A PROBLEM ON THE RIESZ–DUNFORD OPERATOR CALCULUS AND CONVEX UNIVALENT FUNCTIONS

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Introduction. In his paper [**3**], Ky Fan asked whether if f is a convex univalent function in the unit disk, with $f(0)=0$ and $f'(0)=1$, then is it true that the set of $f(A)$ is a convex set of operators, when A runs through all proper contractions on a Hilbert space? We answer this question in the negative.

Let H be a complex Hilbert space. Let A be an operator (i.e. a bounded linear transformation) on H and let $\sigma(A)$ denote its spectrum. If $f(z)$ is a function analytic in a neighborhood G of $\sigma(A)$, then $f(A)$ will denote the operator on H defined by the usual Riesz–Dunford integral [**2**, p. 568]

$$f(A)=\frac{1}{2\pi i}\int_C f(z)(zI-A)^{-1}\,dz,$$

where I stands for the identity operator on H, C is a suitable finite family of positively oriented simple closed rectifiable contours.

As usual, an operator A is called a contraction or a proper contraction, if its norm $\|A\|\le 1$ or $\|A\|<1$ respectively.

Let $\Delta=\{z:|z|<1\}$ be the unit disk and let $K(\Delta)$ be the class of all convex univalent functions f normalized by $f(0)=0$ and $f'(0)=1$. Let $M(\Delta)$ be the class of all functions analytic in Δ; then by a theorem of Brickman–MacGregor–Wilken [**1**], we know that the extreme points of the set $K(\Delta)$ in the vector space $M(\Delta)$ are precisely the functions of the form

$$e_\theta(z)=z(1-e^{i\theta}z)^{-1},\quad\text{where}\quad 0\le\theta<2\pi.$$

In [**3**, Theorem 8], Ky Fan proved that the set of all $e_\theta(A)$ is a convex set of operators, when A runs through all proper contractions on a Hilbert space H. He then asked as to whether the same convexity is true for a function $f\in K(\Delta)$ instead of the extreme points. Furthermore, he proved that if f is a starlike function then its operator range $f(A)$ is also starlike [**3**, Theorem 7]. From this, one might conjecture that the operator range $f(A)$ is convex if f is convex. This however is false as will be seen from the following result. (For the definition of the Schwarz function, see [**4**, p. 385]).

THEOREM. *The Schwarz function*

$$s(z)=\int_0^z(1-t^4)^{-1/2}\,dt=z+\sum_1^\infty\frac{1\,.\,3\,\ldots\,(2n-1)}{2^n n!(4n+1)}\,z^{4n+1}$$

is convex and univalent in Δ. Its operator range $s(A)$ is starlike but not convex.

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Proof. The Schwarz function s maps Δ conformally onto the square with vertices at $\pm s(1)$ and $\pm is(1)$, so that it is convex and univalent in Δ. To prove that the operator range $s(A)$ is not convex, we let

$$A_1 = r\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = r\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{where} \quad 0<r<1.$$

Then the norm $\|A_1\| = \|A_2\| = r < 1$. By a simple computation, we find that

$$s(A_1) = \begin{bmatrix} 0 & s(r) \\ s(r) & 0 \end{bmatrix} \quad \text{and} \quad s(A_2) = \begin{bmatrix} 0 & s(r) \\ -s(r) & 0 \end{bmatrix}.$$

If the assertion were false, then there would be a proper contraction A such that

$$s(A) = \tfrac{1}{2}(s(A_1) + s(A_2)) = \begin{bmatrix} 0 & s(r) \\ 0 & 0 \end{bmatrix} = B.$$

Since the function $w = s(z)$ is univalent and $s(0) = 0$, it follows that the inverse $z = s^{-1}(w)$ is analytic at the origin and can be expanded as

$$z = s^{-1}(w) = w + \sum_{2}^{\infty} a_n w^n.$$

This yields $A = s^{-1}(B) = B$, because $B^n = 0$ for $n > 1$.

Clearly, the function s is continuous on the closure $\bar{\Delta}$ and the value $s(1) > 1$. By choosing r sufficiently close to 1, we obtain the norm $\|A\| = \|B\| = s(r) > 1$, a contradiction. This completes the proof.

REFERENCES

1. L. Brickman, T. H. MacGregor, and D. R. Wilken, Convex hulls of some classical families of univalent functions. *Trans. Amer. Math. Soc.* **156,** (1971) 91–107.

2. N. Dunford and J. T. Schwartz, *Linear operators, Part I: General theory.* (Interscience, 1958).

3. Ky Fan, Analytic functions of a proper contraction, *Math. Z.* **160,** (1978) 275–290.

4. E. Hille, Analytic function theory II. (Ginn, 1962).

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