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ON NEW CLASSES OF EXTREME SHOCK MODELS AND SOME GENERALIZATIONS

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Abstract

In extreme shock models, only the impact of the current, possibly fatal shock is usually taken into account, whereas in cumulative shock models, the impact of the preceding shocks is accumulated as well. A shock model which combines these two types is called a 'combined shock model'. In this paper we study new classes of extreme shock models and, based on the obtained results and model interpretations, we extend these results to several specific combined shock models. For systems subject to nonhomogeneous Poisson processes of shocks, we derive the corresponding survival probabilities and discuss some meaningful interpretations and examples.

Keywords: Extreme shock model; combined shock model; wear; virtual age; probability approximation

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1. Introduction

Consider a system subject to the nonhomogeneous Poisson process (NHPP) of shocks $N(t), t \geq 0$, with rate $\nu(t)$ and arrival (waiting) times T_i, $i = 1, 2, \ldots$. As usual, $N(t)$, $t \geq 0$, will also denote the corresponding counting process. Let our system be, for simplicity, 'absolutely reliable' in the absence of shocks. Assume that each shock (regardless of its number) results in its failure (and, therefore, in the termination of the corresponding NHPP of shocks) with probability $p(t)$ and is harmless (i.e. has no effect) with probability $q(t) = 1 - p(t)$. This setting is often referred to as the *extreme shock model* (see, e.g. Gut and Hüsler (2005)). Denote by T_S the time to termination (failure) of the process. It is well known that

$$\mathrm{P}(T_S \geq t) \equiv \bar{F}_S(t) = \exp\left\{-\int_0^t p(u)\nu(u)\,\mathrm{d}u\right\}, \tag{1}$$

and, therefore, the corresponding failure rate function $\lambda_S(t)$ is

$$\lambda_S(t) = p(t)\nu(t), \qquad t \geq 0. \tag{2}$$

For convenience, in what follows we will refer to this extreme shock model as the '$p(t) \Leftrightarrow q(t)$ model'. The formal proof of (1) in the mathematically equivalent interpretation for systems with perfect (probability $p(t)$) and minimal (probability$1 - p(t)$) repair can be found in

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Beichelt and Fischer (1980) and Block *et al.* (1985). The crucial assumptions for deriving the failure rate (2) are the assumptions that the shock process is the NHPP and that the probability $p(t)$ *also* does not depend on the history of the shock process.

We will relax now for a short while these assumptions and consider the orderly point process with the conditional (complete) intensity function (CIF) $\nu(t \mid H(t))$ (see Cox and Isham (1980) and Anderson *et al.* (1993)), where $H(t)$ is the history of the process up to t. (For the NHPP under consideration, $H(t) \equiv N(s),\ 0 \le s < t$.) Accordingly, let the probability of termination under a single shock be adjusted in a similar way and, therefore, also depend on this history, i.e. $p(t \mid H(t))$. Denote by T_S the corresponding lifetime. It is clear that in accordance with our assumptions, the conditional probability of termination in the infinitesimal interval of time can be written in the following form (see Finkelstein (2008, Chapter 8)):

$$\mathrm{P}(T_S \in [t, t + \mathrm{d}t) \mid T_S \ge t, H(t)) = p(t \mid H(t))\nu(t \mid H(t))\,\mathrm{d}t.$$

The only way for $p(t \mid H(t))\nu(t \mid H(t))$ to become a 'full-fledged' failure rate that corresponds to lifetime T_S (see (2)) and, therefore, for exponential representation (1) to hold, is when there is no dependence on $H(t)$ for both multipliers on the right-hand side. It is obvious that elimination of this dependence for the second multiplier uniquely leads to the NHPP. In our paper we will consider this case. However, specific types of dependence on history in the first multiplier will be retained and this will give rise to the new classes of extreme shock models. Note that, e.g. Gut and Hüsler (2005) considered the case of the renewal process of shocks (and constant probability p), but even for this simplest history of the shock process, only asymptotic results (as $t \to \infty$) could be obtained.

The effect of different shocks in practice is usually accumulated in some way and this leads to considering accumulated shock models (see, e.g. Sumita and Shanthikumar (1985) and Gut and Hüsler (2005)). In this paper we will suggest a general approach that allows us, under reasonable assumptions, to combine extreme shock models with some specific accumulated shock models.

The paper is organized as follows. In Section 2, new classes of extreme shock models are considered and the corresponding survival functions are derived. In Section 3, based on the results obtained in Section 2, the combined shock model of Cha and Finkelstein (2009) is generalized to the case when the wear increments incurred by each shock are independent but not necessarily identically distributed. In Section 4, some numerical results are presented. Finally, in Section 5, some concluding remarks are given.

2. New classes of extreme shock models

2.1. Model A

As mentioned in the introduction, we will consider the NHPP of shocks with rate $\nu(t)$ and history-dependent termination probability $p(t \mid H(t)) = p(t \mid N(s),\ 0 \le s < t)$. Let this be the simplest history case, i.e. the number of shocks $N(t)$ that our system has experienced in $[0, t)$. This seems to be a reasonable assumption, as each shock can contribute to 'weakening' of the system by increasing the probability $p(t \mid H(t)) \equiv p(t, N(t))$ and, therefore, the function $p(t, N(t))$ is usually increasing in $n(t)$ (for each realization, $N(t) = n(t)$). To obtain the following result, we must assume the specific form of this function. It is more convenient to consider the corresponding probability of survival. Let

$$q(t, n(t)) \equiv 1 - p(t, n(t)) = q(t)\rho(n(t)), \tag{3}$$

where $\rho(n(t))$ is a decreasing function of its argument (for each fixed t). Thus, the survival

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probability at each shock decreases as the number of survived shocks in $[0, t)$ increases. The multiplicative form of (3) will be important for us as it will be 'responsible' for the vital independence to be discussed later.

