*Glasgow Math. J.* **45** (2003) 353–372. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089503001290. Printed in the United Kingdom

# SIMPLE (−1, −1) BALANCED FREUDENTHAL KANTOR TRIPLE SYSTEMS

ALBERTO ELDUQUE*
*Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain*
*e-mail: elduque@unizar.es*

NORIAKI KAMIYA**
*Center for Mathematical Sciences, The University of Aizu, 965-8580 Aizu-Wakamatsu, Japan*
*e-mail: kamiya@u-aizu.ac.jp*

and SUSUMU OKUBO***
*Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA*
*e-mail: okubo@pas.rochester.edu*

(Received 8 April, 2002; accepted 8 November, 2002)

**Abstract.** The simple finite dimensional $(-1,-1)$ balanced Freudenthal Kantor triple systems over fields of characteristic zero are classified.

2000 *Mathematics Subject Classification.* 17A40, 17B20, 17B25.

**1. Introduction.** In 1954, H. Freudenthal [**10**] constructed the exceptional simple Lie algebras of types $E_7$ and $E_8$ by means of the exceptional simple Jordan algebras. The construction of $E_8$ has been extended in several ways to give 5-graded Lie algebras

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

starting with some nonassociative algebras or triple systems, which appear as the component $\mathfrak{g}_1$.

The concept of $(\varepsilon,\delta)$-Freudenthal Kantor triple system covers many of these systems:

Definition 1.1 [**29**]. Let $\varepsilon, \delta = \pm 1$. A vector space $V$ over a field $F$, endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \mapsto xyz$, is said to be a $(\varepsilon,\delta)$-*Freudenthal Kantor triple system* ($(\varepsilon,\delta)$-FKTS for short) if the following two conditions are satisfied

(i) $[l_{a,b}, l_{c,d}] = l_{l_{a,b}c,d} + \varepsilon l_{c,l_{b,a}d}$,

(ii) $l_{d,c}k_{a,b} - \varepsilon k_{a,b}l_{c,d} = k_{k_{a,b}c,d}$

for any $a, b, c, d \in V$, where $l_{a,b}, k_{a,b} : V \to V$ are given by $l_{a,b}c = abc$, $k_{a,b}c = acb - \delta bca$.

Thus a $(-1, 1)$-FKTS is exactly a generalized Jordan triple system of second order in the sense of Kantor [**20**] (if $k = 0$ this is just a Jordan triple system), while a

*Supported by the Spanish MCYT (BFM 2001-3239-C03-03).
**Supported in part by a Ministry of Education of Japan grant (no. 10640024).
***Supported in part by U.S. Department of Energy Grant no. DE-F402-ER 40685.

$(1, -1)$-FKTS with $k = 0$ is an anti-Jordan triple system (see [**9**] for the definition of anti-Jordan pair $(U^+, U^-)$; when $U^+ = U^-$ one gets an anti-Jordan triple system).

An $(\varepsilon, \delta)$-FKTS $V$ is said to be *balanced* ($(\varepsilon, \delta)$-BFKTS for short) if there exists a nonzero bilinear form $( \mid ) : V \times V \to F$ such that $k_{a,b} = (a \mid b)1_V$ for any $a, b \in V$ ($1_V$ denotes the identity map on $V$). Since $k_{a,b} = -\delta k_{b,a}$ by its own definition, $( \mid )$ is either symmetric ($\delta = -1$) or skew-symmetric ($\delta = 1$). On the other hand, condition (ii) in Definition 1.1 gives here that $( \mid )$ is either symmetric or skew-symmetric according to $\varepsilon$ being $-1$ or 1, so that $\varepsilon = \delta$ in case $V$ is balanced.

Any $(1, 1)$-BFKTS becomes, by means of minor modifications of its triple product, a symplectic ternary algebra [**8**], a symplectic triple system [**28**] or a Freudenthal triple system [**21**], and conversely. The simple finite dimensional Freudenthal triple systems were classified in [**21**], with some restrictions which are satisfied if the ground field is algebraically closed, and this amounts to a classification of the simple $(1, 1)$-BFKTS (and of the symplectic ternary algebras [**8**]). The related 5-graded Lie algebras satisfy that $\mathfrak{g}_{\pm 2}$ is one dimensional.

Further properties of $(\varepsilon, \delta)$-FKTS's can be found in [**12**–**18**, **24**] and the references therein.

Our aim in this paper is to obtain the classification of the finite dimensional simple $(-1, -1)$-BFKTS's over fields of characteristic 0. To achieve this, the classification [**11**] of the finite dimensional simple Lie superalgebras over algebraically closed fields of characteristic 0 will be used, but we will have to look at the known relationship between $(-1, -1)$-FKTS's and 5-graded Lie superalgebras [**27**] in a different way, suitable to our needs. This will be done in Section 2. The relevant examples of $(-1, -1)$-BFKTS's will be given in Section 3 and, finally, Section 4 will provide the promised classification (Theorem 4.3), which asserts that the simple finite dimensional $(-1, -1)$-BFKTS's fall into six classes, three of them with arbitrarily large dimension: orthogonal, unitarian and symplectic types; and another three classes of four dimensional ($D_\mu$-type), seven dimensional (G-type) and eight dimensional systems (F-type).

Using Definition 1.1, the defining relations for a $(-1, -1)$-BFKTS are

$$ab(xyz) = (abx)yz - x(bay)z + xy(abz), \tag{1.1}$$

$$abx + bax = (a \mid b)x = axb + bxa, \tag{1.2}$$

for any $a, b, x, y, z \in V$, where $( \mid )$ is a nonzero symmetric bilinear form. Over fields of characteristic $\neq 2$, put $\langle \mid \rangle = \frac{1}{2}( \mid )$ and then (1.2) is equivalent to

$$xxy = \langle x \mid x \rangle y = xyx \tag{1.3}$$

for any $x, y \in V$.

The main motivation for the classification of the simple $(-1, -1)$-BFKTS's was provided by the recent paper [**19**] by two of the authors, where the exceptional simple classical Lie superalgebras were constructed by using the last three classes mentioned above (D, G and F types). These triple systems are closely related to quaternion and octonion algebras. A different construction of the exceptional simple classical Lie superalgebras has been given in [**2**] by means of a generalized Tits' construction (which also uses quaternion and octonion algebras).

**2. $(-1, -1)$ balanced Freudenthal Kantor triple systems and Lie superalgebras.** The relationship between $(-1, -1)$-BFKTS and Lie superalgebras has been studied in [**19**]. A more useful approach for us is obtained as indicated by the next Theorem.

THEOREM 2.1. *Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra over a field $F$ of characteristic $\neq 2$ such that $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2(F) \oplus \mathfrak{d}$ (direct sum of ideals) and $\mathfrak{g}_{\bar{1}} = U \otimes_F V$, where $U$ is the two dimensional module for $\mathfrak{sl}_2(F)$ and $V$ is a module for $\mathfrak{d}$. Let $\varphi$ be a nonzero skew symmetric form on $U$, so that we may identify $\mathfrak{sl}_2(F) = \mathfrak{sp}(U, \varphi)$ and for any $a, b \in U$ consider the map $\varphi_{a,b} \in \mathfrak{sl}_2(F)$ given by*

$$\varphi_{a,b}(c) = \varphi(c, a)b + \varphi(c, b)a$$

*for any $c \in U$. Then the product of odd elements in $\mathfrak{g}$ is given by*

$$[a \otimes u, b \otimes v] = \langle u \mid v\rangle \varphi_{a,b} + \varphi(a, b)d_{u,v} \tag{2.1}$$

*for any $a, b \in U$ and $u, v \in V$, where $\langle\,|\,\rangle$ is a symmetric bilinear form and $V \times V \to \mathfrak{d}$, $(x, y) \mapsto d_{x,y}$, is a skew symmetric bilinear map that satisfy*

$$\langle d(x) \mid y\rangle + \langle x \mid d(y)\rangle = 0, \tag{2.2a}$$

$$[d, d_{x,y}] = d_{d(x),y} + d_{x,d(y)}, \tag{2.2b}$$

$$d_{x,y}(y) = \langle y \mid x\rangle y - \langle y \mid y\rangle x, \tag{2.2c}$$

*for any $x, y \in V$ and $d \in \mathfrak{d}$.*

*Conversely, let $V$ be a vector space endowed with a symmetric bilinear form $\langle\,|\,\rangle : V \times V \to F$ and a skew symmetric bilinear map $V \times V \to \mathrm{End}_F(V)$ ($(u, v) \mapsto d_{u,v}$). Assume that:*

$$\langle d_{u,v}(x) \mid y\rangle + \langle x \mid d_{u,v}(y)\rangle = 0, \tag{2.3a}$$

$$[d_{u,v}, d_{x,y}] = d_{d_{u,v}(x),y} + d_{x,d_{u,v}(y)}, \tag{2.3b}$$

$$d_{x,y}(y) = \langle y \mid x\rangle y - \langle y \mid y\rangle x, \tag{2.3c}$$

*for any $u, v, x, y \in V$. Let $\mathfrak{d}$ be span$\{d_{u,v} : u, v \in V\}$ (a Lie subalgebra of $\mathrm{End}_F(V)$ by (2.3b)) and let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be the superalgebra where $\mathfrak{g}_{\bar{0}}$ is the Lie algebra $\mathfrak{sl}_2(F) \oplus \mathfrak{d} = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{d}$, $\mathfrak{g}_{\bar{1}}$ is the $\mathfrak{g}_{\bar{0}}$-module $U \otimes_F V$ and where the product of odd elements is given by (2.1). Then $\mathfrak{g}$ is a Lie superalgebra.*

*Proof.* Since $\mathrm{Hom}_{\mathfrak{sp}(U,\varphi)}(U \otimes_F U, F)$ is spanned by the form $\varphi$ and $\mathrm{Hom}_{\mathfrak{sp}(U,\varphi)}(U \otimes_F U, \mathfrak{sp}(U, \varphi))$ is spanned by the symmetric map $a \otimes b \mapsto \varphi_{a,b}$, formula (2.1) follows. Formulae (2.2a) and (2.2b) follow from the Jacobi superidentity applied to the elements $d \in \mathfrak{d}$ and $a \otimes x, b \otimes y \in U \otimes_F V$ and (2.2c) follows from the Jacobi superidentity applied to three odd elements.

The converse is a straightforward computation. □

With $V$, $V \times V \to \mathrm{End}_F(V)$, $(x, y) \mapsto d_{x,y}$, and $\langle\,|\,\rangle$ as before, consider the triple product in $V$ given by

$$xyz = d_{x,y}z + \langle x \mid y\rangle z \tag{2.4}$$

for any $x, y, z \in V$. Conditions (2.3a–c) translate into

$$xxy = \langle x \mid x\rangle y = xyx, \tag{2.5a}$$

$$uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz), \tag{2.5b}$$

$$\langle uvx \mid y\rangle = \langle x \mid vuy\rangle, \tag{2.5c}$$

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

for any $u, v, x, y, z \in V$. Let us check (2.5b) for instance. For this, denote by $l_{x,y}$ the map $z \mapsto xyz$ for any $x, y, z \in V$, then for any $u, v, x, y \in V$

$$\begin{aligned}[l_{u,v}, l_{x,y}] &= [d_{u,v}, d_{x,y}] \qquad \text{(since } l_{u,v} - d_{u,v} \text{ is scalar)}\\ &= d_{d_{u,v}(x),y} + d_{x,d_{u,v}(y)}\\ &= l_{d_{u,v}(x),y} - \langle d_{u,v}(x) \mid y\rangle + l_{x,d_{u,v}(y)} - \langle x \mid d_{u,v}(y)\rangle\\ &= l_{d_{u,v}(x),y} - l_{x,d_{v,u}(y)}\\ &= l_{uvx,y} - \langle u \mid v\rangle l_{x,y} - l_{x,vuy} + \langle v \mid u\rangle l_{x,y}\\ &= l_{uvx,y} - l_{x,vuy}\end{aligned}$$

and this is equivalent to (2.5b). Conversely, conditions (2.5a–c) give conditions (2.3a–c), if (2.4) is used to define $d_{x,y}$ for $x, y \in V$.

