Bull. Austral. Math. Soc. 53c40, 53c15
Vol. 75 (2007) [1–16]

REAL HYPERSURFACES WITH η-PARALLEL SHAPE OPERATOR IN COMPLEX TWO-PLANE GRASSMANNIANS

Young Suk Choi and Young Jin Suh

In this paper we give a characterisation of $\mathfrak{D}$-invariant real hypersurfaces of type A; that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ or a ruled real hypersurface foliated by complex hypersurfaces which includes a maximal totally geodesic submanifold $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of η-parallel shape operator.

0. Introduction

In the geometry of real hypersurfaces in non-flat complex space forms $M_m(c)$ or in quaternionic space forms there have been many characterisations of model hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$, of type A_0, A_1, A_2 and B in complex hyperbolic space $\mathbb{C}H^m$ or A_1, A_2, B in quaternionic projective space $\mathbb{H}P^m$, which are completely classified by Cecil and Ryan [**4**], Kimura [**6**], Berndt [**1**], Martinez and Pérez [**7**] respectively. Among them there are only a few characterisations of homogeneous real hypersurfaces of type B in complex projective space $\mathbb{C}P^m$. For example, the condition that the shape operator A and the structure tensor ϕ satisfy $A\phi + \phi A = k\phi$, $k = const$, is a model characterisation of this kind of type B, which is a tube over a real projective space $\mathbb{R}P^m$ in $\mathbb{C}P^m$ (see Yano and Kon [**14**]).

On the other hand, real hypersurfaces of type A_1 or A_2 in $\mathbb{C}P^m$ and those of type A_0, A_1 or A_2 in $\mathbb{H}P^m$ mentioned above respectively are said to be of type A. Okumura [**9**] for $c > 0$, Montiel and Romero [**8**] for $c < 0$ has given respectively a characterisation of real hypersurfaces of type A with the condition that the structure tensor ϕ and the shape operator A commute with each other.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing

Received 17th May, 2006
This work was supported by grant Proj. No. R14-2002-003-01001-0 from Korea Research Foundation, Korea 2006. The present authors would like to express their gratitude to the referee for his careful reading of the manuscript and useful comments to develop this paper.

J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperKähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M that $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M. The almost contact structure vector field ξ mentioned above is defined by $\xi = -JN$, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$.

The first result in this direction is the classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying both conditions. Namely, Berndt and the second author [**2**] have proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In Theorem A the vector ξ contained in the one-dimensional distribution $[\xi]$ is said to be a *Hopf* vector when it becomes a principal vector for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Moreover in such a situation M is said to be a *Hopf* hypersurface. Besides of this, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ also admits the 3-dimensional distribution $\mathfrak{D}^\perp$, which is spanned by *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$.

On the other hand, in [**3**] Berndt and the second author consider the geometric condition that the shape operator A of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor, that is, $A\phi = \phi A$, which is equivalent to $\mathcal{L}_\xi g = 0$, where $\mathcal{L}_\xi$ denotes the *Lie* derivative along the direction of the Reeb vector field ξ and g a Riemannian metric on M induced from the metric of $G_2(\mathbb{C}^{m+2})$. This condition also has the geometric meaning that the flow of the Reeb vector field ξ is isometric. From such a view point, they proved that a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with isometric flow is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Moreover, the second author [**12**] has given a characterisation of such a tube by the *Lie* derivative of the second fundamental tensor A of M in $G_2(\mathbb{C}^{m+2})$ along the direction of the Reeb vector field ξ.

Now let us consider a distribution T_0 defined in such a way that $T_0(x) = \{X \in T_xM \mid X \perp \xi\}$ for any point x of M in $G_2(\mathbb{C}^{m+2})$. Then it can be easily proved in section 3 that real hypersurfaces of type A and ruled real hypersurfaces satisfy the following formula on the distribution T_0

$$(*) \qquad g\big((A\phi - \phi A)X, Y\big) = 0,$$

for any X, Y in T_0.

If the shape operator A satisfies

$$(**) \qquad g\big((\nabla_X A)Y, Z\big) = 0$$

for any X, Y and Z in T_0, we say that the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ is said to be η-parallel. Moreover, the formula (**) has a geometric meaning that every geodesic γ on M, considered as a curve in $G_2(\mathbb{C}^{m+2})$, orthogonal to the Reeb vector field ξ, has constant first curvature along γ.