The survival function of the system's lifetime T_S is given by the following theorem.

Theorem 1. *Let* $m(t) \equiv \mathrm{E}[N(t)] = \int_0^t \nu(x)\,\mathrm{d}x$ *and* $\Psi(n) \equiv \prod_{i=0}^{n} \rho(i)$ $(\rho(0) \equiv 1)$. *Suppose that the inverse function* $m^{-1}(t)$ *exists. Then*

$$\mathrm{P}(T_S \geq t) = \mathrm{E}[\Psi(N_{q\nu}(t))] \exp\bigg\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\bigg\}, \tag{4}$$

where $\{N_{q\nu}(t),\ t \geq 0\}$ *follows the NHPP with rate* $q(t)\nu(t)$.

Proof. Obviously, conditioning on the process (in each realization) gives

$$\mathrm{P}(T_S \geq t \mid N(s),\ 0 \leq s < t) = \prod_{i=0}^{N(t)} q(T_i)\rho(i),$$

where, formally, $q(T_0) \equiv 1$ and $\rho(0) \equiv 1$ corresponds to the case when $N(t) = 0$. Also, by convention, $\prod_{i=1}^{n} (\cdot)_i \equiv 1$ for $n = 0$. Then the corresponding expectation is

$$\mathrm{P}(T_S \geq t) = \mathrm{E}\bigg[\prod_{i=1}^{N(t)} q(T_i)\rho(i)\bigg].$$

Define $N^*(t) \equiv N(m^{-1}(t)),\ t \geq 0$, and $T_j^* \equiv m(T_j),\ j \geq 1$. It is known that $\{N^*(t),\ t \geq 0\}$ is a stationary Poisson process with rate 1 (see, e.g. Çinlar (1975, Section 4.7)) and that T_j^*, $j \geq 1$, are the times of occurrence of shocks in the new time scale. Let $s = m(t)$. Then

$$\mathrm{E}\bigg[\prod_{i=1}^{N(t)} q(T_i)\rho(i)\bigg] = \mathrm{E}\bigg[\prod_{i=1}^{N^*(s)} q(m^{-1}(T_i^*))\rho(i)\bigg] = \mathrm{E}\bigg[\mathrm{E}\bigg[\prod_{i=1}^{N^*(s)} q(m^{-1}(T_i^*))\rho(i) \;\bigg|\; N^*(s)\bigg]\bigg].$$

The joint distribution of $(T_1^*, T_2^*, \ldots, T_n^*)$ given $N^*(s) = n$ is the same as the joint distribution of $(V_{(1)}, V_{(2)}, \ldots, V_{(n)})$, where $V_{(1)} \leq V_{(2)} \leq \cdots \leq V_{(n)}$ are the order statistics of independent and identically distributed (i.i.d.) random variables $V_1, V_2, \ldots, V_n$ that are uniformly distributed in the interval $[0, s] = [0, m(t)]$. Thus,

$$\begin{aligned}
\mathrm{E}\bigg[\prod_{i=1}^{N^*(s)} q(m^{-1}(T_i^*))\rho(i) \;\bigg|\; N^*(s) = n\bigg] &= \prod_{i=1}^{n} \rho(i)\,\mathrm{E}\bigg[\prod_{i=1}^{n} q(m^{-1}(T_i^*)) \;\bigg|\; N^*(s) = n\bigg] \\
&= \prod_{i=1}^{n} \rho(i)\,\mathrm{E}\bigg[\prod_{i=1}^{n} q(m^{-1}(V_{(i)}))\bigg] \\
&= \prod_{i=1}^{n} \rho(i)\,\mathrm{E}\bigg[\prod_{i=1}^{n} q(m^{-1}(V_i))\bigg] \\
&= \prod_{i=1}^{n} \rho(i)(\mathrm{E}[q(m^{-1}(V_1))])^n \\
&= \prod_{i=1}^{n} \rho(i)(\mathrm{E}[q(m^{-1}(sU))])^n,
\end{aligned}$$

where $U \equiv V_1/s = V_1/m(t)$ is a random variable uniformly distributed in the unit interval [0,1].

Therefore,

$$\mathrm{E}[q(m^{-1}(sU))] = \int_0^1 q(m^{-1}(su))\,\mathrm{d}u = \int_0^1 q(m^{-1}(m(t)u))\,\mathrm{d}u = \frac{1}{m(t)}\int_0^t q(x)\nu(x)\,\mathrm{d}x.$$

Hence,

$$\mathrm{E}\bigg[\prod_{i=1}^{N^*(s)} q(m^{-1}(T_i^*))\rho(i) \;\bigg|\; N^*(s)=n\bigg] = \prod_{i=1}^{n}\rho(i)\bigg(\frac{1}{m(t)}\int_0^t q(x)\nu(x)\,\mathrm{d}x\bigg)^n.$$

Using $\Psi(n) \equiv \prod_{i=1}^n \rho(i)$,

$$\begin{aligned}
\mathrm{P}(T_S \geq t) &= \mathrm{E}\bigg[\prod_{i=1}^{N(t)} q(T_i)\rho(i)\bigg] \\
&= \sum_{n=0}^{\infty}\Psi(n)\bigg(\frac{1}{m(t)}\int_0^t q(x)\nu(x)\,\mathrm{d}x\bigg)^n \frac{(m(t))^n}{n!}\mathrm{e}^{-m(t)} \\
&= \exp\bigg\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\bigg\}\sum_{n=0}^{\infty}\Psi(n)\frac{(\int_0^t q(x)\nu(x)\,\mathrm{d}x)^n}{n!} \\
&\quad\times \exp\bigg\{-\int_0^t q(x)\nu(x)\,\mathrm{d}x\bigg\} \\
&= \mathrm{E}[\Psi(N_{q\nu}(t))]\exp\bigg\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\bigg\},
\end{aligned}$$

where $\{N_{q\nu}(t),\ t \geq 0\}$ follows the NHPP with rate $q(t)\nu(t)$. This completes the proof.