Conditions (2.5a) and (2.5b) are just the defining conditions (1.3) and (1.1) of a $(-1, -1)$-BFKTS, while condition (2.5c) is a consequence of (2.5a–b) [**13**]. We include a proof of this fact by completeness:

Take $x = y$ in (2.5b) and use (2.5a) to get

$$\begin{aligned}\langle x \mid x\rangle uvz &= (uvx)xz - x(vux)z + \langle x \mid x\rangle uvz\\ &= (uvx)xz + ((vux)xz - 2\langle x \mid vux\rangle z) + \langle x \mid x\rangle uvz\\ &= 2\langle u \mid v\rangle xxz - 2\langle x \mid vux\rangle z + \langle x \mid x\rangle uvz\end{aligned}$$

and this shows that $\langle x \mid vux\rangle = \langle u \mid v\rangle\langle x \mid x\rangle$ for any $x, u, v \in V$. Linearizing this one obtains that $\langle x \mid vuy\rangle + \langle y \mid vux\rangle = 2\langle u \mid v\rangle\langle x \mid y\rangle$ for any $x, y, u, v \in V$, whence

$$\langle x \mid vuy\rangle = \langle 2\langle u \mid v\rangle x - vux \mid y\rangle = \langle uvx \mid y\rangle,$$

as desired. In the same way, (2.3a) follows from (2.3b) and (2.3c).

Because of (2.3a–b), $\mathfrak{d} = d_{V,V}$ is a Lie algebra of derivations of the $(-1, -1)$-BFKTS, which will be said to be the Lie algebra of inner derivations of $V$.

Given a vector space $V$ endowed with a nonzero symmetric bilinear form $\langle \mid \rangle$ and a skew symmetric map $V \times V \to \mathrm{End}_F(V)$, $(x, y) \mapsto d_{x,y}$ for any $x, y \in V$, satisfying conditions (2.3), denote by $\mathfrak{g}(V)$ the Lie superalgebra constructed in Theorem 2.1. Also, consider the triple product $xyz$ defined on $V$ by (2.4) and the triple product given by $\{xyz\} = d_{x,y}(z)$ for any $x, y, z \in V$.

THEOREM 2.2. *Under the hypotheses above, the following conditions are equivalent:*
(i) $\langle \mid \rangle$ *is nondegenerate,*
(ii) $(V, \{xyz\})$ *is a simple triple system,*
(iii) $(V, xyz)$ *is a simple triple system,*
(iv) $\mathfrak{g}(V)$ *is a simple Lie superalgebra.*

*Proof.* Assume that (i) is satisfied and let $I$ be a nonzero ideal of the triple system $(V, \{xyz\})$. Then for any $x \in I$ and $y \in V$, $\{xyy\} = d_{x,y}(y) = -\langle y \mid y\rangle x + \langle x \mid y\rangle y \in I$, by (2.3c), and hence $\langle x \mid y\rangle y \in I$ for any $y \in V$. Since $\langle \mid \rangle$ is nondegenerate, there is a basis of $V$ formed by elements $y$ with $\langle x \mid y\rangle \neq 0$ and this shows that $I = V$. Conversely, $V^\perp = \{x \in V : \langle x \mid V\rangle = 0\}$ is an ideal of $(V, \{xyz\})$ because of (2.3a) and the linearization of (2.3c). Hence (ii) implies (i).

Similarly, condition (i) and the linearization of (2.5a) imply (iii), and conversely (iii) implies (i) since $V^\perp$ is an ideal of $(V, xyz)$ because of (2.5a) and (2.5c).

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

Now assume that (i) is satisfied and that $0 \neq I = I_{\bar{0}} \oplus I_{\bar{1}}$ is an ideal of the Lie superalgebra $\mathfrak{g}(V)$. By $\mathfrak{sl}_2(F)$-invariance, $I_{\bar{1}} = U \otimes_F W$ for a subspace $W$ of $V$. Let $x \in V$ and $y \in W$ with $\langle x \mid y \rangle \neq 0$, then for any $a \in U$, $[a \otimes x, a \otimes y] = -\langle x \mid y \rangle \varphi_{a,a}$, so $\varphi_{a,a} \in I_{\bar{0}}$ for any $a$ and $\mathfrak{sl}_2(F) \subseteq I_{\bar{0}}$. But then $\mathfrak{g}_{\bar{1}} = [\mathfrak{sl}_2(F), \mathfrak{g}_{\bar{1}}] \subseteq I$ and $\mathfrak{g}_{\bar{0}} = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq I$, so $I = \mathfrak{g}$. Otherwise $W = 0$, so $I_{\bar{1}} = 0$, but then it is easy to show that $I = 0$.

Conversely, the graded subspace $d_{V,V^\perp} \oplus (U \otimes_F V^\perp)$ is an ideal of $\mathfrak{g}(V)$, so (iv) implies (i). □

Since the nondegeneracy of a bilinear form is preserved under scalar extensions, it immediately follows that:

COROLLARY 2.3. *With the same notation as above, if* $\langle\,|\,\rangle$ *is nondegenerate, then* $(V, \{xyz\})$, $(V, xyz)$ *and* $\mathfrak{g}(V)$ *are central simple.*

**3. Examples.** This section is devoted to constructing the examples of simple $(-1, -1)$ balanced Freudenthal Kantor triple systems that will appear in the classification. Throughout this section, the ground field $F$ will be assumed of characteristic $\neq 2$.

**3.1. Hermitian type.** Let $R$ be a unital separable associative algebra over $F$ of degree $\leq 2$. Therefore, $R$ is, up to isomorphism, either the ground field $F$, $F \times F$, a quadratic separable field extension $K$ of $F$ or a quaternion algebra $Q$ over $F$. In any case, $R$ is endowed with an involution of the first kind, $x \mapsto \bar{x}$, such that $x + \bar{x}$, $x\bar{x} = \bar{x}x \in F$ for any $x \in R$. Let $V$ be a left module over $R$ endowed with a nondegenerate hermitian form $h : V \times V \to R$. That is, $h$ is $F$-bilinear and satisfies for any $x, y \in V$ and $r \in R$:

$$\begin{aligned} h(rx, y) &= rh(x, y), \\ h(x, y) &= \overline{h(y, x)}, \\ h(x, V) &= 0 \text{ if and only if } x = 0. \end{aligned} \tag{3.1}$$

Then the symmetric bilinear form $V \times V \to F$ defined by means of

$$\langle x \mid y \rangle = \frac{1}{2}(h(x, y) + h(y, x)), \tag{3.2}$$

for any $x, y \in V$, is nondegenerate and determines $h$.

Define now the triple product on $V$ by means of

$$xyz = h(z, x)y - h(z, y)x + h(x, y)z, \tag{3.3}$$

for any $x, y, z \in V$.

It is clear that $xxy = h(x, x)y = \langle x \mid x \rangle y = xyx$ for any $x, y \in V$ and a straightforward computation shows that this triple product satisfies (2.5b) too. Therefore $V$ is a $(-1, -1)$-BFKTS which will be said to be of *hermitian type*. Depending on $\dim_F R$ being either 1, 2 or 4, $V$ will be said to be of *orthogonal*, *unitarian* or *symplectic* type, respectively, for reasons that will become clear later on.

Let us compute the Lie algebra $\mathfrak{d} = d_{V,V}$ in this case. Assume first that $R = F$, the ground field, then $d_{x,y} = \langle - \mid x \rangle y - \langle - \mid y \rangle x =: \sigma_{x,y}$ for any $x, y \in V$, and these maps span the orthogonal Lie algebra $\mathfrak{o}(V)$. From the construction in [**11**, Supplement to 2.1.2], $\mathfrak{g}(V)$ is the orthosymplectic Lie superalgebra $\mathfrak{osp}(V \oplus U)$. A word of caution

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

is needed here: the multiplication of odd elements in [**11**, Supplement to 2.1.2] should read $[a \otimes c, b \otimes d] = -(a, b)_0 c \circ d + (c, d)_1 a \wedge b$ (a minus sign has been added).

Now, in case $R$ is a quadratic étale algebra, that is, either $K = F \times F$ or $K$ is a quadratic field extension of $F$, then for any $x, y \in V$,

$$d_{x,y} = h_{x,y} + h_0(x, y)1_V, \tag{3.4}$$

where

$$h_{x,y} = h(-, x)y - h(-, y)x \tag{3.5}$$

and

$$h_0(x, y) = h(x, y) - \langle x \mid y\rangle = \frac{1}{2}(h(x, y) - h(y, x)). \tag{3.6}$$

Note that

$$h_{x,y} \in \mathfrak{u}(V, h) = \{f \in \mathrm{End}_K(V) : h(f(x), y) + h(x, f(y)) = 0 \text{ for any } x, y \in V\}.$$

Since $\overline{h_0(x, y)} = -h_0(x, y)$, it follows that $\mathfrak{d} \subseteq \mathfrak{u}(V, h)$.

In the split case: $K = F \times F = Fe_1 \oplus Fe_2$, for orthogonal idempotents $e_1$ and $e_2$ ($\bar{e}_1 = e_2$), let $W = e_1 V$ and $\tilde{W} = e_2 V$. Then $h(W, W) = 0 = h(\tilde{W}, \tilde{W})$ and for any $a \in W$ and $u \in \tilde{W}$, $h(a, u) \in Fe_1$. Hence there is a bilinear nondegenerate form $( \mid ) : W \times \tilde{W} \to F$, such that $h(a, u) = (a \mid u)e_1$ for any $a \in W$ and $u \in \tilde{W}$. This bilinear form determines $h$ and allows us to identify $\tilde{W}$ with the dual $W^*$. Therefore we may assume that $V = W \times W^*$, with the natural structure of module over $K = F \times F$, and with $h((a, \alpha), (b, \beta)) = (\beta(a), \alpha(b))$ for any $a, b \in W$ and $\alpha, \beta \in W^*$. Moreover, in this case $\mathfrak{u}(V, h)$ is isomorphic to $\mathfrak{gl}(W)$ by means of the isomorphism that takes any $f \in \mathrm{End}_F(W) = \mathfrak{gl}(W)$ to the endomorphism of $V = W \times W^*$ given by $(a, \alpha) \mapsto (f(a), -\alpha \circ f)$. Through this isomorphism, $h_{(a,0),(0,\alpha)}$ corresponds to the endomorphism of $W$ given by $c \mapsto -\alpha(c)a$, and hence $d_{(a,0),(0,\alpha)}$ corresponds to $c \mapsto -\alpha(c)a + \frac{1}{2}\alpha(a)c$. If the dimension of $W$ is not 2, this shows that $\mathfrak{d} = \mathfrak{u}(V, h) \cong \mathfrak{gl}(W)$, while if the dimension is 2, $\mathfrak{d} = \mathfrak{su}(V, h) \cong \mathfrak{sl}(W)$.

By scalar extension, we have that $\mathfrak{d} = \mathfrak{u}(V, h)$ if $\dim_K V \neq 2$ ($\dim_F V \neq 4$) and $\mathfrak{d} = \mathfrak{su}(V, h)$ if $\dim_K V = 2$.

