-1}}{(2k)!}F_{2k-1} \tag{9}$$

where

$$S \equiv S_\omega = \frac{1}{2\pi i}\int_{\Gamma-\gamma} e^{\omega\lambda}(e^{\omega\lambda}-1)^{-1}R(\lambda, D)\,d\lambda, \tag{10}$$

$$K = \frac{1}{2\pi i}\int_{\Gamma-\gamma} \lambda^{-1}R(\lambda, D)\,d\lambda, \tag{11}$$

$$F_l = \frac{1}{2\pi i}\int_{\Gamma-\gamma} \lambda^l R(\lambda, D)\,d\lambda. \tag{12}$$

The general theory of the operational calculus for closed operators ([4], pages 190–193; 200; 208–209) shows that J_0, J, S, K and F_l belong to $\mathfrak{B}(\mathfrak{A})$, and

$$DK = J \supseteq KD, \qquad K(\mathfrak{A}) \subseteq \mathfrak{D}, \qquad J(\mathfrak{A}) \subseteq \bigcap_{l=1}^{\infty} \operatorname{dom}(D^l), \tag{13}$$

$$KJ = K = JK, \qquad SJ = S = JS, \tag{14}$$

$$F_l = D^l J \supseteq JD^l \quad \text{for } l = 1, 2, \ldots. \tag{15}$$

Introduce the closed vector subspace

$$\mathfrak{J} = J(\mathfrak{A}) = \operatorname{ran}(J) \tag{16}$$

and let $\bar{D}$, $\bar{K}, \ldots$ denote the restrictions of D, $K, \ldots$ to $\mathfrak{J}$.

2.1 Lemma. *The range of K is $\mathfrak{J}$. The restrictions $\bar{D}$ and $\bar{K}$ are one-one maps onto $\mathfrak{J}$ and are inverses of each other. They belong to $\mathfrak{B}(\mathfrak{J})$.*

The proof uses the techniques of the operational calculus, and does not depend upon D being a derivation.

We should like $\mathfrak{J}$ also to be a subalgebra of $\mathfrak{A}$, but unfortunately this is not the case in general. A counterexample is mentioned in Section 5. As a device to get past this difficulty, introduce the operator

$$J_1 = J + J_0 = \frac{1}{2\pi i}\int_{\Gamma} R(\lambda, D)\,d\lambda, \tag{17}$$

for which $J_1^2 = J_1$ and

$$JJ_1 = J = J_1J, \qquad J_0J_1 = J_0 = J_1J_0, \qquad JJ_0 = 0 = J_0J. \tag{18}$$

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2.2 Lemma. *Since D is a derivation, J_0 and J_1 are averaging operators on $\mathfrak{A}$, that is, for all $x, y \in \mathfrak{A}$*

(19) $$J_0(J_0x \cdot y) = J_0x \cdot J_0y = J_0(x \cdot J_0y),$$

(20) $$J_1(J_1x \cdot y) = J_1x \cdot J_1y = J_1(x \cdot J_1y).$$

Moreover

(21) $$J_1(J_0x \cdot y) = J_0x \cdot J_1y, \qquad J_1(x \cdot J_0y) = J_1x \cdot J_0y$$

and

(22) $$J_0(J_1x \cdot y) = J_0(x \cdot J_1y).$$

Proof. Let $x, y \in \mathfrak{A}$. Using (2) we note that

(23) $$R(\lambda, D)x \cdot R(\mu, D)y = R(\lambda + \mu, D)\{R(\lambda, D)x \cdot y + x \cdot R(\mu, D)y\}$$

whenever $\lambda, \mu, \lambda + \mu \in \mathrm{Res}(D)$. With δ, ρ as before introduce the circles, all with positive orientation,

(24) $$\gamma_1: |\lambda| = \frac{1}{2}\delta, \qquad \gamma_2: |\mu| = \frac{1}{4}\delta,$$
$$\Gamma_1: |\lambda| = 3\rho, \qquad \Gamma_2: |\mu| = 2\rho.$$

For $i, j = 0, 1$, $J_ix \cdot J_jy$ can be written using (23) as

(25) $$\frac{1}{(2\pi i)^2}\int_{B_2} d\mu \int_{B_1} d\lambda R(\lambda + \mu, D)(x \cdot R(\mu, D)y)$$
$$+ \frac{1}{(2\pi i)^2}\int_{B_1} d\lambda \int_{B_2} d\mu R(\lambda + \mu, D)(R(\lambda, D)x \cdot y),$$

by making appropriate choices of $B_1 = \gamma_1$ or Γ_1, $B_2 = \gamma_2$ or Γ_2. This simplifies to give the required equation among (19)–(21). A variant of this method proves (22).