On the other hand, we say that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}$-invariant if $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, that is, the distribution $\mathfrak{D}$ is invariant by the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.

Now in this paper we want to give a complete classification of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ satisfying both conditions (*) and (**) as follows:

Theorem. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying the condition (*) and (**). If the distribution $\mathfrak{D}$ is invariant by the shape operator, then M is locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ or to a ruled real hypersurface foliated by complex hypersurfaces which includes a maximal totally geodesic submanifold $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarise basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2] and [**3**]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabiliser isomorphic to $K = S\big(U(2) \times U(m)\big) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, $-B$ restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even

https://doi.org/10.1017/S0004972700038934 Published online by Cambridge University Press

space. For computational reasons we normalise g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{C}P^3$ with constant holomorphic sectional curvature eight we shall assume $m \geq 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = su(m) \oplus su(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the centre of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the centre $\mathfrak{R}$ induces a Kähler structure J and the $su(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$; there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see **[3, 10, 11, 12, 13]**).

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$; that is, a hypersurface in $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression for $\overline{R}$, the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\}\end{aligned}$$

$$+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}$$
$$+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu .$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\ &\phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\ &\phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\ &\phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.\end{aligned}\tag{2.1}$$

Now let us put

$$JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX,\tag{2.2}$$

$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX,\tag{2.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y=&-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y\\ &+\eta_\nu(Y)AX-g(AX,Y)\xi_\nu.\end{aligned}\tag{2.4}$$

Summing up these formulas, we find the following

(2.5)

$$\begin{aligned}\nabla_X(\phi_\nu\xi)&=\nabla_X(\phi\xi_\nu)\\ &=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\ &=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX-g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.\end{aligned}$$

Moreover, from $JJ_\nu=J_\nu J$, $\nu=1,2,3$, it follows that

$$\phi\phi_\nu X=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu.\tag{2.6}$$

3. Proof of Main Theorem

Before giving the proof of our Main Theorem let us investigate the question "What kind of hypersurfaces including hypersurfaces mentioned in Theorem A satisfy the formulas (*) and (**)." In other words, we would like to know whether there exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying both conditions (*) and (**).

First in this section we shall show that only a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ satisfies the formula (*). Next, it can be easily checked that such hypersurfaces also satisfy the formula (**) from the expression of the derivative of the shape operator A of this type (see Berndt and the second author [**3**]). That is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ has η-parallel shape operator.

Now in order to solve such a problem let us recall a Proposition given by Berndt and the second author [**2**] as follows:

For a tube of type A in Theorem A we have the following.

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2$) or four(otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}cot\left(\sqrt{8}r\right)\ ,\ \beta = \sqrt{2}cot\left(\sqrt{2}r\right)\ ,\ \lambda = -\sqrt{2}tan\left(\sqrt{2}r\right),\ \mu = 0$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2,\ m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathfrak{C}^\perp\xi = \mathfrak{C}^\perp N, \\ T_\lambda &= \{X \mid X\perp\mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X \mid X\perp\mathbb{H}\xi, JX = -J_1X\}. \end{aligned}$$

Then for such a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ we may put $\xi = \xi_1, \phi_1\xi, \phi_2\xi, \phi_3\xi \in \mathfrak{D}^\perp$. So $\xi \in T_\alpha$ and $\xi_2, \xi_3 \in T_\beta$.

In paper [**3**] we have proved that the shape operator A of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, that is, the Reeb flow on M is isometric. Then naturally the tube satisfies the formula (*).