Finally, assume that $R$ is a quaternion algebra $Q$. Again $d_{x,y} = h_{x,y} + h_0(x, y)1_V$, but now $h_{x,y}$ is $Q$-linear, while $h_0(x, y)1_V$ is not in general, since the center of $Q$ is $F$. It is easily checked here that $[h_{x,y}, h_{u,v}] = h_{h_{x,y}(u),v} + h_{u,h_{x,y}(v)}$ for any $x, y, u, v \in V$, and thus $h_{V,V} = \mathrm{span}\left\{h_{x,y} : x, y \in V\right\}$ is a Lie algebra contained in

$$\mathfrak{sp}(V, h) = \{f \in \mathrm{End}_Q(V) : h(f(x), y) + h(x, f(y)) = 0 \text{ for any } x, y \in V\},$$

and $\mathfrak{d} = d_{V,V}$ is contained in $\mathfrak{sp}(V, h) \oplus Q_0 1_V$, where $Q_0 = [Q, Q]$ is the set of skew symmetric elements in $Q$ relative to its involution, which form a three dimensional simple Lie algebra.

Again, consider the split case: $Q = \mathrm{End}_F(U)$ for a two dimensional vector space $U$ endowed with a nonzero skew symmetric bilinear map $\varphi$ which induces the involution in $Q$. Standard arguments of complete reducibility as a module over $Q$ show that $V = U \otimes_F W$ for some vector space $W$ over $F$. For any $q \in Q_0 = \mathfrak{sl}(U) = \mathfrak{sp}(U, \varphi)$

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

and for any $x, y \in V$,

$$\langle qx \mid y\rangle = \frac{1}{2}(h(qx, y) + h(y, qx)) = \frac{1}{2}(qh(x, y) + \overline{qh(x, y)})$$
$$= \frac{1}{2}(h(x, y)q + \overline{h(x, y)q}) = -\langle x \mid qy\rangle,$$

so $Q_0$ embeds into the orthogonal Lie algebra $\mathfrak{o}(V, \langle\,|\,\rangle)$ and, therefore, $\langle\,|\,\rangle$ is invariant under the action of $\mathfrak{sl}(U) = \mathfrak{sp}(U, \varphi)$. But, up to scalars, $\varphi$ is the unique bilinear form on $U$ which is $\mathfrak{sp}(U, \varphi)$-invariant, so $\langle a \otimes u \mid b \otimes v\rangle = \frac{1}{2}\varphi(a, b)\psi(u, v)$, for any $a, b \in U$, $u, v \in W$, for a skew-symmetric nondegenerate bilinear form $\psi : W \times W \to F$.

Since the hermitian form $h$ is completely determined by $\langle\,|\,\rangle$, it turns out that $h : V \times V \to Q = \mathrm{End}_F(U)$ is given by $h(a \otimes u, b \otimes v) = \psi(u, v)\varphi(-, b)a$, for any $a, b \in U$ and $u, v \in W$. Note that $h$ thus defined is hermitian and

$$\frac{1}{2}(h(a \otimes u, b \otimes v) + h(b \otimes v, a \otimes u)) = \frac{1}{2}\psi(u, v)(\varphi(-, b)a - \varphi(-, a)b).$$

But $\varphi(a, b)c + \varphi(b, c)a + \varphi(c, a)b = 0$ for any $a, b, c \in U$, so

$$\frac{1}{2}(h(a \otimes u, b \otimes v) + h(b \otimes v, a \otimes u)) = \frac{1}{2}\psi(u, v)\varphi(a, b)1_V.$$

Hence, for any $a, b \in U$ and $u, v \in W$:

$$h_0(a \otimes u, b \otimes v) = \frac{1}{2}(h(a \otimes u, b \otimes v) - h(b \otimes v, a \otimes u)) \quad \text{(see (3.6))}$$
$$= \frac{1}{2}\psi(u, v)(\varphi(-, b)a + \varphi(-, a)b) = \frac{1}{2}\psi(u, v)\varphi_{a,b},$$

and thus, for any $a, b, c \in U$ and $u, v, w \in W$:

$$h_{a\otimes u, b\otimes v}(c \otimes w) = \psi(w, u)\varphi(b, a)c \otimes v - \psi(w, v)\varphi(a, b)c \otimes u$$
$$= -\varphi(a, b)c \otimes (\psi(w, u)v + \psi(w, v)u) = -\varphi(a, b)c \otimes \psi_{u,v}(w).$$

Therefore, $h_{V,V} = \mathfrak{sp}(V, h) := \{f \in \mathrm{End}_Q(V) : h(f(x), y) + h(x, f(y)) = 0$ for any $x, y \in V\} \cong \mathfrak{sp}(W, \psi)$ (which acts on $V = U \otimes_F W$ in a natural way: on the second factor). Moreover, from (3.4),

$$d_{a\otimes u, b\otimes v} = h_{a\otimes u, b\otimes v} + h_0(a \otimes u, b \otimes v)1_V = \frac{1}{2}\varphi_{a,b} \otimes \psi(u, v)1_W - \varphi(a, b)1_U \otimes \psi_{u,v},$$

so $\mathfrak{d} = d_{V,V} = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{sp}(W, \psi) = \mathfrak{sl}(U) \oplus \mathfrak{sp}(W, \psi)$.

For general $Q$, again extending scalars we arrive at $h_{V,V} = \mathfrak{sp}(V, h)$ (which is a simple Lie algebra of type $C$) and $\mathfrak{d}$ is the direct sum of the three dimensional simple Lie algebra $Q_0$ and of the simple Lie algebra $\mathfrak{sp}(V, h)$.

Summarizing the above discussion:

PROPOSITION 3.1. *Let $R$ be a unital separable associative algebra of degree $\leq 2$ over a field $F$ of characteristic $\neq 2$, and let $V$ be a left module over $R$ endowed with a nondegenerate hermitian form $h : V \times V \to R$. Endow $V$ with the structure of a simple $(-1, -1)$-BFKTS of hermitian type (with associated symmetric bilinear form given by*

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

*$\langle x \mid y\rangle = \frac{1}{2}(h(x, y) + h(y, x))$ for any $x, y \in V$) and let $\mathfrak{d} = d_{V,V}$ be the associated Lie algebra of inner derivations. Then:*

(i) *If $R = F$, then $\mathfrak{d} = \mathfrak{o}(V, \langle\,|\,\rangle)$.*

(ii) *If $R = K$ is a quadratic étale algebra, then $\mathfrak{d} = \mathfrak{u}(V, h)$ unless* $\dim_F(V) = 4$. *In this latter case, $\mathfrak{d} = \mathfrak{su}(V, h)$.*

(iii) *If $R$ is a quaternion algebra $Q$, then $\mathfrak{d} \cong Q_0 \oplus \mathfrak{sp}(V, h)$, where $\mathfrak{sp}(V, h)$ acts naturally on $V$, and the simple three dimensional Lie algebra $Q_0$ acts by left multiplication on the $Q$ module $V$.*

**3.2. $\mathbf{D}_\mu$-type.** Let $V$ be a four dimensional vector space, endowed with a nondegenerate symmetric bilinear form $\langle\,|\,\rangle$. Let $\Phi$ be a nonzero skew symmetric multilinear form: $\Phi : V \times V \times V \times V \to F$. Define a skew symmetric triple product $[xyz]$ on $V$ by means of:

$$\Phi(x, y, z, t) = \langle [xyz] \mid t\rangle, \tag{3.7}$$

for any $x, y, z, t \in V$. The proof of the next result is left to the reader.

LEMMA 3.2. *With the hypotheses above, there exists a nonzero scalar $\mu \in F$ such that*

$$\langle [a_1a_2a_3] \mid [b_1b_2b_3]\rangle = \mu \det(\langle a_i \mid b_j\rangle), \tag{3.8}$$

*for any $a_i, b_i \in V$ ($i = 1, 2, 3$).*

Now, for any such $V$ and $\Phi$, and for any $x, y \in V$, consider the endomorphism $d_{x,y} \in \mathrm{End}_F(V)$ defined by

$$d_{x,y}z = [xyz] + \langle z \mid x\rangle y - \langle z \mid y\rangle x. \tag{3.9}$$

As shown in [**22**, §5], conditions (2.3a–b) are satisfied, so if the triple product $xyz$ on $V$ is defined by means of

$$xyz = [xyz] + \langle z \mid x\rangle y - \langle z \mid y\rangle x + \langle x \mid y\rangle z. \tag{3.10}$$

for any $x, y, z \in V$, then $V$ becomes a $(-1, -1)$-BFKTS, which will be said to be of $\mathrm{D}_\mu$-type.

Assume for a while that the scalar $\mu$ in (3.8) is a square, $\mu = \nu^2$, $0 \neq \nu \in F$, and that $\langle\,|\,\rangle$ represents 1. Then, by [**4**, Theorem 2], $V$ is endowed with a binary multiplication that makes it a quaternion algebra $Q$ over $F$, with involution $x \mapsto \bar{x}$ such that $x\bar{x} = \langle x \mid x\rangle$ for any $x \in V$, satisfying

$$\nu^{-1}[xyz] = x\bar{y}z - \langle x \mid y\rangle z + \langle z \mid x\rangle y - \langle z \mid y\rangle x$$

for any $x, y, z \in V$. Therefore, for any $x, y, z \in V$, (3.9) shows that:

$$\begin{aligned} d_{x,y}(z) &= \nu x\bar{y}z + (1+\nu)(\langle z \mid x\rangle y - \langle z \mid y\rangle x) - \nu\langle x \mid y\rangle z \\ &= \nu x\bar{y}z + \frac{1+\nu}{2}((x\bar{z} + z\bar{x})y - x(\bar{y}z + \bar{z}y)) - \frac{\nu}{2}(x\bar{y} + y\bar{x})z \\ &= \left(\nu x\bar{y} - \frac{1+\nu}{2}x\bar{y} - \frac{\nu}{2}(x\bar{y} + y\bar{x})\right) z + \frac{1+\nu}{2}z\bar{x}y \\ &= \left(-\frac{1}{2}x\bar{y} - \frac{\nu}{2}y\bar{x}\right) z + \frac{1+\nu}{2}z\bar{x}y = \frac{\nu - 1}{4}(x\bar{y} - y\bar{x})z + \frac{1+\nu}{4}z(\bar{x}y - \bar{y}x), \end{aligned}$$

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

because $\bar{x}y + \bar{y}x = x\bar{y} + y\bar{x} = 2\langle x \mid y\rangle \in F$, so $\bar{x}y - \bar{y}x = 2\bar{x}y - (x\bar{y} + y\bar{x})$. Hence, for any $x, y \in V$, $d_{x,y} = L_p - R_q$, with $p = \frac{\nu-1}{4}(x\bar{y} - y\bar{x})$, $q = -\frac{\nu+1}{4}(\bar{x}y - \bar{y}x) \in Q_0$, where $L$ and $R$ denote left and right multiplications in $V = Q$. Therefore, if $\mu = 1$ ($\nu = \pm 1$), $\mathfrak{d} = d_{V,V}$ is isomorphic to the three dimensional simple Lie algebra $Q_0$. However, if $\mu \neq 0, 1$ ($\nu \neq 0, \pm 1$), then $\mathfrak{d} = L_{Q_0} \oplus R_{Q_0}$, a direct sum of two copies of the three dimensional Lie algebra $Q_0$, which coincides with the orthogonal Lie algebra $\mathfrak{o}(V, \langle\,|\,\rangle)$. Moreover, in this latter case, [**2**, Lemma 3.1 and its proof], $\mathfrak{g}(V)$ is a form of the simple Lie superalgebra $\Gamma(-\frac{1}{2}, \frac{1-\nu}{4}, \frac{1+\nu}{4})$ (notation as in [**26**, pp. 16–17]). That is, it is a form of $D(2, 1; \frac{\nu-1}{2})$ (see also [**19**]).