Partly as a corollary of this lemma we have

2.3 Lemma. *$J_0(\mathfrak{A})$ and $J_1(\mathfrak{A})$ are closed subalgebras of $\mathfrak{A}$, containing e when $\mathfrak{A}$ has identity e, and invariant under D. Also*

(26) $$J_0(\mathfrak{A}) \cup \mathfrak{J} \subseteq J_1(\mathfrak{A}) \subseteq \mathfrak{D},$$

(27) $$\mathfrak{J} \cap J_0(\mathfrak{A}) = (0).$$

If $x, y \in \mathfrak{J}$ then $xy \in J_1(\mathfrak{A})$; $e \notin \mathfrak{J}$.

We know that $\bar{D}$ is nearly a derivation on $\mathfrak{J}$; precisely,

(28) $$D(xy) = \bar{D}x \cdot y + x \cdot \bar{D}y \quad \text{for all } x, y \in \mathfrak{J}.$$

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The next result shows that its inverse $\bar{K}$ is nearly an antiderivation on $\mathfrak{J}$.

2.4 Lemma. *For all* $x, y \in \mathfrak{A}$,

$$J(Kx \cdot Ky) = K(Kx \cdot Jy + Jx \cdot Ky), \tag{29}$$

so for $x, y \in \mathfrak{J}$,

$$J(\bar{K}x \cdot \bar{K}y) = K(\bar{K}x \cdot y + x \cdot \bar{K}y). \tag{30}$$

If $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}$ then $\bar{K}$ is an antiderivation on $\mathfrak{J}$. More generally, if $\mathfrak{K}$ is any subalgebra of $\mathfrak{A}$ contained in $\mathfrak{J}$ and invariant under K then $K \mid \mathfrak{K}$ is an antiderivation.

Proof. The elements Kx, Ky belong to $\mathfrak{D}$ by (13), so $Kx \cdot Ky \in \mathfrak{D}$ by assumption on $\mathfrak{D}$; apply KD, to get (29). If $x, y \in \mathfrak{J}$ then (29) becomes (30). If $\mathfrak{J}$ is a subalgebra then

$$\begin{aligned} J(Kx \cdot Ky) &= J(JKx \cdot JKy) \quad \text{by (14)}, \\ &= JKx \cdot JKy = Kx \cdot Ky. \end{aligned}$$

3. S_ω and the Euler-Maclaurin formula

Equation (9), restricted to $\mathfrak{J}$, can now be written

$$\bar{S}_\omega = \frac{1}{\omega}\bar{K} + \frac{1}{2}\bar{I} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k \omega^{2k-1}}{(2k)!} \bar{D}^{2k-1}. \tag{31}$$

It remains to show that $\bar{S}_\omega$, which belongs to $\mathfrak{B}(\mathfrak{J})$ when $0 < |\omega| < 2\pi/\nu(D)$, satisfies the functional equation (4) or a closely similar equation. Since the problem is essentially one in $\mathfrak{B}(\mathfrak{J})$ we reduce it as far as possible to that algebra. Let subscripts be used to denote the algebra with respect to which spectra and resolvents are taken. It is not difficult to show that

$$\mathrm{Sp}_{\mathfrak{B}(\mathfrak{J})}(\bar{D}) \subseteq \mathrm{Sp}_{\mathfrak{B}(\mathfrak{A})}(D) \setminus \{0\} = \sigma \tag{32}$$

and

$$R_{\mathfrak{B}(\mathfrak{J})}(\lambda, \bar{D}) = R_{\mathfrak{B}(\mathfrak{A})}(\lambda, D) \mid \mathfrak{J}; \tag{33}$$

therefore restricting to $\mathfrak{J}$ in (10) gives

$$\bar{S} = S \mid \mathfrak{J} = SJ \mid \mathfrak{J} = \frac{1}{2\pi i} \int_{\Gamma - \gamma} e^{\omega\lambda}(e^{\omega\lambda} - 1)^{-1} R(\lambda, \bar{D})\, d\lambda. \tag{34}$$

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Since the operator $\bar{D}$ is bounded and $e^{\omega\lambda}$ and $(e^{\omega\lambda}-1)^{-1}$ are holomorphic and nonvanishing on σ we conclude that

(35) $$\bar{S}_\omega = e^{\omega\bar{D}}(e^{\omega\bar{D}} - \bar{J})^{-1},$$

where $e^{\omega\bar{D}}$ is the sum in $\mathfrak{B}(\mathfrak{J})$ of the exponential series, and that $\bar{S}_\omega$ is a regular element of $\mathfrak{B}(\mathfrak{J})$.

