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SEQUENCES AND BASES IN p-BANACH SPACES

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Necessary and sufficient conditions are given for an infinite dimensional subspace of a p-Banach space X with basis to contain a basic sequence which can be extended to a basis of X.

In [1] it is proved that if X is a Banach space with a basis and $(y_n)_{n=1}^{\infty}$ is a regular sequence which converges to zero coordinatewise, then $(y_n)_{n=1}^{\infty}$ has a subsequence which can be extended to a basis of X and so every infinite dimensional subspace Y of X contains a basic sequence which can be extended to a basis of X. Our purpose is to study these properties in p-Banach spaces.

If X is a real linear space and $0 < p \leq 1$, a p-norm $||\cdot||$ on X is a map from X into $[0,+\infty)$ which satisfies the following conditions:

a) $||x|| = 0$ if and only if $x = 0$.

b) $||tx|| = |t|^p \, ||x||$ if $x \in X$ and $t \in \mathbb{R}$.

c) $||x+y|| \leq ||x|| + ||y||$ if $x,y \in X$.

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With the distance $d(x,y) = ||x-y||$, X becomes a metric space and if $(X, ||\cdot||)$ is complete, it is called p-Banach space (Banach space if $p=1$). The p-th root of a p-norm is a quasi-norm $|||\cdot|||$ and satisfies

a) $|||x||| = 0$ if and only if $x = 0$.

b) $|||tx||| = |t|\ |||x|||$ if $x \in X$ and $t \in \mathbb{R}$

c) $|||x+y||| \leqq C(|||x||| + |||y|||)$, where $C \geqq 1$ is a positive number which does not depend on x and y.

In particular, the space ℓ_p $(0 < p < 1)$ of all real sequences $(x_n)_{n=1}^{\infty}$ such that

$$||(x_n)|| = \sum_{n=1}^{\infty} |x_n|^p < \infty$$

is a p-Banach space.

Let $(X, ||\cdot||)$ be a p-Banach space with topological dual X^*, and suppose that X^* separates the points of X (this condition will be assumed throughout this paper). We can provide X^* with the dual norm

$$||x^*||^* = \sup_{||x|| \leqq 1} |x^*(x)|$$

and $(X^*, ||\cdot||^*)$ becomes a Banach space. The inclusion map from $(X, ||\cdot||)$ into $(X^{**}, ||\cdot||^{**})$ is continuous, more precisely, $||x||^{**} \leqq ||x||^{1/p}$ for every x.

If $(X, ||\cdot||)$ is a p-Banach space, the convex hulls of the balls of $(X, ||\cdot||)$ form a basis of zero neighbourhoods of a locally convex topology which is the finest locally convex topology on X whose dual is X^*, that is, the Mackey topology of the dual pair $\langle X, X^* \rangle$. This is usually called the Mackey topology of $(X, ||\cdot||)$, and can be defined by the norm induced by the bidual $(X^{**}, ||\cdot||^{**})$ (See [3]).

If Y is a closed subspace of $(X, ||\cdot||)$, $(Y, ||\cdot||)$ is a p-Banach space for which the bidual norm $||\cdot||_Y^{**}$ defines the Mackey topology. In general, the Mackey topology of Y is stronger than the topology induced by the Mackey topology of X. It is easy to see, using duality arguments, that both topologies coincide if and only if Y has the Hahn Banach extension property (HBEP): every $y^* \in Y^*$ is the restriction to Y of some $x^* \in X^*$.

A sequence $(z_n)_{n=1}^{\infty}$ is called block basic sequence with respect to a basis $(x_n)_{n=1}^{\infty}$ if there exists a strictly increasing sequence of positive integers $(m_n)_{n=1}^{\infty}$ such that

$$z_n = \sum_{i=m_{n-1}+1}^{m_n} a_i x_i \qquad (*)$$

where $(a_n)_{n=1}^{\infty}$ is a sequence of scalars. And a sequence $(y_n)_{n=1}^{\infty}$ is called regular if $\inf_n ||y_n|| > 0$. If $(x_n)_{n=1}^{\infty}$ is a bounded and regular basis of a p-Banach space, then $\inf_n ||x_n||^{**} > 0$. (See [2, Proposition 3.2.iii])

Let $(z_n)_{n=1}^{\infty}$ be a block basic sequence as $(*)$. If $(y_n)_{n=1}^{\infty}$ is a sequence in X with $Y_{m_j} = z_j$ and $y_n \in [x_i]_{i=m_{j-1}+1}^{m_j}$ whenever $m_{j-1} < n \leqq m_j$, then $(y_n)_{n=1}^{\infty}$ is called block extension of $(z_n)_{n=1}^{\infty}$. Morrow [5] has proved that a bounded and regular block basic sequence $(z_n)_{n=1}^{\infty}$ in a p-Banach space X has a block extension that is a basis of X if and only if $\inf_n ||z_n||^{**} > 0$.

The proof of the following lemma is similar to the one known for the Banach case (see [4, 1.a.9]).

LEMMA 1. *Let* $(x_n)_{n=1}^{\infty}$ *be a normalized basis of a p-Banach space* X *with basis constant* K. *Let* $(y_n)_{n=1}^{\infty}$ *be a sequence in* X *with* $\sum_{n=1}^{\infty} ||x_n - y_n|| < \frac{1}{2K}$. *Then* $(y_n)_{n=1}^{\infty}$ *is a basis of* X *which is equivalent to* $(x_n)_{n=1}^{\infty}$.