Simply by extending scalars, we obtain:

PROPOSITION 3.3. *Let $V$ be a four dimensional vector space over a field $F$ of characteristic $\neq 2$ with a nondegenerate symmetric bilinear form $\langle\,|\,\rangle$. Let $\Phi$ be a nonzero skew symmetric 4-linear form and let the triple product $[xyz]$ be defined by means of $\langle [xyz] \mid t\rangle = \Phi(x, y, z, t)$ for any $x, y, z, t \in V$. Let $0 \neq \mu \in F$ be given by (3.8). Endow $V$ with the structure of a simple $(-1, -1)$-BFKTS by means of (3.10) and let $\mathfrak{d} = d_{V,V}$ be the corresponding Lie algebra of inner derivations. Then:*

(i) *If $\mu = 1$, then $\mathfrak{d}$ is a three dimensional simple ideal of the orthogonal Lie algebra $\mathfrak{o}(V, \langle\,|\,\rangle)$.*

(ii) *If $\mu \neq 0, 1$, then $\mathfrak{d}$ coincides with the orthogonal Lie algebra $\mathfrak{o}(V, \langle\,|\,\rangle)$.*

There is some overlapping in the types considered up to now.

To begin with, let $V$ be any four dimensional simple $(-1, -1)$-BFKTS and let $[xyz]$ be defined by $[xyz] = xyz - \langle z \mid x\rangle y + \langle z \mid y\rangle x - \langle x \mid y\rangle z$, for any $x, y, z \in V$. Because of (2.5a), $[xyz]$ is skew symmetric on its arguments. In case $[xyz]$ is identically zero, we are in presence of a system of orthogonal type. Otherwise, this is a system of D-type. This means that the systems of hermitian type with $R = K$ or $Q$ and with $\dim_F V = 4$ are systems of D-type. Let us check which $\mu$'s are involved in these cases. To do so, it is enough to consider the split cases.

Assume $K = F \times F$ and $V = W \times W^*$ with $h((a, \alpha), (b, \beta)) = (\beta(a), \alpha(b))$ for any $a, b \in W$ and $\alpha, \beta \in W$ and with $\dim_F W = 2$. Take $a, b \in W$ and $\alpha, \beta \in W^*$ with $\alpha(a) = 1 = \beta(b)$, $\alpha(b) = 0 = \beta(a)$. Then with $(a_1, \alpha_1) = (a, 0)$, $(a_2, \alpha_2) = (0, \alpha)$ and $(a_3, \alpha_3) = (b, \beta)$,

$$\det(\langle (a_i, \alpha_i) \mid (a_j, \alpha_j)\rangle) = \begin{vmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{1}{4},$$

while $[(a, 0)(0, \alpha)(b, \beta)] = \frac{1}{2}(b, \beta)$ and

$$\langle [(a, 0)(0, \alpha)(b, \beta)] \mid [(a, 0)(0, \alpha)(b, \beta)]\rangle = \frac{1}{4}\langle (b, -\beta) \mid (b, -\beta)\rangle = -\frac{1}{4}.$$

Hence, $\mu = 1$ in this case. (This can also be deduced directly from the size of the Lie algebras $\mathfrak{d}$.)

Assume now that $R = Q$ is a quaternion algebra and $\dim_F V = 4$, then $V$ is a free $Q$-module of rank 1 and hence we may assume that $V = Q$ and that $h(x, y) = \alpha x\bar{y}$ for any $x, y \in Q$, where $0 \neq \alpha = h(1, 1) \in F$. Then for any $x_1, x_2, x_3 \in Q$,

$$[x_1x_2x_3] = h_0(x_3, x_1)x_2 - h_0(x_3, x_2)x_1 + h_0(x_1, x_2)x_3$$

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

where $h_0(x, y) = \frac{1}{2}(h(x, y) - h(y, x)) = \alpha(x\bar{y} - y\bar{x}) \in Q_0$. By skew symmetry of $h_0$,

$$[x_1x_2x_3] = \frac{1}{2}\sum_{\sigma} h_0\big(x_{\sigma(1)}, x_{\sigma(2)}\big)x_{\sigma(3)} = \frac{\alpha}{2}(-1)^{\sigma} x_{\sigma(1)}\bar{x}_{\sigma(2)}x_{\sigma(3)}$$

where the sum is over all the permutations of $1, 2, 3$. Take $x_1 = 1$, $x_2$, and $x_3$ mutually orthogonal to get $\langle 1 \mid 1\rangle = h(1, 1) = \alpha$, $\det(\langle x_i \mid x_j\rangle) = \alpha\langle x_2 \mid x_2\rangle\langle x_3 \mid x_3\rangle$, while $[x_1x_2x_3] = -3\alpha x_2x_3$ since $x_2x_3 = -x_3x_2$, $\bar{x}_i = -x_i$ for $i = 2, 3$, and $\bar{1} = 1$. Thus, $\langle [x_1x_2x_3] \mid [x_1x_2x_3]\rangle = 9\alpha^3(x_2x_3)\overline{(x_2x_3)} = 9\alpha\langle x_2 \mid x_2\rangle\langle x_3 \mid x_3\rangle$, and $\mu = 9$ in this case.

A final overlap occurs if $V$ is of hermitian type with $R = K$ quadratic and with $\dim_F V = 2$. As above, $[x_1x_2x_3] = \frac{1}{2}\sum_{\sigma} h_0(x_{\sigma(1)}, x_{\sigma(2)})x_{\sigma(3)}$ for any $x_1, x_2, x_3 \in V$. By skew symmetry and dimension, this is zero, and therefore we are in the situation of $R = F$. We summarize the above arguments in the following remark, whose last part follows from the structure of the Lie algebras of inner derivations.

Remark 3.4.
– The simple $(-1, -1)$-BFKTS $V$ of unitarian type and $\dim_F V = 2$ are also of orthogonal type.
– The simple $(-1, -1)$-BFKTS $V$ of unitarian type and $\dim_F V = 4$ are of $\mathrm{D}_1$-type.
– The simple $(-1, -1)$-BFKTS $V$ of symplectic type and $\dim_F V = 4$ are of $\mathrm{D}_9$-type.
– There are no more overlaps among different types.

**3.3. G-type.** Let $C$ be an eight-dimensional Cayley-Dickson (or octonion) algebra over $F$ with norm $n$ and trace $t$. Let $C_0$ be the set of trace zero elements. For any $x, y \in C$, the linear map

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \tag{3.11}$$

(where $L_x$ and $R_x$ denote the left and right multiplication by $x$) is known to be a derivation of $C$ [**25**, Ch. III.8], and hence it leaves $C_0$ invariant. Consider then, for any $0 \neq \alpha \in F$, the nondegenerate symmetric bilinear form and the triple product on $V = C_0$ given by $\langle x \mid y\rangle = -2\alpha t(xy)$ and $xyz = \alpha(D_{x,y}(z) - 2t(xy)z)$, for any $x, y, z \in V$. Since $D_{x,y}$ is a derivation and

$$\begin{aligned} D_{x,y}(y) &= xy^2 - y(xy) + xy^2 - (xy)y + y^2x - (yx)y = 4y^2x - 2(xy + yx)y \\ &= -4n(y)x - 2t(xy)y = -\langle y \mid y\rangle x + \langle x \mid y\rangle y, \end{aligned}$$

where we have used that $x^2 = -n(x)1 = -\frac{1}{2}t(x^2)1$ for any $x \in V = C_0$; it follows from (2.3) that $V$ is a $(-1, -1)$-BFKTS (see also [**19**]), which will be said to be of G-type. It is clear here that the Lie algebra $\mathfrak{d}$ is the span of the $D_{x,y}$'s, which is precisely the Lie algebra of derivations of the Cayley-Dickson algebra $C$ in case the characteristic is $\neq 3$ [**25**, Ch. III.8], a simple Lie algebra of type $G_2$. (If the characteristic is 3, then this is a seven dimensional simple Lie algebra which is a form of $\mathfrak{psl}(7)$ [**1**].)

**3.4. F-type.** Let $X$ be a 3-fold vector cross product on a vector space $V$ of dimension 8, endowed with a nondegenerate symmetric bilinear form $\langle\,|\,\rangle$ . That is, $X$ is a trilinear map $X : V \times V \times V \to V$, $(a, b, c) \mapsto X(a, b, c)$, satisfying (see [**4**], [**23**,

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

Ch. 8] and the references therein):

$$\begin{gathered}\langle X(a_1,a_2,a_3)\mid a_i\rangle = 0 \text{ for any } i=1,2,3,\\ \langle X(a_1,a_2,a_3)\mid X(a_1,a_2,a_3)\rangle = \det(\langle a_i\mid a_j\rangle),\end{gathered}\tag{3.12}$$

for any $a_1, a_2, a_3 \in V$.

It is known that (3.12) implies the skew symmetry of $X$. Moreover, $X$ satisfies:

$$\begin{aligned}&\langle X(a_1,a_2,a_3)\mid X(b_1,b_2,b_3)\rangle\\ &\quad= \det(\langle a_i\mid b_j\rangle) + \epsilon\sum_{\sigma\text{ even}}\sum_{\tau\text{ even}}\big\langle a_{\sigma(1)}\,\big|\, b_{\tau(1)}\big\rangle\Phi\big(a_{\sigma(2)},a_{\sigma(3)},b_{\tau(2)},b_{\tau(3)}\big)\end{aligned}\tag{3.13}$$

for any $a_i, b_i \in V$ $(i = 1, 2, 3)$, where $\Phi(a, b, c, d) = \langle a \mid X(b, c, d)\rangle$ for any $a, b, c, d \in V$, and $\epsilon = \pm 1$. In case $\epsilon = 1$ (resp. $-1$), $X$ is said to be of type I (resp. II). Also, if $\dim_F V = 8$ and $X$ is of type I, then $-X$ is of type II, and conversely.

Assume now that the characteristic of the ground field $F$ is $\neq 2, 3$. Given a 3-fold vector cross product $X$ of type I, define $d_{x,y} \in \mathrm{End}_F(V)$, $x, y \in V$, by means of:

$$d_{x,y}z = \frac{1}{3}X(x,y,z) + \langle z\mid x\rangle y - \langle z\mid y\rangle x\,.\tag{3.14}$$

As shown in [**22**, §5], condition (2.3b) is satisfied, so if the triple product $xyz$ on $V$ is defined by means of

$$xyz = \frac{1}{3}X(x,y,z) + \langle z\mid x\rangle y - \langle z\mid y\rangle x + \langle x\mid y\rangle z.\tag{3.15}$$

for any $x, y, z \in V$, then $V$ becomes a $(-1,-1)$-BFKTS, which will be said to be of F-type.