It is obvious that $A^*(t) \equiv \mathrm{E}[\Psi(N_{q\nu}(t))] \leq 1$ and

$$A^*(t) = \sum_{n=0}^{\infty}\Psi(n)\frac{(\int_0^t q(x)\nu(x)\,\mathrm{d}x)^n}{n!}\exp\bigg\{-\int_0^t q(x)\nu(x)\,\mathrm{d}x\bigg\}.$$

Therefore, for the given values of parameters, $A^*(t)$ can be approximately obtained numerically. Taking into account the multiplicative form of $\Psi(n)$, we could expect that the number of terms, n^*, in this sum for the reasonable approximation is not very large (for the values of the corresponding probabilities, this make sense in practice).

Example 1. Let $\rho(i) = \rho^{i-1}$, $i = 1, 2, \ldots$. Then $\Psi(n) \equiv \rho^{n(n-1)/2}$ and

$$\begin{aligned}
\mathrm{P}(T_S \geq t) &= \sum_{n=0}^{\infty}\rho^{n(n-1)/2}\frac{(\int_0^t q(x)\nu(x)\,\mathrm{d}x)^n}{n!}\exp\bigg\{-\int_0^t q(x)\nu(x)\,\mathrm{d}x\bigg\} \\
&\quad\times \exp\bigg\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\bigg\} \\
&= \sum_{n=0}^{\infty}\rho^{n(n-1)/2}\frac{(\int_0^t q(x)\nu(x)\,\mathrm{d}x)^n}{n!}\exp\bigg\{-\int_0^t \nu(x)\,\mathrm{d}x\bigg\}. \qquad (5)
\end{aligned}$$

In Section 4 we will show numerically that $A^*(t)$ in this case is well approximated by the finite sum with a relatively small value of n^*.

Before discussing the obtained result, it is useful to make the following remark, which reinterprets the well-known fact about the splitting of the Poisson process.

Remark 1. Let $\{N(t),\ t \geq 0\}$ be the NHPP with intensity function (rate) $\nu(t)$. If an event occurs at time t, it is classified as a type-I event with probability $p(t)$ and as a type-II event with the complementary probability $1-p(t)$, as in our initial $p(t) \Leftrightarrow q(t)$ model in the introduction. Then $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ are the independent NHPPs with rates $p(t)\nu(t)$ and $q(t)\nu(t)$, respectively, and $N(t) = N_1(t) + N_2(t)$. Accordingly, e.g. given that there have been no type-I events in $[0, t)$, the process $\{N(t),\ t \geq 0\}$ reduces to $\{N_2(t),\ t \geq 0\}$, as in our specific case when a type-I event (fatal shock) leads to the termination of the process (failure). Therefore, in order to describe the lifetime to the (first) termination, it is obviously sufficient to consider $\{N_2(t),\ t \geq 0\}$, and not the original $\{N(t),\ t \geq 0\}$.

We will use a similar reasoning for the more general $p(t \mid H(t)) \Leftrightarrow q(t \mid H(t))$ model considered above, although interpretation of the types of event will be slightly different in this case. In the following, in accordance with our previous notation, $N_2(t) = N_{q\nu}(t)$ and the arrival times of this process will be denoted by $T_{(q\nu)1}, T_{(q\nu)2}, \ldots$.

The multiplicative form of the specific result in (4) indicates that it might also be obtained and interpreted via the following general reasoning, which can be useful for probabilistic analysis of various extensions of standard extreme shock models. Considering the classical $p(t) \Leftrightarrow q(t)$ extreme shock model, assume that there can be other additional causes of termination dependent either directly on a history of the point process (as in Model A) or on some other variables, e.g. as for the marked point process, when each event is ‘characterized’ by some variable (e.g. damage or wear). The latter case will be considered in Section 3. Just for the sake of definiteness of presentation, let us call this ‘initial’ cause of failure, which corresponds to the $p(t) \Leftrightarrow q(t)$ model, the *main* or the *critical cause of failure* (termination) and the shock that leads to this event, *the critical shock* (type-I event). However, distinct from the $p(t) \Leftrightarrow q(t)$ model, the type-II events, which follow the Poisson process with rate $q(t)\nu(t)$, can also now result in failure.

Let, as previously, T_S be the corresponding time to failure of the system, and let $E_C(t)$ denote the event that no critical shock has occurred until time t in the absence of other causes of failures. Then, obviously,

$$\mathrm{P}(T_S \geq t \mid E_C(t)) = \frac{\mathrm{P}(T_S \geq t, E_C(t))}{\mathrm{P}(E_C(t))} = \frac{\mathrm{P}(T_S \geq t)}{\mathrm{P}(E_C(t))},$$

and, thus,

$$\mathrm{P}(T_S \geq t) = \mathrm{P}(T_S \geq t \mid E_C(t))\,\mathrm{P}(E_C(t)),$$

where

$$\mathrm{P}(E_C(t)) = \mathrm{P}(N_1(t) = 0) = \exp\biggl\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\biggr\}. \tag{6}$$