3.1 Lemma. *For all $x, y \in \mathfrak{A}$ and $0 < |\omega| < 2\pi/\nu(D)$ the operator $S \equiv S_\omega$ in* (10) *satisfies*

(36) $$J(Sx\cdot Sy) = S(Sx\cdot Jy + Jx\cdot Sy - Jx\cdot Jy),$$

so for $x, y \in \mathfrak{J}$,

(37) $$J(\bar{S}x\cdot\bar{S}y) = S(\bar{S}x\cdot y + x\cdot\bar{S}y - xy).$$

If $\mathfrak{R}$ is any subalgebra of $\mathfrak{A}$ which is contained in $\mathfrak{J}$ and invariant under S then $S \mid \mathfrak{R}$ is a summation operator.

Proof. It suffices to prove (37) since (36) then follows. The fact that $\mathfrak{J}$ may not be a subalgebra and $e^{\omega D}$ is possibly undefined makes the proof slightly delicate. First note that $e^{\omega\bar{D}}$ is the restriction to $\mathfrak{J}$ of $e^{\omega DJ_1}$ on $J_1(\mathfrak{A})$; here $DJ_1 \in \mathfrak{B}(\mathfrak{A})$, and $DJ_1 \mid J_1(\mathfrak{A})$ is a derivation on the subalgebra $J_1(\mathfrak{A})$, by (20). Therefore

$$e^{\omega DJ_1}(xy) = e^{\omega DJ_1}(x)\cdot e^{\omega DJ_1}(y) \quad \text{for all } x, y \in J_1(\mathfrak{A})$$

(the Leibnitz formula) and so

$$e^{\omega DJ_1}(ab) = e^{\omega\bar{D}}(a)\cdot e^{\omega\bar{D}}(b) \quad \text{for all } a, b \in \mathfrak{J}$$

(see 2.3). Also note from (14) that for any $x \in \mathfrak{A}$,

$$Sx = SJx = (e^{\omega\bar{D}} - \bar{J})^{-1} e^{\omega\bar{D}} Jx.$$

Let $x, y \in \mathfrak{J}$. Write

$$a = (e^{\omega\bar{D}} - \bar{J})^{-1}x, \qquad b = (e^{\omega\bar{D}} - \bar{J})^{-1}y,$$

so that $a, b \in \mathfrak{J}$ and

$$e^{\omega\bar{D}}a = \bar{S}x, \qquad e^{\omega\bar{D}}b = \bar{S}y.$$

Let

$$z = J(\bar{S}x\cdot\bar{S}y) - S(\bar{S}x\cdot y + x\cdot\bar{S}y - xy),$$

so that also $z \in \mathfrak{J}$. We have

$$\begin{aligned}(e^{\omega\bar{D}} - \bar{J})z = (e^{\omega DJ_1} - J)J(e^{\omega\bar{D}}a \cdot e^{\omega\bar{D}}b) \\ - (e^{\omega\bar{D}} - \bar{J})S\{e^{\omega\bar{D}}a \cdot (e^{\omega\bar{D}}b - b) + (e^{\omega\bar{D}}a - a) \cdot e^{\omega\bar{D}}b \\ -(e^{\omega\bar{D}}a - a) \cdot (e^{\omega\bar{D}}b - b)\} \\ = (e^{\omega DJ_1} - J)Je^{\omega DJ_1}(ab) - e^{\omega DJ_1}J\{e^{\omega DJ_1}(ab) - ab\} = 0,\end{aligned}$$

and therefore $z = 0$.

We have now established the main result, assuming [*]. If instead $[*]_0$ is the case then the theory is the same except for the simplification that $J_0 = 0$, $J_1 = J$ and therefore $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}$. The results can be formulated as follows.