Since $d_{x,y}$ is a derivation of the triple system and it is skew symmetric relative to $\langle\,|\,\rangle$, it follows that $d_{x,y}$ is a derivation of the 3-fold vector cross product $X$. According to [**4**, Theorem 12], if $e$ is an element of $V$ with $\langle e\mid e\rangle \neq 0$, $W = \{v \in V : \langle e \mid x\rangle = 0\}$, and $q$ is the nondegenerate quadratic form on $V$ defined by $q(v) = -\langle e\mid e\rangle^{-1}\langle v\mid v\rangle$, then the Lie algebra of derivations of $X$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(W, q)$. Actually, $V$ has the structure of an eight dimensional Cayley-Dickson algebra $C$ with unit $1 = e$, so that there is an scalar $0 \neq \alpha \in F$ such that $X(a, b, c) = \alpha((a\bar{b})c + (a\mid c)b - (b\mid c)a - (a\mid b)c)$ and $\langle a\mid b\rangle = \alpha(a\mid b)$, for any $a, b, c \in V = C$. Here $x \mapsto \bar{x}$ denotes the involution and $(a\mid a) = a\bar{a}$ is the norm of $C$. Note that $\alpha = \langle e\mid e\rangle$. Hence for any $x, y \in V$, $d_{x,y}$ is a derivation of the $3C$-product given by $(a\bar{b})c$ (see [**4**]). But for any $x, y, z \in V = C$, $\frac{3}{\alpha}d_{x,y}(z) = (x\bar{y})z + 4(z\mid x)y - 4(z\mid x)x - (x\mid y)z$, in particular, for a traceless $x$ $(\bar{x} = -x)$, $\frac{3}{\alpha}d_{e,x}(y) = -xy + 2x(y+\bar{y}) - 2(x\bar{y} - yx) = xy + 2yx = (L+2R)_x(y)$, that is, $d_{e,x} = (L+2R)_x$, where $L$ and $R$ denote the left and right multiplications in $C$. But these operators generate the Lie algebra of derivations of the triple product given by $(a\bar{b})c$ [**7**, **4**] (see also [**5**]), so we conclude that $\mathfrak{d}$ is isomorphic to $\mathfrak{o}(W, q)$.

Note that in [**19**] it is already proved that, after scalar extension, $\mathfrak{d}$ is isomorphic to $\mathfrak{o}(7)$, by an explicit computation.

**4. Classification.** Given a $(-1,-1)$-BFKTS $V$ over a field of characteristic $\neq 2$, in Section 2 a simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}(V)$ has been defined that contains a copy $\mathfrak{s} = \mathfrak{s}(V)$ of $\mathfrak{sl}_2(F)$, which is an ideal of $\mathfrak{g}_{\bar{0}}$ that is complemented by the ideal $\mathfrak{d} = \mathfrak{d}(V) = d_{V,V}$. In this situation $\mathfrak{d} = \{d \in \mathfrak{g}_{\bar{0}} : [d, \mathfrak{s}] = 0\}$ is completely determined by $\mathfrak{g}$ and $\mathfrak{s}$. Moreover, as a module for $\mathfrak{g}_{\bar{0}}$, $\mathfrak{g}_{\bar{1}}$ is the tensor product of the two dimensional irreducible module for $\mathfrak{s}$ and the module $V$ for $\mathfrak{d}$.

Consider a ground field $F$ of characteristic $\neq 2$ and the pairs $(\mathfrak{g}, \mathfrak{s})$, where $\mathfrak{g}$ is a Lie superalgebra over $F$ and $\mathfrak{s}$ is a complemented ideal of $\mathfrak{g}_{\bar{0}}$ isomorphic to $\mathfrak{sl}_2(F)$. Two such pairs $(\mathfrak{g}^1, \mathfrak{s}^1)$, $(\mathfrak{g}^2, \mathfrak{s}^2)$ are said to be isomorphic if there is an isomorphism of Lie superalgebras $\phi : \mathfrak{g}^1 \to \mathfrak{g}^2$ such that $\phi(\mathfrak{s}^1) = \mathfrak{s}^2$.

Given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and a nonzero scalar $\alpha$, the new Lie superalgebra defined over $\mathfrak{g}$ with the new product $[\ ,\ ]_\alpha$ given, for homogeneous elements, by

$$\begin{cases} [x,y]_\alpha = \alpha[x,y] & \text{if both } x \text{ and } y \text{ are odd} \\ [x,y]_\alpha = [x,y] & \text{otherwise} \end{cases}$$

will be denoted by $\mathfrak{g}_\alpha$. Also, given a $(-1,-1)$-BFKTS $V$, we will denote by $V_\alpha$ the new $(-1,-1)$-BFKTS defined on $V$ but with the new product given by $(xyz)_\alpha = \alpha xyz$, and new symmetric bilinear form given by $\langle x \mid y\rangle_\alpha = \alpha\langle x \mid y\rangle$, for any $x, y, z \in V$. From the definitions, it is clear that $\mathfrak{g}(V_\alpha) = \mathfrak{g}(V)_\alpha$. Two $(-1,-1)$-BFKTS $V^1$ and $V^2$ will be said to be *equivalent* in case there is a nonzero scalar $\alpha$ such that $V^1$ and $V^2_\alpha$ are isomorphic.

THEOREM 4.1. *Let $V^1$ and $V^2$ be two $(-1,-1)$-BFKTS's. Then $V^1$ is equivalent to $V^2$ if and only if $(\mathfrak{g}(V^1), \mathfrak{s}(V^1))$ is isomorphic to $(\mathfrak{g}(V^2), \mathfrak{s}(V^2))$.*

*Proof.* Let $\mathfrak{g}^i = \mathfrak{g}(V^i)$ and $\mathfrak{d}^i = \mathfrak{d}(V^i) = d_{V^i,V^i}$ for $i = 1, 2$. Also, $\mathfrak{s}(V^1) = \mathfrak{s}(V^2) = \mathfrak{sp}(U, \varphi)$ as in Section 2. Thus $\mathfrak{g}^i{}_{\bar{0}} = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{d}^i$ and $\mathfrak{g}^i{}_{\bar{1}} = U \otimes_F V^i$, for $i = 1, 2$. Let $\Phi : \mathfrak{g}^1 \to \mathfrak{g}^2$ be an isomorphism such that it restricts to an automorphism of $\mathfrak{sp}(U, \varphi)$. But any automorphism $\xi$ of $\mathfrak{sp}(U, \varphi)$ can be extended as in [**6**, proof of Lemma 2.1] to an isomorphism from $\mathfrak{g}^2$ onto $\mathfrak{g}^2_\alpha$ for some nonzero scalar $\alpha$ and, therefore, we may (and will) assume that $\Phi$ is the identity on $\mathfrak{sp}(U, \varphi)$. Since $\mathfrak{d}^i$ is the centralizer of $\mathfrak{sp}(U, \varphi)$ in $\mathfrak{g}^i{}_{\bar{0}}$, $i = 1, 2$, $\Phi$ restricts to an isomorphism $\Psi : \mathfrak{d}^1 \to \mathfrak{d}^2$. Also, $\Phi$ restricts then to an isomorphism of $\mathfrak{sp}(U, \varphi)$-modules $\Phi_{\bar{1}} : U \otimes_F V^1 \to U \otimes_F V^2$. Since $U$ is absolutely irreducible as a module for $\mathfrak{sp}(U, \varphi)$, there is an isomorphism of vector spaces $\psi : V^1 \to V^2$ such that $\Phi(a \otimes x) = a \otimes \psi(x)$ for any $a \in U$ and $x \in V^1$.

Now, for any $x, y, z \in V^1$ and any $a \in U$, we have $a \otimes \psi(d_{x,y}(z)) = \Phi([d_{x,y}, a \otimes z]) = [\Psi(d_{x,y}), a \otimes \psi(z)] = a \otimes \Psi(d_{x,y})(\psi(z))$, so

$$\psi(d_{x,y}(z)) = \Psi(d_{x,y})(\psi(z)), \tag{4.1}$$

for any $x, y, z \in V^1$. Also, for any $a, b \in U$ and $x, y \in V^1$ we have $\Phi([a \otimes x, b \otimes y]) = [a \otimes \psi(x), b \otimes \psi(y)] = \langle \psi(x) \mid \psi(y)\rangle \varphi_{a,b} + \varphi(a,b) d_{\psi(x),\psi(y)}$, but also $\Phi([a \otimes x, b \otimes y]) = \Phi(\langle x \mid y\rangle \varphi_{a,b} + \varphi(a,b) d_{x,y}) = \langle x \mid y\rangle \varphi_{a,b} + \varphi(a,b)\Psi(d_{x,y})$. So

$$\begin{cases} \Psi(d_{x,y}) = d_{\psi(x),\psi(y)}, \\ \langle \psi(x) \mid \psi(y)\rangle = \langle x \mid y\rangle, \end{cases} \tag{4.2}$$

for any $x, y \in V^1$, which, together with (4.1), shows that $\psi$ is an isomorphism between the triple systems $V^1$ and $V^2$.

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

For the converse, if $V^1$ and $V^2$ are equivalent, there is a $0 \neq \alpha \in F$ such that $V^1$ and $V^2_\alpha$ are isomorphic. From here it is easy to deduce that the pairs $(\mathfrak{g}(V^1), \mathfrak{s}(V^1))$ and $(\mathfrak{g}(V^2_\alpha), \mathfrak{s}(V^2))$ are isomorphic. But $\mathfrak{g}(V^2_\alpha)$ is isomorphic to $\mathfrak{g}(V^2)$ by means of an isomorphism taking $\mathfrak{s}(V^2)$ into itself (see [**6**, proof of Lemma 2.1]). □

In order to classify the simple $(-1,-1)$-BFKTS of finite dimension over a field of characteristic zero, we will first assume that the ground field $F$ is algebraically closed. Following Theorems 2.1, 2.2 and 4.1, we will determine, up to isomorphism, the pairs $(\mathfrak{g}, \mathfrak{s})$, where $\mathfrak{g}$ is a simple finite dimensional Lie superalgebra and $\mathfrak{s}$ is an ideal of $\mathfrak{g}_{\bar 0}$ isomorphic to $\mathfrak{sl}(2)$:

THEOREM 4.2. *Let $F$ be an algebraically closed field of characteristic zero. The following list exhausts, up to isomorphism, the pairs $(\mathfrak{g}, \mathfrak{s})$, where $\mathfrak{g}$ is a simple finite dimensional Lie superalgebra over $F$ $(\mathfrak{g}_{\bar 1} \neq 0)$ and $\mathfrak{s}$ is a three dimensional simple ideal of $\mathfrak{g}_{\bar 0}$.*

(i) $\mathfrak{g} = \mathfrak{sl}(m, 2)$*, $m \geq 3$, and $\mathfrak{s}$ is the* (*unique*) *ideal of $\mathfrak{g}_{\bar 0}$ isomorphic to* $\mathfrak{sl}(2)$.

(ii) $\mathfrak{g} = \mathfrak{psl}(2, 2)$ *and $\mathfrak{s}$ is any of the two simple ideals of* $\mathfrak{g}_{\bar 0}$.

(iii) $\mathfrak{g} = \mathfrak{osp}(m, 2)$*, $m \geq 1$, $m \neq 4$, so that $\mathfrak{g}_{\bar 0} = \mathfrak{o}(m) \oplus \mathfrak{sp}(2)$, and $\mathfrak{s}$ is the copy of* $\mathfrak{sp}(2)$.

(iv) $\mathfrak{g} = \mathfrak{osp}(4, 2r)$*, $r \geq 2$, so that $\mathfrak{g} = \mathfrak{o}(4) \oplus \mathfrak{sp}(2r)$ and $\mathfrak{s}$ is any of the two simple simple ideals of* $\mathfrak{o}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$.

(v) $\mathfrak{g} = D(2, 1; \alpha)$*, $\alpha \neq 0, -1$, so that $\mathfrak{g}_{\bar 0} = \mathfrak{sp}(U, \varphi) \oplus \mathfrak{sp}(U, \varphi) \oplus \mathfrak{sp}(U, \varphi)$, $U$ being a two dimensional vector space and $\varphi$ a nonzero skew symmetric bilinear form on $U$, $\mathfrak{g}_{\bar 1} = U \otimes_F U \otimes_F U$, with the natural multiplication in $\mathfrak{g}_{\bar 0}$ and natural action of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$ in which the $i^{\text{th}}$ copy of $\mathfrak{sp}(U, \varphi)$ acts on the $i^{\text{th}}$ factor of $U$, and with the multiplication of odd elements given by:*

$$[u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3] = \varphi(u_2, v_2)\varphi(u_3, v_3)\varphi_{u_1,v_1} + \alpha\varphi(u_1, v_1)\varphi(u_3, v_3)\varphi_{u_2,v_2} - (1+\alpha)\varphi(u_1, v_1)\varphi(u_2, v_2)\varphi_{u_3,v_3}$$

*for any $u_i, v_i \in U$, $i = 1, 2, 3$. Here $\mathfrak{s}$ is the first copy of* $\mathfrak{sp}(U, \varphi)$.