Therefore, in accordance with our previous reasoning (Remark 1) and notation, we can describe $\mathrm{P}(T_S \geq t \mid E_C(t))$ in terms of the process $\{N_{q\nu}(t),\ t \geq 0\}$ (and not in terms of the original process$\{N(t),\ t \geq 0\}$) in the following general form to be specified for the forthcoming models:

$$\mathrm{P}(T_S \geq t \mid E_C(t)) = \mathrm{E}[\mathbf{1}(\Psi(N_{q\nu}(t), \Theta) \in S) \mid E_C(t)].$$

Here $\mathbf{1}(\cdot)$, as usual, is the corresponding indicator, Θ is a set of random variables that are ‘responsible’ for other causes of failure (see later), $\Psi(N_{q\nu}(t), \Theta)$ is a real-valued function of

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$(N_{q\nu}(t), \Theta)$ which represents the state of the system at time t (given $E_C(t)$ i.e. no critical shock has occurred), and S is a set of real values which defines the survival of the system in terms of $\Psi(N_{q\nu}(t), \Theta)$. That is, if the critical shock has not occurred, the system survives when $\Psi(N_{q\nu}(t), \Theta) \in S$.

We now return to Model A with the foregoing general reasoning in mind. In order to apply it, we have to reinterpret Model A as follows. Suppose first that the system is composed of two parts in series and that each shock affects only one component. If it hits the first component (with probability $p(t)$), it directly causes its (and the system's) failure (the critical shock). On the other hand, if it hits the second component (with probability $q(t)$) then it fails with probability $1 - \rho(n(t))$ and survives with probability $\rho(n(t))$. This interpretation nicely conforms with the two independent causes of failure described by (3). Note that, in fact, we are speaking about the *conditional independence* of causes of failure (on condition that a shock from the Poisson process with rate $\nu(t)$ has occurred).

Another (and probably more practical) interpretation is as follows. Assume that there are some parts of a system (component 1) that are critical to *only*, e.g. the shock's level of severity (failure with probability $p(t)$) and that other parts (component 2) are critical to *only* the accumulation of damage (failure with probability $1-\rho(n(t))$). Assuming the series structure and the corresponding independence, we arrive at the survival (on shock) probability (3).

We can define now the function $\Psi(N_{q\nu}(t), \Theta)$ for Model A. Suppose that there have been no critical shocks in $[0, t)$, and let $\varphi_i = 1$ if the second component survives the ith shock and $\varphi_i = 0$, $i = 1, 2, 3, \ldots, N(t)$, otherwise. Then

$$\Psi(N_{q\nu}(t), \Theta) = \prod_{i=1}^{N_{q\nu}(t)} \varphi_i$$

and $S = \{1\}$. Therefore, as events $E_C(t)$ and $\Psi(N_{q\nu}(t), \Theta) \in S$ are 'related' to only the first (main) and the second causes of failure, respectively, and these causes of failure are independent, we have

$$\begin{aligned}
\mathrm{P}(T_S \geq t \mid E_C(t)) &= \mathrm{E}[\mathbf{1}(\Psi(N_{q\nu}(t), \Theta) \in S) \mid E_C(t)] \\
&= \mathrm{E}[\mathbf{1}(\Psi(N_{q\nu}(t), \Theta) \in S)] \\
&= \mathrm{E}\bigg[\mathbf{1}\bigg(\prod_{i=1}^{N_{q\nu}(t)} \varphi_i = 1\bigg)\bigg] \\
&= \mathrm{E}\bigg[\mathrm{P}\bigg(\prod_{i=1}^{N_{q\nu}(t)} \varphi_i = 1 \;\bigg|\; N_{q\nu}(t)\bigg)\bigg] \\
&= \mathrm{E}\bigg[\prod_{i=1}^{N_{q\nu}(t)} \rho(i)\bigg].
\end{aligned}$$

Combining this equation with (6), we arrive at the original result in (4).

In the following subsection, we will see that the utility of the above reasoning is not limited to the special case considered above and that the suggested approach can be useful in more general situations.

2.2. Model B

In this subsection we consider a different type of extreme shock model, which is, in fact, a generalization of Model A. In Subsection 2.1, the second cause of failure (termination) was due to the number of noncritical shocks, no matter what the severity of these shocks was. Now we will count only those shocks (to be called 'dangerous') with severity larger than some level κ. Assume that the second cause of failure 'materializes' only when the number of dangerous shocks exceeds some random level M. That is, given $M = m$, in the absence of critical shocks, the system fails as soon as it experiences the $(m + 1)$th dangerous shock.

Assume that the shock's severity is a random variable with cumulative distribution function (CDF) $G(t)$ and that the survival function for M, $\mathrm{P}(M > l)$, $l = 0, 1, 2, \dots$, is also given. Suppose that there have been no critical shocks until time t, and let φ_i be the indicator random variable ($\varphi_i = 1$ if the ith shock is dangerous and $\varphi_i = 0$ otherwise). Then, as previously,

$$\Psi(N_{q\nu}(t), \Theta) = \mathbf{1}\bigg(M \geq \sum_{i=1}^{N_{q\nu}(t)} \varphi_i\bigg)$$

and $S = \{1\}$. Thus,

$$\begin{aligned}
\mathrm{P}(T_S \geq t \mid E_C(t)) &= \mathrm{E}[\mathbf{1}(\Psi(N_{q\nu}(t), \Theta) \in S)] \\
&= \mathrm{E}\bigg[\mathbf{1}\bigg(M \geq \sum_{i=1}^{N_{q\nu}(t)} \varphi_i\bigg)\bigg] \\
&= \mathrm{P}\bigg(M \geq \sum_{i=1}^{N_{q\nu}(t)} \varphi_i\bigg) \\
&= \mathrm{E}\bigg[\mathrm{P}\bigg(M \geq \sum_{i=1}^{N_{q\nu}(t)} \varphi_i \,\bigg|\, N_{q\nu}(t)\bigg)\bigg],
\end{aligned}$$