3.2 THEOREM. *Let $\mathfrak{A}$ be a complex Banach algebra, and let D be a closed derivation on $\mathfrak{A}$ whose domain $\mathfrak{D}$ is a subalgebra and whose spectrum is compact, not equal to* $\{0\}$, *and either does not contain* 0 *or else has* 0 *as an isolated point. Let J be the residue idempotent* (6) *for the spectral set* $\mathrm{Sp}(D)\setminus\{0\}$, *assumed nonempty, and let* $\mathfrak{J} = J(\mathfrak{A})$. *Let* $\omega \in \mathbf{C}$, $0 < |\omega| < 2\pi/\nu(D)$.

Then $\mathfrak{J}$ is a closed vector subspace of $\mathfrak{A}$ not containing the identity of $\mathfrak{A}$ (if $\mathfrak{A}$ has an identity), but $\mathfrak{J} \subseteq \mathfrak{D}$, and there exist operators K and S_ω in $\mathfrak{B}(\mathfrak{A})$ such that

(i) *D, K, S_ω restrict on $\mathfrak{J}$ to regular elements of $\mathfrak{B}(\mathfrak{J})$, and $K \mid \mathfrak{J}$ is the inverse of $D \mid \mathfrak{J}$.*

(ii) *For $x, y \in \mathfrak{J}$, operators K and S_ω satisfy respectively the identities*

$$J(Kx \cdot Ky) = K(Kx \cdot y + x \cdot Ky), \tag{38}$$

$$J(S_\omega x \cdot S_\omega y) = S_\omega(S_\omega x \cdot y + x \cdot S_\omega y - xy). \tag{39}$$

(iii) *Let $\mathfrak{K}$ be a subalgebra of $\mathfrak{A}$ contained in $\mathfrak{J}$; if $\mathfrak{K}$ is invariant under K then $K \mid \mathfrak{K}$ is an antiderivation, if $\mathfrak{K}$ is invariant under S_ω then $S_\omega \mid \mathfrak{K}$ is a summation operator.*

(iv) *For all $x \in \mathfrak{J}$ and $0 < |\omega| < 2\pi/\nu(D)$ the Euler-Maclaurin sum formula*

$$S_\omega x = \frac{1}{\omega}Kx + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{B_k\omega^{2k-1}}{(2k)!}D^{2k-1}x \tag{40}$$

holds. The series converges in the operator norm of $\mathfrak{B}(\mathfrak{J})$.

(v) *The operators S_ω and K with these properties are given, on $\mathfrak{J}$, by* (10) *and* (11) *respectively.*

(vi) *If* $0 \notin \mathrm{Sp}(D)$ *then $\mathfrak{J}$ is a subalgebra of $\mathfrak{A}$, and K and S_ω are an antiderivation and a summation operator respectively on $\mathfrak{J}$.*

4. Remarks on the theorem

4.1 An example in which there exists a nontrivial subalgebra $\mathfrak{K}$ as in 3.2(iii), invariant under both K and S_ω, is given in [8], namely, the case when $\mathfrak{A}$ is the algebra $\mathfrak{M}_p$ of all $p \times p$ matrices over $\mathbf{C}$ and D is an inner derivation determined by a regular diagonalizable matrix.

4.2 The formal connection between D and S_ω, when they are restricted to the closed subspace $\mathfrak{J}$ (which is invariant under them) is the simple relation (35) between bounded operators,

$$\bar{S}_\omega = \left(I - e^{-\omega \bar{D}}\right)^{-1}. \tag{41}$$

If $\mathfrak{J}$ is a subalgebra then (41) is an instance of the formula (remarked by F. V. Atkinson [1], page 16)

$$T = (I - H)^{-1} \tag{42}$$

relating a homomorphism H and a summation operator T, both bounded, on a Banach algebra. When the algebra is unital then H must be nilpotent (Miller [7], page 519); (41) is an example where the algebra is not unital and H is not nilpotent.

As a statement about operators in $\mathfrak{B}(\mathfrak{J})$, equation (41) can be written

$$e^{\omega\bar{D}}\left(e^{\omega\bar{D}} - \bar{I}\right)^{-1} = \frac{1}{\omega}\bar{D}^{-1} + \frac{1}{2}\bar{I} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k \omega^{2k-1}}{(2k)!} \bar{D}^{2k-1}; \tag{43}$$

thus it is the result simply of the formal substitution of $\bar{D}$ for λ in (7).

4.3 Just as the Riemann integral of a function is the limit of a sequence of approximating sums constructed using equally spaced partition points (see (46) below), so too K results as a uniform limit from S_ω, namely

$$K = \lim_{\omega \to 0} \omega S_\omega. \tag{44}$$

This follows from (9); the limit is in the norm of $\mathfrak{B}(\mathfrak{A})$.