Now let us check whether such kind of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ have η-parallel shape operator or not. Then by the expression for the shape operator A given in [**3**] we know the following for any $X, Y, Z \in T_0$

$$g\big((\nabla_X A)Y, Z\big) = -\sum_{\nu=1}^{3}\{\eta_\nu(Y)g(\phi_\nu X, Z) - \eta_\nu(\phi Y)g(\phi\phi_\nu X, Z) \\ -2\eta_\nu(\phi X)g(\phi\phi_\nu Y, Z)\} \\ -\sum_{\nu=1}^{3}\big\{g(\phi_\nu X, Y)\eta_\nu(Z) + g(\phi_\nu\phi X, Y)g(\phi_\nu\xi, Z)\big\}.$$

From this, together with the formula (2.1), we know $g\big((\nabla_X A)Y, Z\big) = 0$ for any X, Y and $Z \in \mathfrak{D}$. Moreover, it can be easily proved that

$$g\big((\nabla_{\xi_2} A)\xi_2, \xi_2\big) = 0,\ g\big((\nabla_{\xi_2} A)\xi_2, \xi_3\big) = 0,\ g\big((\nabla_X A)\xi_2, \xi_3\big) = 0,$$

and $g\big((\nabla_{\xi_2} A)\xi_3, X\big) = 0$ for any $X \in \mathfrak{D}$. This means that the shape operator A of a tube over $G_2(\mathbb{C}^{m+1})$ is η-parallel.

We now turn to the main theorem. Let us suppose that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies the condition (*)

$$g\big((A\phi - \phi A)X, Y\big) = 0 \tag{3.1}$$

for any X, Y in $T_0 = \{X \in T_x M \mid X \perp \xi\}$.

From this, differentiating and using the formulas in section 2, we have for any X, Y and Z in T_0

$$\text{(3.2)}\quad g\big((\nabla_X A)Y, \phi Z\big) + g\big((\nabla_X A)Z, \phi Y\big) \\ = \eta(AY)g(X, AZ) + \eta(AZ)g(Y, AX) + g(X, A\phi Y)g(Z, V) + g(X, A\phi Z)g(Y, V).$$

On the other hand, by using the equation of Codazzi we have for any X, Y and Z in T_0

$$g\big((\nabla_X A)Y, \phi Z\big) - g\big((\nabla_Y A)X, \phi Z\big) \\ = \sum\nolimits_\nu\{\eta_\nu(X)g(\phi_\nu Y, \phi Z) - \eta_\nu(Y)g(\phi_\nu X, \phi Z) - 2g(\phi_\nu X, Y)\eta_\nu(\phi Z)\} \\ + \sum\nolimits_\nu\{\eta_\nu(\phi X)g(\phi_\nu Y, Z) - \eta_\nu(\phi Y)g(\phi_\nu X, Z)\}.$$

Then from this, taking the cyclic sum of (3.1), subtracting the third one from the sum of the first and the second formulas and using (3.2), we have

$$
\begin{aligned}
(3.3)\qquad 2g\big((\nabla_X A)Y, \phi Z\big) &- \textstyle\sum_\nu \{\eta_\nu(X)g(\phi_\nu Y, \phi Z) - \eta_\nu(Y)g(\phi_\nu X, \phi Z) \\
&- 2g(\phi_\nu X, Y)\eta_\nu(\phi Z)\} \\
&- \textstyle\sum_\nu \{\eta_\nu(\phi X)g(\phi_\nu Y, Z) - \eta_\nu(\phi Y)g(\phi_\nu X, Z)\} \\
&+ \textstyle\sum_\nu \{\eta_\nu(X)g(\phi_\nu Z, \phi Y) - \eta_\nu(Z)g(\phi_\nu X, \phi Y) \\
&- 2g(\phi_\nu X, Z)\eta_\nu(\phi Y)\} \\
&+ \textstyle\sum_\nu \{\eta_\nu(\phi X)g(\phi_\nu Z, Y) - \eta_\nu(\phi Z)g(\phi_\nu X, Y)\} \\
&+ \textstyle\sum_\nu \{\eta_\nu(Y)g(\phi_\nu Z, \phi X) - \eta_\nu(Z)g(\phi_\nu Y, \phi X) \\
&- 2g(\phi_\nu Y, Z)\eta_\nu(\phi X)\} \\
&+ \textstyle\sum_\nu \{\eta_\nu(\phi Y)g(\phi_\nu Z, X) - \eta_\nu(\phi Z)g(\phi_\nu Y, X)\} \\
&= 2\eta(AZ)g(AX, Y) \\
&+ 2g(X, V)g(Y, A\phi Z) + 2g(Y, V)g(X, A\phi Z),
\end{aligned}
$$