where,

$$\begin{aligned}
&\mathrm{P}\bigg(M \geq \sum_{i=1}^{N_{q\nu}(t)} \varphi_i \,\bigg|\, N_{q\nu}(t) = n\bigg) \\
&\quad = \mathrm{P}(M > n \mid N_{q\nu}(t) = n) \\
&\qquad + \sum_{m=0}^{n} \mathrm{P}\bigg(M \geq \sum_{i=1}^{n} \varphi_i \,\bigg|\, N_{q\nu}(t) = n,\ M = m\bigg) \mathrm{P}(M = m \mid N_{q\nu}(t) = n) \\
&\quad = \mathrm{P}(M > n) + \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{l} \overline{G}(\kappa)^l G(\kappa)^{n-l}\, \mathrm{P}(M = m) \\
&\quad = \mathrm{P}(M > n) + \sum_{l=0}^{n} \sum_{m=l}^{n} \binom{n}{l} \overline{G}(\kappa)^l G(\kappa)^{n-l}\, \mathrm{P}(M = m) \\
&\quad = \mathrm{P}(M > n) + \sum_{l=0}^{n} \binom{n}{l} \overline{G}(\kappa)^l G(\kappa)^{n-l} (\mathrm{P}(M \geq l) - \mathrm{P}(M \geq n + 1)) \\
&\quad = \sum_{l=0}^{n} \binom{n}{l} \overline{G}(\kappa)^l G(\kappa)^{n-l}\, \mathrm{P}(M \geq l).
\end{aligned}$$

Thus, similar to the derivations of the previous section,

$$\mathrm{P}(T_S \ge t \mid E_C(t)) = \sum_{n=0}^{\infty}\bigg[\sum_{l=0}^{n}\mathrm{P}(M \ge l)\binom{n}{l}\overline{G}(\kappa)^l G(\kappa)^{n-l}\bigg] m_q(t)^n \frac{\exp\{-m_q(t)\}}{n!},$$

where $m_q(t) \equiv \int_0^t q(x)\nu(x)\,\mathrm{d}x$, and, finally, we have

$$\mathrm{P}(T_S \ge t) = \exp\bigg\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\bigg\}\sum_{n=0}^{\infty}\bigg[\sum_{l=0}^{n}\mathrm{P}(M \ge l)\binom{n}{l}\overline{G}(\kappa)^l G(\kappa)^{n-l}\bigg] \times m_q(t)^n \frac{\exp\{-m_q(t)\}}{n!}.$$

Remark 2. When the expression for $\mathrm{P}(T_S \ge t \mid E_C(t))$ involves not only the number of shocks, $N_{q\nu}(t)$, but also the filtration generated by $(N_{q\nu}(s),\ 0 \le s \le t)$, the computation becomes intensive and the results might not be useful in practice. For example, consider another type of generalized extreme shock model with two causes of failure. The first cause is the failure from a critical shock and the second cause is the failure which occurs when two consecutive shocks are 'too close' (see Finkelstein (2008, pp. 201–205) for details on the second cause of failure). The latter means that the system did not recover from the consequences of the previous shock. It is clear that these causes of failure are conditionally independent (on the condition that a shock from the Poisson process with rate $\nu(t)$ has occurred). Suppose that there have been *no critical shocks in* $[0, t)$, and let φ_i be the time needed for recovery from the $(i-1)$th shock. It is natural to assume that the φ_i, $i = 1, 2, \dots,$ are i.i.d. random variables with common CDF $\vartheta(t)$ and independent of the shock process. Then the ith shock causes immediate failure of the system, with probability $1 - \vartheta(T_i - T_{i-1})$, and is harmless to the system with probability $\vartheta(T_i - T_{i-1})$, where $T_0 \equiv 0$. In this case we define

$$\Psi(N_{q\nu}(t), T_{(q\nu)1}, T_{(q\nu)2}, \dots, T_{(q\nu)N_{(q\nu)}(t)}, \Theta) = \prod_{i=1}^{N_{q\nu}(t)} \mathbf{1}(\varphi_i < T_{(q\nu)i} - T_{(q\nu)i-1}),$$

$S = \{1\}$, and then

$$\mathrm{P}(T_S \ge t \mid E_C(t)) = \mathrm{E}[\mathbf{1}(\Psi(N_{q\nu}(t), T_{(q\nu)1}, T_{(q\nu)2}, \dots, T_{(q\nu)N_{(q\nu)}(t)}, \Theta) \in S) \mid E_C(t)].$$

Using similar procedures as above, it can be shown that

$$\begin{aligned}\mathrm{P}(T_S \ge t \mid E_C(t)) &= \mathrm{E}\bigg[\prod_{i=0}^{N_{q\nu}(t)} \vartheta(T_{(q\nu)i} - T_{(q\nu)i-1})\bigg]\\ &= \bigg(\exp\{-m_q(t)\} \\ &\quad + \sum_{n=1}^{\infty}\bigg[\int_0^{m_q(t)}\int_0^{u_n}\cdots\int_0^{u_3}\int_0^{u_2}\prod_{i=1}^{n}\vartheta(m_q^{-1}(u_i) - m_q^{-1}(u_{i-1})) \\ &\qquad \times \mathrm{d}u_1\,\mathrm{d}u_2\cdots\mathrm{d}u_{n-1}\,\mathrm{d}u_n\bigg]\exp\{-m_q(t)\}\bigg),\end{aligned}$$

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and, thus,

$$\mathrm{P}(T_S \geq t) = \exp\left\{-\int_0^t p(x)\nu(x)\,\mathrm{d}x\right\}\mathrm{P}(T_S \geq t \mid E_C(t)).$$

However, this result contains high-dimensional multiple integrations and is not really useful for reliability applications. Thus, the corresponding stochastic simulation could be better than the computational approximation based on these integrations.