Similarly (41) gives

$$\bar{D} = \lim_{\omega \to 0} \omega^{-1} \bar{S}_\omega^{-1}; \tag{45}$$

this time the limit is in $\mathfrak{B}(\mathfrak{J})$.

4.4 It is clearly to be expected that an Euler-Maclaurin formula can be generated starting from a given antiderivation or a given summation operator, instead of from a derivation as we have done here. The latter will be the subject of a later paper [12].

4.5 It is not clear how the theorem can be applied to the differentiation operator on algebras of functions. The appropriate operators are $(Dx)(t) = x'(t)$,

$$(46) \qquad (Kx)(t) = \int_0^t x(u)\,du, \qquad (S_\omega x)(t) = \sum_{r=0}^{[t/\omega]} x(t - r\omega);$$

see [8], Section 1. However for the most natural algebras $\mathrm{Sp}(D)$ is either $\varnothing$, or is unbounded: it is necessary to formulate the problem in a Banach algebra for which $\mathrm{Sp}(D)$ satisfies $[*]$ or $[*]_0$. The classical formula can be obtained indirectly by applying the theorem to the Banach algebra $\mathfrak{X}_\Omega$ of generalized functions, described in [14].

5. Inner derivations

In [8] the author has given an Euler-Maclaurin formula for an inner derivation $D_g x = gx - xg$ where g is a regular element of unital noncommutative $\mathfrak{A}$, with a finite spectrum consisting of simple poles of $R(\lambda, D_g)$. In this case the derivation is of course bounded, and $J_1 = I$. The present discussion and Theorem 3.2 cover the more general case where $\mathrm{Sp}(g)$ is any finite set which may include 0. If $\mathrm{Sp}(g)$ were infinite, its difference set $\mathrm{Sp}(g) - \mathrm{Sp}(g)$ would have an accumulation point at 0, and since $\mathrm{Sp}(D_g) \subseteq \mathrm{Sp}(g) - \mathrm{Sp}(g)$, the premise in 3.2 that 0 be isolated in $\mathrm{Sp}(D)$ might not hold.

Suppose that $\mathrm{Sp}(g)$ is finite and equals $\{\alpha_1, \alpha_2, \ldots, \alpha_f\}$ say, where the α's are distinct and one may be 0. From spectral theory ([4], page 179) it is known that

$$(47) \qquad g = \alpha_1 j_1 + \alpha_2 j_2 + \cdots + \alpha_f j_f + c$$

where the j's are nonzero idempotents, $\sum_{r=1}^{f} j_r = e$ and $j_r j_s = 0$ for $r \neq s$, and j_r, g, c all commute. Assume as in [8] that $c = 0$. Then

$$(48) \qquad R(\lambda, D_g) = \sum_{r=1}^{f} \sum_{s=1}^{f} (\lambda - (\alpha_r - \alpha_s))^{-1} j_r x j_s,$$

so from (6),

$$(49) \qquad J_0 x = \sum_r j_r x j_r, \qquad Jx = \sum_{r} \sum_{\substack{s \\ r \neq s}} j_r x j_s,$$

$$\mathfrak{I} = \{x : j_r x j_r = 0 \text{ for } r = 1, 2, \ldots, f\}.$$

Thus for any $x, y \in \mathfrak{A}$,

$$Jx \cdot Jy = \sum_q \sum_r \sum_{\substack{s \\ q \neq r \\ r \neq s}} j_q x j_r x j_s,$$

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so $\mathfrak{J}$ in this case can fail to be a subalgebra of $\mathfrak{A}$. In [8] we introduced the vector subspace

$$\mathfrak{K} = \{x \in \mathfrak{A} : j_r x j_s = 0 \text{ if } (r, s) \notin \Omega\}, \quad \text{where } \Omega = \{(r, s) : |\alpha_r| > |\alpha_s|\};$$

for $x \in \mathfrak{K}$ it is possible to write

$$Kx = g^{-1}x + g^{-2}xg + g^{-3}xg^2 + \cdots. \tag{50}$$

Here $\mathfrak{K}$ is a closed subalgebra of $\mathfrak{A}$ contained in $\mathfrak{J}$ and invariant under D_g, K, S_ω. It is also contained in the set of quasinilpotents of $\mathfrak{A}$.

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