(vi) $\mathfrak{g} = G(3)$ *and $\mathfrak{s}$ is the* (*unique*) *ideal of $\mathfrak{g}_{\bar 0}$ isomorphic to* $\mathfrak{sl}(2)$.

(vii) $\mathfrak{g} = F(4)$ *and $\mathfrak{s}$ is the* (*unique*) *ideal of $\mathfrak{g}_{\bar 0}$ isomorphic to* $\mathfrak{sl}(2)$.

(viii) $\mathfrak{g} = \mathfrak{sp}(3, 2r)$*, $r \geq 1$, and $\mathfrak{s}$ is the copy of $\mathfrak{o}(3)$ in* $\mathfrak{g}_{\bar 0}$.

*Moreover, different choose of the simple ideal $\mathfrak{s}$ in* (ii) *or* (iv) *give isomorphic pairs and two pairs in* (v) *corresponding to the values $\alpha_1$ and $\alpha_2$ are isomorphic if and only if either $\alpha_1 = \alpha_2$ or $\alpha_1 + \alpha_2 = -1$.*

*Proof.* A careful look at the list of simple Lie superalgebras in [**11**, Theorem 5] shows that the semisimple part of $\mathbf{W}(n)_{\bar 0}$ $(n \geq 2)$, of $\mathbf{S}(n)$ $(n \geq 3)$ and of $\tilde{\mathbf{S}}(n)$ $(n \geq 3)$, is isomorphic to $\mathfrak{sl}(n)$ [**11**, Propositions 3.1.1 and 3.3.1], while $\mathbf{W}(2)$ is isomorphic to $\mathfrak{sl}(1, 2)$. Also, the semisimple part of $\mathbf{H}(n)$ $(n \geq 4)$ is isomorphic to $\mathfrak{o}(n)$ [**11**, Proposition 3.3.6], while $\mathbf{H}(4)$ is isomorphic to $\mathfrak{psl}(2, 2)$. Hence, it is enough to deal with the classical algebras. One checks easily that the simple classical Lie superalgebras with $\mathfrak{g}_{\bar 0}$ containing a three dimensional simple ideal are those listed above. Since $\mathfrak{osp}(4, 2)$ is isomorphic to $D(2, 1; 1)$, this has been excluded from (iii) and included in (v), and since $\mathfrak{sl}(1, 2)$ is isomorphic to $\mathfrak{osp}(2, 2)$, this has been included in (iii).

The last assertion about cases (ii) and (iv) is clear. Also, the Lie algebras in (v) are the ones denoted by $\Gamma(1, \alpha, -(1+\alpha))$ in [**26**, p. 16–17]. Here we have three

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

copies of $\mathfrak{sl}(2)$ in $\mathfrak{g}_{\bar{0}}$, but there are isomorphisms preserving the three copies from $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ $(\sigma_1 + \sigma_2 + \sigma_3 = 0)$ onto $\Gamma(\eta\sigma_1, \eta\sigma_2, \eta\sigma_3)$ for any $0 \neq \eta \in F$, and also natural isomorphisms permuting the three copies of $\mathfrak{sl}(2)$ (and the corresponding $\sigma_i$'s). Therefore, the distinguished copy of $\mathfrak{sl}(2)$ can always be taken to be the first one. Finally, if there is an isomorphism from $\Gamma(1, \alpha, -1-\alpha)$ onto $\Gamma(1, \beta, -1-\beta)$ that takes the first copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \alpha, -1-\alpha)$ to the first copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \beta, -1-\beta)$, then it takes the second copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \alpha, -1-\alpha)$ to either the second or the third copy of $\mathfrak{sl}(2)$ in $\Gamma(1, \beta, -1-\beta)$, whence the last assertion of the Theorem. □

Now we are ready for our main Theorem, it asserts that the examples in Section 3 exhaust all the simple $(-1, -1)$-BFKTS's:

THEOREM 4.3. *Let $V$ be a finite dimensional simple $(-1, -1)$-BFKTS over a field $F$ of characteristic zero with associated symmetric bilinear form $\langle\,|\,\rangle$. Either:*

(i) *The multiplication in $V$ is given by*

$$xyz = \langle z \mid x\rangle y - \langle z \mid y\rangle x + \langle x \mid y\rangle z,$$

*for any $x, y, z \in V$ (orthogonal type), or*

(ii) *There is a quadratic étale algebra $K$ over $F$ such that $V$ is a free $K$-module of rank at least 3, endowed with a hermitian form $h : V \times V \to K$ such that*

$$\begin{cases} \langle x \mid y\rangle = \frac{1}{2}(h(x, y) + h(y, x)), \\ xyz = h(z, x)y - h(z, y)x + h(x, y)z, \end{cases}$$

*for any $x, y, z \in V$ (unitarian type).*

(iii) *There is a quaternion algebra $Q$ over $F$ such that $V$ is a free left $Q$-module of rank $\geq 2$, endowed with a hermitian form $h : V \times V \to Q$ such that*

$$\begin{cases} \langle x \mid y\rangle = \frac{1}{2}(h(x, y) + h(y, x)) \\ xyz = h(z, x)y - h(z, y)x + h(x, y)z \end{cases}$$

*for any $x, y, z \in V$ (symplectic type).*

(iv) $\dim_F V = 4$ *and there is a nonzero skew symmetric multilinear form $\Phi : V \times V \times V \times V \to F$ such that for any $x, y, z \in V$:*

$$xyz = [xyz] + \langle z \mid x\rangle y - \langle z \mid y\rangle x + \langle x \mid y\rangle z,$$

*where $[xyz]$ is defined by means of $\Phi(x, y, z, t) = \langle [xyz] \mid t\rangle$ for any $x, y, z, t \in V$. In this case, there is a nonzero scalar $\mu \in F$ such that (3.8) holds ($D_\mu$-type).*

(v) $\dim_F V = 7$ *and there is an eight dimensional Cayley-Dickson algebra $C$ over $F$ with trace $t$ and a nonzero scalar $\alpha \in F$ such that $V = C_0 = \{x \in C : t(x) = 0\}$, and for any $x, y, z \in V$:*

$$\begin{cases} \langle x \mid y\rangle = -2\alpha t(xy) \\ xyz = \alpha(D_{x,y}(z) - 2t(xy)z) \end{cases}$$

*where $D_{x,y}$ is the inner derivation of $C$ given by (3.11) (G-type).*

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

(vi) $\dim_F V = 8$ *and* $(V, \langle\,|\,\rangle)$ *is endowed with a* 3*-fold vector cross product* $X$ *of type I such that*

$$xyz = \frac{1}{3}X(x, y, z) + \langle z \mid x\rangle y - \langle z \mid y\rangle x + \langle x \mid y\rangle z$$

*for any* $x, y, z \in V$. *(F-type.)*

*Moreover, two triple systems in different items cannot be isomorphic and:*

(i′) *Two triple systems of orthogonal type are isomorphic if and only if the corresponding symmetric bilinear forms are isometric.*

(ii′) *Two triple systems of unitarian type* $V_1$ *and* $V_2$*, with associated quadratic étale algebras* $K_1$ *and* $K_2$ *and hermitian forms* $h_1$ *and* $h_2$*, are isomorphic if and only if the hermitian pairs* $(V_1, h_1)$ *and* $(V_2, h_2)$ *are isomorphic; that is, there is an isomorphism of* $F$*-algebras* $\sigma : K_1 \to K_2$ *and a linear bijection* $\varphi : V_1 \to V_2$ *such that* $h_2(\varphi(x), \varphi(y)) = \sigma(h_1(x, y))$ *for any* $x, y \in V_1$.

(iii′) *Two triple systems of symplectic type* $V_1$ *and* $V_2$*, with associated quaternion algebras* $Q_1$ *and* $Q_2$ *and hermitian forms* $h_1$ *and* $h_2$*, are isomorphic if and only if the hermitian pairs* $(V_1, h_1)$ *and* $(V_2, h_2)$ *are isomorphic.*

(iv′) *Two triple systems of* $D_\mu$*-type, with associated scalars* $\mu_1$ *and* $\mu_2$*, are isomorphic if and only if the corresponding symmetric bilinear forms are isometric and* $\mu_1 = \mu_2$.

(v′) *Two triple systems of G-type, with associated Cayley-Dickson algebras* $C_1$ *and* $C_2$ *and scalars* $\alpha_1$ *and* $\alpha_2$*, are isomorphic if and only if so are* $C_1$ *and* $C_2$ *and* $\alpha_1 = \alpha_2\gamma^2$ *for some* $0 \neq \gamma \in F$.

(vi′) *Two triple systems of F-type* $V_1$ *and* $V_2$*, with associated type I* 3*-fold vector cross products* $X_1$ *and* $X_2$*, are isomorphic if and only if so are the triple systems* $(V_1, X_1)$ *and* $(V_2, X_2)$.

*Proof.* First, the new triple product defined on $V$ by $[xyz] = xyz - \langle z \mid x\rangle y + \langle z \mid y\rangle x - \langle x \mid y\rangle z$ for any $x, y, z \in V$ is skew symmetric because of (2.5a). If this is identically zero, $V$ is of orthogonal type. Otherwise, if the dimension of $V$ is 4, $V$ is of $\mathrm{D}_\mu$-type.

Hence, in what follows, assume that $\dim_F V \neq 4$. Then, after extending scalars to an algebraic closure $\bar{F}$ of $F$, if $\bar{V} = \bar{F} \otimes_F V$, $(\mathfrak{g}(\bar{V}), \mathfrak{s}(\bar{V}))$ is one of the pairs considered in cases (i), (iii), (iv), (vi) or (vii) in Theorem 4.2. Note that case (viii) does not appear since there $\mathfrak{g}_{\bar{1}}$ is a direct sum of adjoint modules for $\mathfrak{s}$ instead of a direct sum of two dimensional irreducible modules.

Because of Theorem 4.1 and the computations in Section 3, and since the classical Lie superalgebras other than $D(2, 1; \alpha)$'s are determined by its even part and the structure of $\mathfrak{g}_{\bar{1}}$ as a $\mathfrak{g}_{\bar{0}}$-module [**11**, Proposition 2.1.4], it follows that case (i) in Theorem 4.2 corresponds to the unitarian type with $\bar{K} = \bar{F} \times \bar{F}$ and $\dim_F V \geq 6$, case (iii) in 4.2 corresponds to the orthogonal type, case (iv) to the symplectic type and $\dim_F V \geq 8$ and cases (vi) and (vii) to G and F types.

Therefore, it is enough to deal with the forms over $F$ of the simple $(-1, -1)$-BFKTS's over $\bar{F}$ considered in Section 3 with dimension $\neq 4$.