3. Extended 'combined' shock models

In this section, using the approach suggested in the previous sections, the combined shock model of Cha and Finkelstein (2009) is extended to the case when the wear increments W_i, $i = 1, 2, \ldots$, are nonidentically distributed. For this purpose, we first discuss a new 'simplified' interpretation of the failure model studied in our previous work.

3.1. New interpretation of the model

First, we briefly revisit the setting (slightly simplified) considered in Cha and Finkelstein (2009). As before, the system is subject to $\{N(t),\ t \geq 0\}$—an orderly point process of shocks with arrival times T_i, $i = 1, 2, \ldots$. Assume that the ith shock is critical (type I) with probability $p(T_i)$, and, with probability $q(T_i) = 1 - p(T_i)$, the shock increases the wear of the system by a random increment $W_i \geq 0$. As before, let T_S denote the corresponding lifetime of the system. In accordance with this setting, the random wear of the system at time t is (given that no critical shock has occurred until time t)

$$W(t) = \sum_{i=0}^{N_q(t)} W_i,$$

where $N_q(t)$ is the 'thinned' original process with thinning probability $q(t)$. Failure occurs when the critical shock occurs or $W(t)$ reaches the boundary R. Observe that, under the condition that no critical shocks have occurred until time t, $N_q(t) \equiv N(t)$ (all events from the original process are events from the thinned process—see also Remark 1). Therefore,

$$\begin{aligned}
&\mathrm{P}(T_S \geq t \mid N(s),\ 0 \leq s \leq t;\ W_1, W_2, \ldots, W_{N(t)};\ R) \\
&\quad = \mathrm{P}(E_C(t) \mid N(s),\ 0 \leq s \leq t;\ W_1, W_2, \ldots, W_{N(t)};\ R) \\
&\qquad \times \mathrm{P}(W(t) \leq R \mid N(s),\ 0 \leq s \leq t;\ W_1, W_2, \ldots, W_{N(t)};\ R;\ E_C(t)) \\
&\quad = \prod_{i=0}^{N(t)} q(T_i)\,\mathbf{1}\left(\sum_{i=0}^{N(t)} W_i \leq R\right), \qquad (7)
\end{aligned}$$

where, as previously, $E_C(t)$ denotes the event that no critical shocks have occurred in $[0, t)$, $q(T_0) \equiv 1$, and, as throughout the paper, this probability should be understood conditionally on the corresponding realizations of $N(t)$, W_i, $i = 1, 2, \ldots, N(t)$, and R.

Assume now that $\{N(t),\ t \geq 0\}$ is the NHPP with rate $\nu(t)$, the W_i are the i.i.d. random variables, and that R is exponentially distributed with parameter λ. Then $\mathrm{P}(T_S \geq t)$ can be obtained explicitly by direct derivation (see Theorem 1 of Cha and Finkelstein (2009)) as

$$\begin{aligned}
\mathrm{P}(T_S \geq t) &= \mathrm{E}\left[\left(\prod_{i=0}^{N(t)} q(T_i)\right)\exp\left\{-\lambda\sum_{i=0}^{N(t)} W_i\right\}\right] \\
&= \exp\left\{-\int_0^t \nu(x)\,\mathrm{d}x + M_W(-\lambda)\int_0^t q(x)\nu(x)\,\mathrm{d}x\right\}, \qquad t \geq 0, \qquad (8)
\end{aligned}$$

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and, therefore, the corresponding failure rate function is

$$\lambda_S(t) = (1 - M_W(-\lambda)q(t))\nu(t), \tag{9}$$

where $M_W(\cdot)$ is the moment generating function of the W_is.

In view of our reasoning in previous sections, the multiplicative form of (8) suggests the following probabilistic interpretation. A system can fail from (i) the critical shock or (ii) the accumulated wear caused by the shocks. Suppose that the system has survived until time t. Then, as the distribution of the random boundary R is exponential, the accumulated wear until time t, $\sum_{i=0}^{N(t)} W_i$, does not affect the failure process of the component after time t. That is, on the next shock, the probability of the system's failure due to the accumulated wear, given that a critical shock has not occurred, is just $\mathrm{P}(R \le W_{N(t)+1})$, and does not depend on the wear accumulation history, that is,

$$\begin{aligned} &\mathrm{P}(R \ge W_1 + W_2 + \cdots + W_n \mid R > W_1 + W_2 + \cdots + W_{n-1}) \\ &\quad = \mathrm{P}(R > W_n) \quad \text{for all } n = 1, 2, \ldots, W_1, W_2, \ldots, \end{aligned} \tag{10}$$

where $W_1 + W_2 + \cdots + W_{n-1} \equiv 0$ when $n = 1$. Then, finally, each shock results in the immediate failure with probability $p(t) + q(t)\,\mathrm{P}(R \le W_1)$; otherwise, the system survives with probability $q(t)\,\mathrm{P}(R > W_1)$. Although we have two (independent) causes of failure in this case, the second cause (distinct from (3)), also does not depend on the history of the process and, therefore, our initial $p(t) \Leftrightarrow q(t)$ model can be applied after an obvious modification. In accordance with (2), the corresponding failure rate can then be immediately obtained as

$$\begin{aligned} \lambda_S(t) &= (p(t) + q(t)\,\mathrm{P}(R \le W_1))\nu(t) \\ &= (1 - q(t)\,\mathrm{P}(R > W_1))\nu(t) \\ &= (1 - q(t)M_W(-\lambda))\nu(t). \end{aligned} \tag{11}$$

The validity of the above reasoning and interpretation can again be verified by comparing the failure rate function in (11) with that directly derived in (9). This interpretation will also be used in the next subsections for the extension of the model to the case when the W_i, $i = 1, 2, \ldots,$ are nonidentically distributed.