THEOREM 2. *Let* $(x_n)_{n=1}^{\infty}$ *be a basis of a p-Banach space* X *and* $(y_n)_{n=1}^{\infty}$ *a normalized sequence in* X *which converges to zero coordinatewise and such that* $\inf_n ||y_n||^{**} = C > 0$ *. Then* $(y_n)_{n=1}^{\infty}$ *has a subsequence which can be extended to a basis of* X *.*

Proof. We can suppose that $(x_n)_{n=1}^{\infty}$ is normalized. Let K be the basis constant of $(x_n)_{n=1}^{\infty}$. With an usual "gliding hump" method we can find a subsequence $(y_{p_n})_{n=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ and a $(z_n)_{n=1}^{\infty}$ block basic sequence of $(x_n)_n$ as $(*)$ such that:

$$||y_{p_n} - z_n|| < \frac{C}{K.2^{n+1}}$$

Since $||y_{p_n} - z_n||^{**} \leq ||y_{p_n} - z_n||^{1/p} < \frac{C}{4}$, we deduce that $\inf_n ||z_n||^{**} > 0$ and therefore a $(u_n)_{n=1}^{\infty}$ block extension of $(z_n)_{n=1}^{\infty}$ exists, which is a basis of X . Let $(v_n)_{n=1}^{\infty}$ be the sequence defined by

$$v_n = \begin{cases} u_n & \text{if } n \neq m_k \text{ for every } k \\ y_{p_k} & \text{if } n = m_k \text{ .} \end{cases}$$

Since

$$\sum_{n=1}^{\infty} ||v_n - u_n|| = \sum_{n=1}^{\infty} ||y_{p_n} - z_n|| < \frac{1}{2K} .$$

$(v_n)_{n=1}^{\infty}$ is a basis of X which is an extension of $(y_{p_n})_{n=1}^{\infty}$ (lemma 1 is used).

THEOREM 3. *Let* X *be a p-Banach space with basis. Let* Y *be an infinite dimensional subspace of* X *. The following are equivalent:*

i) Y *contains a basic sequence which can be extended to a basis of* X *.*

ii) The unit ball of Y *is not relatively compact in* $(X^{**}, ||\cdot||^{**})$ *.*

Proof. i ⇒ ii) Let $(y_n)_{n=1}^{\infty}$ be a basic sequence in X such that a basis of X $(w_n)_{n=1}^{\infty}$ with $w_{p_n} = y_n$ for every $n \in \mathbb{N}$ exists. We can suppose that $(y_n)_{n=1}^{\infty}$ is contained in the unit ball of Y. Let $(w_n^*)_{n=1}^{\infty} \subset X^*$ be the sequence such that $w_n^*(w_m) = \delta_{n,m}$ and we write $y_n^* = w_{p_n}^*$. If a subsequence $(y_{n_j})_{j=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ converges in $(X^{**}, ||\cdot||^{**})$, then the sequence $(z_j)_{j=1}^{\infty}$ with $z_j = y_{n_j} - y_{n_{j+1}}$ converges to zero in $(X^{**}, ||\cdot||^{**})$. But this is impossible because

$$||z_j||^{**} \geq \frac{y_{n_j}^*}{||y_{n_j}^*||}(y_{n_j} - y_{n_{j+1}}) = \frac{1}{||y_{n_j}^*||} \geq \frac{1}{K}$$

where K is the basis constant of $(w_n)_{n=1}^{\infty}$, and so the unit ball of Y is not relatively compact in $(X^{**}, ||\cdot||^{**})$.

ii ⇒ i) If the unit ball of Y is not relatively compact in $(X^{**}, ||\cdot||^{**})$ then a bounded sequence $(z_n)_{n=1}^{\infty}$ in Y exists which does not have any subsequence converging in $(X^{**}, ||\cdot||^{**})$. Replacing $(z_n)_{n=1}^{\infty}$ with a subsequence, if needed, we can suppose that $(z_n)_{n=1}^{\infty}$ is Cauchy-coordinatewise, and also assume that $0 < \inf_n ||z_n - z_{n+1}||^{**}$.

Then if $y_n = z_n - z_{n+1}$, applying theorem 1 $(y_n)_{n=1}^{\infty}$ has a subsequence which can be extended to a basis.

COROLLARY 4. *If Y has an infinite dimensional subspace Z with HBEP then Y contains a sequence which can be extended to a basis of X.*

Proof. If the unit ball of $(Y, ||\cdot||)$ is relatively compact in $(X^{**}, ||\cdot||^{**})$, the same is true for any subspace Z of Y. If Z has the HBEP, the Mackey topology of $(Z, ||\cdot||)$ is defined by $||\cdot||^{**}$ and Z must be finite dimensional.

THEOREM 5. *Let $(X, ||\cdot||)$ be a p-Banach space with basis, such that every infinite dimensional subspace $Y \subset X$ contains a basic sequence which can be extended to a basis of X. Then $(X, ||\cdot||)$ must be locally convex.*

Proof. If $(X, ||\cdot||)$ is not locally convex, we can find a sequence $(x_n)_{n=1}^{\infty}$ in X such that $||x_n|| = 1$ and $||x_n||^{**} < \frac{1}{n}$. Using a "gliding hump" method we can assume that there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ which is a basic sequence. Let $Y = \overline{sp}[x_{n_k}]_{k=1}^{\infty}$.

Since $||x_{n_k}|| = 1$ and $||x_{n_k}||^{**} < \frac{1}{2^{n_k}}$ the inclusion map i: $(Y, ||\cdot||) \longrightarrow (X^{**}, ||\cdot||^{**})$ is compact, and theorem 3 ensures us that Y does not have any basic sequence which can be extended to a basis of X, contradicting the hypothesis.

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