It is clear that if $\bar{V}$ is of orthogonal type, so is $V$. If $\bar{V}$ is of unitarian type with $\dim_F V \geq 6$, then since $\bar{K} = \mathrm{End}_{\bar{\mathfrak{d}}}(\bar{V}) = \bar{F} \otimes_F \mathrm{End}_{\mathfrak{d}}(V)$, $K = \mathrm{End}_{\mathfrak{d}}(V)$ is a quadratic étale algebra over $F$; besides, there is a $\bar{K}$-hermitian form $\bar{h} : \bar{V} \times \bar{V} \to \bar{K}$ such that $xyz = \bar{h}(z, x)y - \bar{h}(z, y)x + \bar{h}(x, y)z$ for any $x, y, z \in \bar{V}$. But if $\{1, i\}$ is an $F$-basis of $K$ with $i^2 = \alpha \in F$, then $\bar{h}(x, y) = \langle x \mid y\rangle - \alpha^{-1}\langle x \mid iy\rangle i$ for any $x, y \in \bar{V}$. Since both

$\langle x \mid y\rangle$ and $\langle x \mid iy\rangle$ are in $F$ in case $x, y \in V$, it follows that $\bar{h}$ restricts to an hermitian form $h : V \times V \to K$ and $V$ is the corresponding simple $(-1,-1)$-BFKTS of unitarian type. A similar argument works in case $\bar{V}$ is of symplectic type and $\dim_F V \geq 8$. In this case $\bar{\mathfrak{d}} = \bar{\mathfrak{b}} \oplus \bar{\mathfrak{s}}$ with $\bar{\mathfrak{s}} \cong \mathfrak{sl}(2, \bar{F}) \ncong \bar{\mathfrak{b}}$, so that $\mathfrak{d} = \mathfrak{b} \oplus \mathfrak{s}$ for a suitable unique ideal $\mathfrak{b}$ and $\bar{Q} = \mathrm{End}_{\bar{\mathfrak{b}}}(\bar{V}) = \bar{F} \otimes_F \mathrm{End}_{\mathfrak{b}}(V)$. Hence $\mathrm{End}_{\mathfrak{b}}(V) = Q$ is a quaternion algebra and $V$ is a free $Q$-module. Now one takes a suitable $F$-basis $\{1, i, j, k\}$ of $Q$ and argues as above.

If $\bar{V}$ is of G-type, then $\mathfrak{d}$ is a form of $G_2$, so there is an eight-dimensional Cayley-Dickson algebra $C$ over $F$ such that $\mathfrak{d} \cong \mathrm{Der}\, C$ and $V$ is, up to isomorphism, its seven dimensional irreducible module for $\mathfrak{d}$, that is $C_0$, the set of traceless elements in $C$. Since $\mathrm{Hom}_{\mathfrak{d}}(V \otimes_F V, \mathfrak{d})$ is one-dimensional, after identifying $V$ with $C_0$ there exists a nonzero $\alpha \in F$ such that $d_{x,y} = \alpha D_{x,y}$ for any $x, y \in C_0 = V$. From here, using (2.3c), it follows that $V$ is of G-type.

Finally, if $\bar{V}$ is of $F$-type, define $X : V \times V \times V \to F$ by $X(x, y, z) = 3(xyz - \langle z \mid x\rangle y + \langle z \mid y\rangle x - \langle x \mid y\rangle z)$, for any $x, y, z \in V$. Then $X$ is a 3-fold vector cross product of type I (because it is so after extending scalars) and hence $V$ is of F-type.

Moreover, two simple $(-1,-1)$-BFKTS's of different types cannot be isomorphic because the corresponding Lie algebras of inner derivations are not. Also note that, because of (2.5a), any isomorphism among two $(-1,-1)$-BFKTS's is an isometry of the corresponding symmetric bilinear forms. Now (i′) is clear and (ii′) (respectively (iii′)) follows from the fact that $K_1$ and $K_2$ (resp. $Q_1$ and $Q_2$) are determined as centralizers of the action of a suitable ideal of the Lie algebra of inner derivations.

Let us check (iv′), so let $(V_i, (xyz)_i)$ be two simple $(-1,-1)$-BFKTS's of $\mathrm{D}_{\mu_i}$-type $(i = 1, 2)$. If $\varphi : V_1 \to V_2$ is an isomorphism, then it is an isometry and thus $\varphi([xyz]_1) = [\varphi(x)\varphi(y)\varphi(z)]_2$ for any $x, y, z \in V_1$. Hence

$$\langle \varphi([x_1x_2x_3]_1) \mid \varphi([x_1x_2x_3]_1)\rangle_2 = \langle [x_1x_2x_3]_1 \mid [x_1x_2x_3]_1\rangle_1 = \mu_1 \det(\langle x_i \mid x_j\rangle_1),$$

but also

$$\begin{aligned}\langle \varphi([x_1x_2x_3]_1) \mid \varphi([x_1x_2x_3]_1)\rangle_2 &= \langle [\varphi(x_1)\varphi(x_2)\varphi(x_3)]_2 \mid [\varphi(x_1)\varphi(x_2)\varphi(x_3)]_2\rangle_2 \\ &= \mu_2 \det(\langle \varphi(x_i) \mid \varphi(x_j)\rangle_2) \\ &= \mu_2 \det(\langle x_i \mid x_j\rangle_1).\end{aligned}$$

Therefore, $\mu_1 = \mu_2$. Conversely, assume that $\varphi : V_1 \to V_2$ is an isometry and that $\mu_1 = \mu_2 = \mu$. Consider $\Phi_i : V_i^4 \to F$ $(i = 1, 2)$ given by $\Phi_i(x_1, x_2, x_3, x_4) = \langle [x_1x_2x_3]_i \mid x_4\rangle_i$. Also, let $\tilde{\Phi}_1 : V_1^4 \to F$ be defined by

$$\tilde{\Phi}_1(x_1, x_2, x_3, x_4) = \Phi_2(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4))$$

for any $x_1, x_2, x_3, x_4 \in F$. Since $\dim_F V_1 = 4$ and both $\Phi_1$ and $\tilde{\Phi}_1$ are skew symmetric, they are proportional, and hence there is a nonzero scalar $\alpha \in F$ such that $\tilde{\Phi}_1 = \alpha\Phi_1$. For any $x_1, x_2, x_3, y_1, y_2, y_3 \in F$:

$$\begin{aligned}\Phi_2(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi([y_1y_2y_3]_1)) &= \tilde{\Phi}_1(x_1, x_2, x_3, [y_1y_2y_3]_1) \\ &= \alpha\Phi_1(x_1, x_2, x_3, [y_1y_2y_3]_1) \\ &= \alpha\mu \det(\langle x_i \mid y_j\rangle_1) \\ &= \alpha\mu \det(\langle \varphi(x_i) \mid \varphi(y_j)\rangle_2) \\ &= \Phi_2(\varphi(x_1), \varphi(x_2)\varphi(x_3), \alpha[\varphi(y_1)\varphi(y_2)\varphi(y_3)]_2),\end{aligned}$$

where we have used (3.8) and the fact that $\varphi$ is an isometry. Thus $\varphi([y_1y_2y_3]_1) = \alpha[\varphi(y_1)\varphi(y_2)\varphi(y_3)]_2$ for any $y_1, y_2, y_3 \in V_1$. But now, again by (3.8), this shows that $\mu \det(\langle y_i \mid y_j\rangle_1) = \alpha^2\mu \det(\langle y_i \mid y_j\rangle_1)$ for any $y_i$'s, so that $\alpha^2 = 1$. If $\alpha = 1$ we are done, otherwise $\alpha = -1$. In this latter case, choose any isometry $\sigma$ of $\langle \mid \rangle_1$ with $\det\sigma = -1$ and consider $\hat\varphi = \varphi\sigma : V_1 \to V_2$. Then if $\hat\Phi_1(x_1, x_2, x_3, x_4) = \Phi_2(\hat\varphi(x_1), \hat\varphi(x_2), \hat\varphi(x_3), \hat\varphi(x_4))$ for any $x_i \in V_1$ $(i = 1, 2, 3, 4)$, we have:

$$\begin{aligned}\hat\Phi_1(x_1, x_2, x_3, x_4) &= \Phi_2(\hat\varphi(x_1), \hat\varphi(x_2), \hat\varphi(x_3), \hat\varphi(x_4))\\ = \tilde\Phi_1(\sigma(x_1), \sigma(x_2), \sigma(x_3), \sigma(x_4)) &= \alpha(\det\sigma)\Phi_1(x_1, x_2, x_3, x_4) = \Phi_1(x_1, x_2, x_3, x_4),\end{aligned}$$

because $\alpha = -1 = \det\sigma$ and $\Phi_1$ is multilinear and alternating. The same argument as above, with $\tilde\Phi_1$ replaced by $\hat\Phi_1$ shows that $\hat\varphi$ is an isomorphism between the two triple systems.

With regard to (v′), if $\varphi : V^1 \to V^2$ is an isomorphism of two triple systems of G-type with associated Cayley-Dickson algebras $C^1$ and $C^2$ and scalars $\alpha_1$ and $\alpha_2$, then $\varphi$ is an isometry of the associated symmetric bilinear forms and for any $x, y, z \in V_1$

$$\varphi(d_{x,y}z) = d_{\varphi(x),\varphi(y)}\varphi(z). \tag{4.2}$$

Also, $\phi : \mathfrak{d}^1 = d_{V^1,V^1} \to \mathfrak{d}^2$: $d \mapsto \varphi d\varphi^{-1}$ is an isomorphism of Lie algebras and $\varphi$ becomes an isomorphism of $\mathfrak{d}^1$-modules, where $V^2$ is a $\mathfrak{d}^1$-module through $\phi$. Since $\mathrm{Hom}_{\mathfrak{d}^1}(\Lambda^2(V^1), V^1)$ is spanned by $x \wedge y \mapsto [x, y] = xy - yx$ (multiplication in $C^1$), there is a nonzero scalar $\mu \in F$ such that

$$\varphi([x, y]) = \mu[\varphi(x), \varphi(y)] \tag{4.3}$$

for any $x, y \in V^1 = C^1_0 = \{z \in C^1 : t(z) = 0\}$. In particular, $\mu\varphi : (C^1_0, [\,,\,]) \to (C^2_0, [\,,\,])$ is an isomorphism of Malcev algebras and hence $C^1$ and $C^2$ are isomorphic (see, for instance, [**3**, (3.1)]). But the associator $(x, y, z) = (xy)z - x(yz)$ in $C^1$ is skew symmetric on its arguments, so for any $x, y, z \in C^1$, $(x, y, z) = -(x, z, y) = (z, x, y) = (y, z, x)$, so that $L_{xy} - L_xL_y = [L_x, R_y] = R_yR_x - R_{xy} = [R_x, L_y]$, hence $ad_{xy} - L_xL_y + R_yR_x = 2[L_x, R_y]$ for any $x, y \in C^1$, where $\mathrm{ad}_x y = [x, y] = (L_x - R_x)(y)$. Permuting $x$ and $y$ and subtracting we get $\mathrm{ad}_{[x,y]} = [L_x, L_y] + [R_x, R_y] + 4[L_x, R_y] = D_{x,y} + 3[L_x, R_y]$. On the other hand,

$$\begin{aligned}[\mathrm{ad}_x, \mathrm{ad}_y] &= [L_x - R_x, L_y - R_y] = [L_x, L_y] + [R_x, R_y] - 2[L_x, R_y]\\ &= D_{x,y} - 3[L_x, R_y],\end{aligned}$$

and from here we conclude that $2D_{x,y} = \mathrm{ad}_{[x,y]} + [\mathrm{ad}_x, \mathrm{ad}_y]$ for any $x, y \in C^1$. Since $d_{x,y} = \alpha_1 D_{x,y}$ and $d_{\varphi(x),\varphi(y)} = \alpha_2 D_{\varphi(x),\varphi(y)}$ for any $x, y \in V^1$, equation (4.3) gives:

$$\begin{aligned}\varphi d_{x,y} &= \frac{\alpha_1}{2}\varphi\big(\mathrm{ad}_{[x,y]} + [\mathrm{ad}_x, \mathrm{ad}_y]\big) = \frac{\alpha_1}{2}\mu^2\big(\mathrm{ad}_{[\varphi(x),\varphi(y)]} + \big[\mathrm{ad}_{\varphi(x)}, \mathrm{ad}_{\varphi(y)}\big]\big)\varphi\\ &= \frac{\alpha_1}{2}\mu^2 D_{\varphi(x),\varphi(y)}\varphi,\end{aligned}$$

while $d_{\varphi(x),\varphi(y)} = \frac{\alpha_2}{2}D_{\varphi(x),\varphi(y)}$ for any $x, y \in V^1$, so that equation (4.2) gives $\alpha_1\mu^2 = \alpha_2$, as desired. Conversely, if $\psi : C^1 \to C^2$ is an isomorphism and $\alpha_2 = \alpha_1\mu^2$, then the map $\varphi : V^1 = C^1_0 \to V^2 = C^2_0$, given by $\varphi(x) = \mu^{-1}\psi(x)$ for any $x \in V^1$, is an isomorphism of triple systems.