It is clear that this reasoning can be applied due to the specific, exponential distribution of the boundary R, which implies the Markov property for the wear 'accumulation'. Note that, in Cha and Finkelstein (2009) the case of a deterministic boundary was also considered and, obviously, the foregoing interpretation 'does not work' for this case.

3.2. Nonidentically distributed wear increments: Model I

We consider the model in (7), but now we assume that the W_i, $i = 1, 2, \ldots,$ are independent but not identically distributed. Obviously, property (10) still holds due to the independence condition and the reasoning of the previous subsection can be partially applied. However, in this case, when the ith shock occurs at time t, the system fails with probability $p(t) + q(t)\,\mathrm{P}(R \le W_i) = (1 - q(t)\,\mathrm{P}(R > W_i))$ and survives this shock with probability $q(t)\,\mathrm{P}(R > W_i)$, $t \ge 0, i = 1, 2, \ldots.$ This is similar to (3) and, therefore, the approach developed in Section 2 with $\rho(i) = \mathrm{P}(R > W_i)$, $i = 1, 2, \ldots,$ is valid. Thus, Theorem 1 holds with the substitution of $\Psi(n) \equiv \prod_{i=0}^{n} \rho(i)$ by $\Psi(n) \equiv \prod_{i=0}^{n} \mathrm{P}(R > W_i)$ $(\mathrm{P}(R > W_0) \equiv 1)$.

Example 2. Consider the simple case when the W_i, $i = 1, 2, \ldots,$ are increasing but deterministic: $W_i = iw$, $i = 1, 2, \ldots,$ where w is a positive constant. Then, from the model considered in Example 1, we have $\rho(i) \equiv \mathrm{P}(R > W_i) = \exp\{-iw\lambda\} = \rho^i$, $i = 1, 2, \ldots,$ where $\rho = \exp\{-w\lambda\}$. Therefore, (5) holds with $\rho^{n(n-1)/2}$ replaced by $\rho^{n(n+1)/2}$.

Example 3. Let the X_i, $i = 1, 2, \ldots$, be the i.i.d. sequence of continuous random variables with CDF $F_X(x)$ and probability density function (PDF) $f_X(x)$. Let the W_i, $i = 1, 2, \ldots$, be stochastically increasing as $W_i = iX_i$, $i = 1, 2, \ldots$. Then $\rho(i) \equiv \mathrm{P}(R > W_i) = \int_0^\infty (\exp\{-\lambda x\})^i f_X(x)\,\mathrm{d}x$ and

$$\Psi(n) \equiv \prod_{i=1}^{n} \biggl(\int_0^\infty (\exp\{-\lambda x\})^i f_X(x)\,\mathrm{d}x \biggr).$$

In this case, using Jensen's inequality, a lower bound for $\mathrm{P}(T_S \geq t)$ can be obtained as (5) with $\rho = \int_0^\infty (\exp\{-\lambda x\}) f_X(x)\,\mathrm{d}x$ and $\rho^{n(n-1)/2}$ replaced by $\rho^{n(n+1)/2}$.

Example 4. Let the X_i, $i = 1, 2, \ldots$, be the i.i.d. sequence of continuous random variables with CDF $F_X(x)$ and PDF $f_X(x)$. Let the W_i, $i = 1, 2, \ldots$, be stochastically increasing as $W_i = \alpha^{i-1} X_i$, $i = 1, 2, \ldots$, where $\alpha > 1$. Then $\rho(i) \equiv \mathrm{P}(R > W_i) = \int_0^\infty (\exp\{-\lambda x\})^{\alpha^{i-1}} \times f_X(x)\,\mathrm{d}x$ and

$$\Psi(n) \equiv \prod_{i=1}^{n} \biggl(\int_0^\infty (\exp\{-\lambda x\})^{\alpha^{i-1}} f_X(x)\,\mathrm{d}x \biggr).$$

3.3. Nonidentically distributed wear increments: Model II

In this subsection we also consider nonidentically distributed but independent wear increments (model (7)). Assume that the wear increment caused by the ith shock is given by $Z(T_i)$, where $Z(t)$ (the wear increment at time t) is a nonnegative continuous random variable with CDF $F_{Z(t)}(x)$ and PDF $f_{Z(t)}(x)$. Assume that these random variables for different values of t are independent of each other. Note that W_i in the previous subsection was dependent on the history via the number i, whereas now we do not have such a dependence and, therefore, the modified $p(t) \Leftrightarrow q(t)$ model can be applied. In practice, $Z(t)$ is often stochastically increasing with t, but our description does not require this assumption. Thus, the accumulated wear in $[0, t)$ is (given that no critical shock has occurred until time t)

$$W(t) = \sum_{i=0}^{N_q(t)} Z(T_i),$$

where, by convention, $W(t) = 0$ when $N_q(t) = 0$. Some simple examples of $W(t)$ are as follows.

(i) If the distribution of $Z(t)$ does not depend on t and is given by $F_Z(x)$, then the model reduces to (7).

(ii) Let $f_{Z(t)}(x) = (1/\mu(t)) \exp\{-(1/\mu(t))x\}$, $x \geq 0$, where $\mu(t)$ is increasing in t. Then it is easy to see that $Z(t_2) \geq_{\mathrm{st}} Z(t_1)$ for $t_2 > t_1$, which implies that $Z(t)$ is stochastically increasing in t.