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

We are left with the isomorphism problem for the $(-1,-1)$-BFKTS's of F-type. For these we need some preliminaries, which have their own independent interest:

LEMMA 4.4. *Let $X$ be a 3-fold vector cross product of type I on an eight dimensional vector space $V$ over a field $F$ of characteristic $\neq 2$, and let $\langle\,|\,\rangle$ be the associated nondegenerate symmetric bilinear form (so that (3.11) is satisfied). Then $\langle\,|\,\rangle$ is determined by $X$.*

*Proof.* Because of (3.12), for any $a, b, c, d \in V$:

$$-\langle d \mid X(a,b,X(a,b,c))\rangle = \langle X(a,b,c) \mid X(a,b,d)\rangle = \begin{vmatrix} \langle a \mid a\rangle & \langle a \mid b\rangle & \langle a \mid d\rangle \\ \langle b \mid a\rangle & \langle b \mid b\rangle & \langle b \mid d\rangle \\ \langle c \mid a\rangle & \langle c \mid b\rangle & \langle c \mid d\rangle \end{vmatrix}$$

$$= \langle \langle a\wedge b \mid a\wedge b\rangle c - \langle a\wedge b \mid a\wedge c\rangle b + \langle a\wedge b \mid b\wedge c\rangle a \mid d\rangle$$

where $\langle a\wedge b \mid u\wedge v\rangle = \begin{vmatrix} \langle a \mid u\rangle & \langle a \mid v\rangle \\ \langle b \mid u\rangle & \langle b \mid v\rangle \end{vmatrix}$ for any $a, b, u, v \in V$. By nondegeneracy of $\langle\,|\,\rangle$, this gives:

$$X(a,b,X(a,b,c)) = \langle a\wedge b \mid c\wedge b\rangle a + \langle a\wedge b \mid a\wedge c\rangle b - \langle a\wedge b \mid a\wedge b\rangle c. \tag{4.4}$$

Hence, for any $a, b, c \in V$, if $d = X(a,b,c)$, then $X(a,b,d) \in Fa + Fb + Fc$ and, similarly (since $d = X(b,c,a) = X(c,a,b)$), $X(b,c,d), X(a,c,d) \in Fa + Fb + Fc$, so that $W = Fa + Fb + Fc + Fd$ is closed under $X$. Let us prove now that for any $0 \neq v \in V$:

$$X(v, V, V) = \{x \in V : \langle v \mid x\rangle = 0\}. \tag{4.5}$$

Because of (3.12), $X(v,V,V) \subseteq \{x \in V : \langle x \mid v\rangle = 0\}$. Now, take $a = v$ and let $b \in V$ linearly independent with $a$ and such that $\langle\,|\,\rangle$ is nondegenerate on $W_b = Fa + Fb$. By (4.4) $c \in X(a,b,V) \subseteq X(v,V,V)$ for any $c \in W_b^{\perp} = \{x \in V : \langle x \mid a\rangle = 0 = \langle x \mid b\rangle\}$. Take any two such $b$'s with different $W_b$'s, then the sum of the $W_b^{\perp}$'s is $\{x \in V : \langle x \mid v\rangle = 0\}$ , so (4.5) follows.

Thus, assume that $X$ is also a 3-fold vector cross product of type I relative to another nondegenerate symmetric bilinear form $(\,|\,)$ on $V$. Then for any $0 \neq u, v \in V$, if $\langle u \mid v\rangle = 0$, then $u \in X(v,V,V)$ by (4.5), so by (3.12), also $(u \mid v) = 0$. The only possibility then is that $(\,|\,) = \alpha\langle\,|\,\rangle$ for some nonzero scalar $\alpha \in F$. But then (3.12) implies that $\alpha^3 = \alpha$, so $\alpha = \pm 1$, and (3.13) that $\alpha = 1$. □

Note that if $X$ is a 3-fold vector cross product of type I on an eight dimensional vector space $V$ relative to the nondegenerate symmetric bilinear form $\langle\,|\,\rangle$, then $X$ is a 3-fold vector cross product of type II relative to $-\langle\,|\,\rangle$. Also note that $\langle\,|\,\rangle$ does not determine $X$, since not every orthogonal transformation relative to $\langle\,|\,\rangle$ is an automorphism of $X$ (**[4]**).

COROLLARY 4.5. *Let $X_i$ be a 3-fold vector cross product on an eight dimensional vector space $V_i$ over a field $F$ of characteristic $\neq 2$ with associated nondegenerate symmetric bilinear form $\langle\,|\,\rangle_i$ $(i = 1, 2)$. Then if $\varphi : (V_1, X_1) \to (V_2, X_2)$ is an isomorphism, then it is also an isometry $\varphi : (V_1, \langle\,|\,\rangle_1) \to (V_2, \langle\,|\,\rangle_2)$.*

Now, the proof of item (vi′) in Theorem 4.3 follows immediately from the Corollary above and this finishes its proof.

## REFERENCES

**1.** P. Alberca Bjerregaard, A. Elduque, C. Martín González and F. J. Navarro Márquez, On the Cartan-Jacobson Theorem, *J. Algebra* **250** (2002), no. 2, 397–407.

**2.** G. Benkart and A. Elduque, The Tits construction and the exceptional simple classical Lie superalgebras, *Quart. J. Math. Oxford*, to appear.

**3.** A. Elduque, On maximal subalgebras of central simple Mal'cev algebras, *J. Algebra* **103** (1986), no. 1, 216–227.

**4.** A. Elduque, On a class of ternary composition algebras, *J. Korean Math. Soc.* **33** (1996), no. 1, 183–203.

**5.** A. Elduque, On triality and automorphisms and derivations of composition algebras, *Linear Algebra Appl.* **314** (2000), no. 1–3, 49–74.

**6.** A. Elduque, Quaternion, octonions, and the forms of the exceptional simple classical Lie superalgebras, submitted. Jordan Theory Preprint Archives: http://mathematik.uibk.ac.at/jordan/ (paper no. 116).

**7.** A. Elduque and H. C. Myung, The reductive pair $(B_3, G_2)$ and affine connections on $S^7$, *J. Pure Appl. Algebra* **86** (1993), no. 2, 155–171.

**8.** J. R. Faulkner and J. C. Ferrar, On the structure of symplectic ternary algebras, *Nederl. Akad. Wetensch. Proc. Ser. A* **75** = *Indag. Math.* **34** (1972), 247–256.

**9.** J. R. Faulkner and J. C. Ferrar, Simple anti-Jordan pairs, *Commun. Algebra* **8** (1980), no. 11, 993–1013.

**10.** H. Freudenthal, Beziehungen der $E_7$ und $E_8$ zur Oktavenebene. I, II, *Nederl. Akad. Wetensch. Proc. Ser. A.* **57** = *Indag. Math.* **16** (1954), 218–230 and 363–380.

**11.** V. G. Kac, Lie superalgebras, *Adv. Math.* **26** (1977), no. 1, 8–96.

**12.** N. Kamiya, A structure theory of Freudenthal-Kantor triple systems, *J. Algebra* **110** (1987), no. 1, 108–123.

**13.** N. Kamiya, A structure theory of Freudenthal-Kantor triple systems II, *Comment. Math. Univ. St. Paul.* **38** (1989), no. 1, 41–60.

**14.** N. Kamiya, A structure theory of Freudenthal-Kantor triple systems. III, *Mem. Fac. Sci. Shimane Univ.* **23** (1989), 33–51.

**15.** N. Kamiya, The construction of all simple Lie algebras over **C** from balanced Freudenthal-Kantor triple systems, in *Contributions to general algebra, 7 (Vienna, 1990)* (Hölder-Pichler-Tempsky, Vienna, 1991), 205–213.

**16.** N. Kamiya, On Freudenthal-Kantor triple systems and generalized structurable algebras, in *Non-associative algebra and its applications (Oviedo, 1993)*, *Math. Appl.* **303** (Kluwer, Dordrecht, 1994), 198–203.

**17.** N. Kamiya, On the Peirce decompositions for Freudenthal-Kantor triple systems, *Commun. Algebra* **25** (1997), no. 6, 1833–1844.

**18.** N. Kamiya and S. Okubo, On $\delta$-Lie supertriple systems associated with $(\epsilon, \delta)$-Freudenthal-Kantor supertriple systems, *Proc. Edinburgh Math. Soc. (2)* **43** (2000), no. 2, 243–260.

**19.** N. Kamiya and S. Okubo, Construction of Lie superalgebras $D(2, 1; \alpha)$, $G(3)$ and $F(4)$ from some triple systems, *Proc. Edinburgh Math. Soc. (2)* **46** (2003), 87–98.

**20.** I. L. Kantor, Models of the exceptional Lie algebras (Russian), *Dokl. Akad. Nauk SSSR* **208** (1973), no. 6, 1276–1279. English translation in *Soviet Math. Dokl.* **14**, no. 1, 254–258.

**21.** K. Meyberg, Eine Theorie der Freudenthalschen Tripelsysteme. I, II, *Nederl. Akad. Wetensch. Proc. Ser. A* **71** = *Indag. Math.* **30** (1968), 162–174 and 175–190.

**22.** S. Okubo, Triple products and Yang-Baxter equation. I. Octonionic and quaternionic triple systems, *J. Math. Phys.* **34** (1993), no. 7, 3273–3291.

**23.** S. Okubo *Introduction to octonion and other non-associative algebras in physics*, Montroll Memorial Lecture Series in Mathematical Physics No. 2 (Cambridge University Press, 1995).

**24.** S. Okubo and N. Kamiya, Quasi-classical Lie superalgebras and Lie super triple systems, *Commun. Algebra* **30** (2002), no. 8, 3825–3850.

**25.** R. D. Schafer, *An introduction to nonassociative algebras*, Pure and Applied Mathematics, Vol. 22 (Academic Press, 1966).

**26.** M. Scheunert, *The theory of Lie superalgebras. An introduction*, Lecture Notes in Mathematics No 716 (Springer-Verlag, 1979).

https://doi.org/10.1017/S0017089503001290 Published online by Cambridge University Press

**27.** K. Yamaguti, Constructions of Lie (super)algebras from Freudenthal-Kantor triple system $U(\epsilon, \delta)$, Proceedings of the 14th ICGTMP, Seoul, 1985 (World Sci. Publishing, Singapore, 1986), 222–225.

**28.** K. Yamaguti and H. Asano, On the Freudenthal's construction of exceptional Lie algebras, *Proc. Japan Acad.* **51** (1975), no. 4, 253–258.

**29.** K. Yamaguti and A. Ono, On representations of Freudenthal-Kantor triple systems $U(\epsilon, \delta)$, *Bull. Fac. School Ed. Hiroshima Univ. Part II* **7** (1984), 43–51.