(iii) Let $Z(t) = h(t)$, where $h(t)$ is a deterministic function. Then the random variable $Z(t)$ has a degenerate distribution with the corresponding mass at $h(t)$, $t \geq 0$.

Based on the interpretation given in Subsection 3.1, the shock that has occurred at time t causes immediate failure of the system (given that it has survived up to t) with probability $p(t) + q(t)\,\mathrm{P}(R \leq Z(t))$; otherwise, the system survives with probability $q(t)\,\mathrm{P}(R > Z(t))$. Then (8) and (9) hold with the substitution of $M_W(-\lambda)$ by the time dependent $M_{Z(t)}(-\lambda)$.

4. Numerical results

In this section we reconsider two examples given in the previous sections and show that numerical approximation is already sufficiently satisfactory for the relatively small number of iterated calculations in the corresponding terms.

Example 5. (*Example 1 revisited.*) Let $\rho \equiv 0.8$, $\nu(t) = 1.0$, $t \geq 0$, and $q(t) = \exp\{-t\}$, $t \geq 0$. Fix n^*, and calculate the approximate probability:

$$\mathrm{P}(T_S \geq t) \approx \sum_{n=0}^{n^*} \rho^{n(n-1)/2} \frac{(\int_0^t q(x)\nu(x)\,\mathrm{d}x)^n}{n!} \exp\biggl\{-\int_0^t \nu(x)\,\mathrm{d}x\biggr\}.$$

The results are summarized in Table 1.

Example 6. (*Example 4 revisited.*) Let $\alpha \equiv 1.2$, $\nu(t) = 1.0$, $t \geq 0$, $q(t) = \exp\{-t\}$, $t \geq 0$, $f_X(x) = \exp\{-x\}$, $x \geq 0$, and $\lambda = 0.1$. Then

$$\rho(i) = \mathrm{P}(R > W_i) = \int_0^\infty (\exp\{-\lambda x\})^{\alpha^{i-1}} f_X(x)\,\mathrm{d}x = \frac{1}{\alpha^{i-1}\lambda + 1}, \qquad i = 1, 2, \ldots.$$

Fix n^*, and calculate the approximate probability:

$$\mathrm{P}(T_S \geq t) \approx \sum_{n=0}^{n^*} \prod_{i=1}^{n} \frac{1}{\alpha^{i-1}\lambda + 1} \frac{(\int_0^t q(x)\nu(x)\,\mathrm{d}x)^n}{n!} \exp\biggl\{-\int_0^t \nu(x)\,\mathrm{d}x\biggr\}.$$

The results are summarized in Table 2.

TABLE 1: Approximate probabilities.

t	n^*							
	1	2	3	4	5	6	8	10
0.2	0.967 141	0.977 902	0.978 319	0.978 328	0.978 328	0.978 328	0.978 328	0.978 328
0.4	0.891 311	0.920 454	0.922 503	0.922 590	0.922 592	0.922 592	0.922 592	0.922 592
0.8	0.696 761	0.751 263	0.757 666	0.758 117	0.758 137	0.758 138	0.758 138	0.758 138
1.2	0.511 670	0.570 503	0.579 274	0.580 059	0.580 103	0.580 105	0.580 105	0.580 105
1.6	0.363 031	0.414 472	0.423 230	0.424 125	0.424 183	0.424 186	0.424 186	0.424 186
2.0	0.252 355	0.292 828	0.300 294	0.301 120	0.301 179	0.301 181	0.301 181	0.301 181

TABLE 2: Approximate probabilities.

t	n^*							
	1	2	3	4	5	6	8	10
0.2	0.953 650	0.964 568	0.965 114	0.965 167	0.965 167	0.965 167	0.965 167	0.965 167
0.4	0.871 221	0.900 789	0.903 630	0.903 829	0.903 840	0.903 841	0.903 841	0.903 841
0.8	0.674 268	0.729 565	0.738 438	0.739 480	0.739 575	0.739 582	0.739 582	0.739 582
1.2	0.492 536	0.552 229	0.564 383	0.566 193	0.566 403	0.566 423	0.566 424	0.566 424
1.6	0.348 382	0.400 575	0.412 712	0.414 777	0.415 050	0.415 079	0.415 081	0.415 081
2.0	0.241 717	0.282 781	0.293 127	0.295 034	0.295 307	0.295 339	0.295 342	0.295 342

As shown by the above numerical examples, the convergence of the survival probability $\mathrm{P}(T_S \geq t)$ is rather 'quick' and an excellent approximation is achieved for relatively small n^*.

5. Concluding remarks

In this paper, some new classes of shock models have been studied. Based on the obtained results and model interpretations, we have suggested a general approach for obtaining survival probabilities in various extreme shock models with two conditionally independent causes of failure. Furthermore, applying this general approach, our previous combined shock model (see Cha and Finkelstein (2009)) has been revisited and extended to the case when the wear increments are not necessarily identically distributed. Some other settings that can arise in practical applications were also considered. Throughout the models considered in this paper, it has been shown that if a failure of certain type is triggered by a sufficient number of Poisson shocks of certain type, then the survivability of a system can be factored in terms of tractable survival probabilities of the system under various failure modes.

The developed approach is based on the assumption that the underlying shock process follows the NHPP pattern. It seems that this assumption cannot be relaxed in the framework of the suggested methodology. However, another crucial assumption of the conditional independence of the causes of failure (termination) can be most likely dropped and the corresponding dependence structure can be studied. More than two causes of termination can be also considered. These are the topics for future work.

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