where we have used the condition (3.1) and the formula $g(\phi\phi_\nu X, Z) = g(\phi_\nu \phi X, Z)$ for any X, Z in T_0. Then by direct calculations we assert the following

$$
\begin{aligned}
(3.4)\quad & g\big((\nabla_X A)Y, \phi Z\big) + \textstyle\sum_\nu \eta_\nu(Y)g(\phi_\nu X, \phi Z) + \sum_\nu g(\phi_\nu X, Y)\eta_\nu(\phi Z) \\
& - 2\textstyle\sum_\nu \eta_\nu(\phi X)g(\phi_\nu Y, Z) - \sum_\nu \eta_\nu(Z)g(\phi_\nu X, \phi Y) - \sum_\nu g(\phi_\nu X, Z)\eta_\nu(\phi Y) \\
& = \eta(AZ)g(AX, Y) + g(X, V)g(Y, A\phi Z) + g(Y, V)g(X, A\phi Z)
\end{aligned}
$$

for any X, Y and Z in T_0. Replacing Z by ϕZ in T_0, we have

$$
\begin{aligned}
(3.5)\qquad g\big((\nabla_X A)Y, Z\big) = \mathfrak{S}_{X,Y,Z}\, g(AX, Y)g(Z, V) &- \textstyle\sum_\nu \eta_\nu(Y)g(\phi_\nu X, Z) \\
&- \textstyle\sum_\nu g(\phi_\nu X, Y)\eta_\nu(Z) - 2\sum_\nu \eta_\nu(\phi X)g(\phi_\nu Y, \phi Z) \\
&- \textstyle\sum_\nu \eta_\nu(\phi Z)g(\phi_\nu X, \phi Y) - \sum_\nu g(\phi_\nu X, \phi Z)\eta_\nu(\phi Y),
\end{aligned}
$$

where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum of the formula with respect to X, Y and Z.

Now let us assume that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has η-parallel second fundamental tensor. Then (3.5) gives that

(3.6)

$$
\begin{aligned}
\mathfrak{S}_{X,Y,Z}\, g(AX, Y)g(Z, V) &= \textstyle\sum_\nu \eta_\nu(Y)g(\phi_\nu X, Z) + \sum_\nu g(\phi_\nu X, Y)\eta_\nu(Z) \\
&+ 2\textstyle\sum_\nu \eta_\nu(\phi X)g(\phi_\nu Y, \phi Z) + \sum_\nu \eta_\nu(\phi Z)g(\phi_\nu X, \phi Y) \\
&+ \textstyle\sum_\nu g(\phi_\nu X, \phi Z)\eta_\nu(\phi Y).
\end{aligned}
$$

Now in order to give our result we are going to prove the following:

PROPOSITION 3.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying the conditions (*) and (**). If the distribution $\mathfrak{D}$ is A-invariant, then $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.*

PROOF: Now let us suppose that $\xi = X_1 + X_2$ for some $X_1 \in \mathfrak{D}$ and $X_2 \in \mathfrak{D}^\perp$. Then $A\xi = AX_1 + AX_2$. This implies

$$\phi A\xi = \phi AX_1 + \phi AX_2. \tag{3.7}$$

Now let us construct a subbundle $\mathfrak{F} = \{X \in T_0 \cap \mathfrak{D} \mid \phi X \in \mathfrak{D}\}$. Then the subbundle $\mathfrak{F}$ is invariant by the structure tensor ϕ. That is, for any $X \in \mathfrak{F}$ we know ϕX also belongs to $\mathfrak{F}$. By using this fact in (3.6), we have the following

$$g(AX,Y)g(Z,V) + g(AY,Z)g(X,V) + g(AZ,X)g(Y,V) = 0.$$

From this, substituting (3.7) and using the fact that the distribution $\mathfrak{D}$ is A-invariant, we have

$$g(AX,Y)g(\phi Z, AX_1) + g(AY,Z)g(\phi X, AX_1) + g(AZ,X)g(\phi Y, AX_1) = 0$$

for any X, Y and Z in $\mathfrak{F}$. Then by putting $Y = Z = X$ in $\mathfrak{F}$ we have

$$g(AX,X)g(\phi X, AX_1) = 0.$$

From this and linearisation we are able to assert that

$$g(AX,Y) = 0 \text{ or } g(\phi X, AX_1) = 0$$

for any $X, Y \in \mathfrak{F}$. These two cases are similar. So let us consider the second case as follows:

By virtue of the A-invariance of the distribution $\mathfrak{D}$, we know that

$$AX_1 \in \mathfrak{D}.$$

On the other hand, since $\phi X \in \mathfrak{F}$, we are able to put $AX_1 \in \mathfrak{D}$ in such a way that

$$AX_1 = a\xi + \sum_i \lambda_i \xi_i + \sum_i \mu_i \phi\xi_i + Y_0,$$

for some $Y_0 \in \mathfrak{D}$ orthogonal to the subbundle $\mathfrak{F}$. From this formula, the A-invariance of the distribution $\mathfrak{D}$ gives that all $\lambda_i = 0$, $i = 1,2,3$. Then we know that the formula

$$a\xi + \sum_i \mu_i \phi_i \xi + Y_0 = aX_1 + aX_2 + \sum_i \mu_i \phi_i X_1 + \sum_i \mu_i \phi_i X_2 + Y_0 \tag{3.8}$$

belongs to the distribution $\mathfrak{D}$. From this, taking an inner product with $X_2 \in \mathfrak{D}^\perp$, then we have

$$0 = ag(X_2, X_2) = a.$$

Then we may put

$$AX_1 = \sum_i \mu_i \phi_i X_1 + \sum_i \mu_i \phi_i X_2 + Y_0,$$

where the left side, and the first and the third terms in the right side belong to the distribution $\mathfrak{D}$.

On the other hand, we know that

$$\sum_i \mu_i \phi_i X_2 \in \mathfrak{D}^\perp.$$

Then it follows that

$$\sum_i \mu_i \phi_i X_2 \in \mathfrak{D} \cap \mathfrak{D}^\perp = 0.$$

Moreover, from this expression it follows that the vectors $\phi_1 X_2$, $\phi_2 X_2$ and $\phi_3 X_2$ cannot be linearly independent vectors, because $X_2 \in \mathfrak{D}^\perp$. So the coefficients μ_i, $i = 1, 2, 3$ cannot be simultaneously vanishing. From this, if we put $X_2 = \xi_2 \in \mathfrak{D}^\perp$, we know that

$$\mu_1 \xi_3 - \mu_3 \xi_1 = 0.$$

This is in contradiction to $\dim \mathfrak{D}^\perp = 3$. Accordingly, we assert that $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. ∎

Now let us suppose that the distribution $\mathfrak{D}$ is invariant by the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Then we consider the following two cases:

Case I. $\xi \in \mathfrak{D}$ and ξ is not principal.

Then by the A-invariancy of the distribution $\mathfrak{D}$ we know

$$A\xi = \alpha\xi + \beta U \in \mathfrak{D}.$$

So the vector $U \in \mathfrak{D}$. Then it follows that the vector $V = \phi A\xi = \beta\phi U$ is orthogonal to $\phi\xi_1, \phi\xi_2$ and $\phi\xi_3$ for a non-vanishing function $\beta \neq 0$ on $\mathfrak{U}$. The formula (3.6), together with $V = Z$ in (3.6), gives that

$$\begin{aligned} g(AX, Y)g(V, V) + g(AY, V)g(X, V) + g(AZ, X)g(Y, V) \\ &= \sum_\nu \eta_\nu(V) g(\phi_\nu X, Y) + \sum_\nu \eta_\nu(\phi V) g(\phi_\nu X, \phi Y) \\ &= \beta \sum_\nu g(\phi\xi_\nu, U) g(\phi_\nu X, Y) \end{aligned}$$

https://doi.org/10.1017/S0004972700038934 Published online by Cambridge University Press

for any $X, Y \in \mathfrak{D}$ orthogonal to $\phi\xi_1, \phi\xi_2$ and $\phi\xi_3$. From this, putting $X = Y = V$ and using $V = \phi A\xi$ orthogonal to $\phi\xi_1, \phi\xi_2$ and $\phi\xi_3$, we have

$$2g(AX, V)g(V, V) + g(AV, V)g(X, V) = 0$$

and

$$g(AV, V)g(V, V) = 0.$$

Since the structure vector ξ is not principal, we have $g(AX, V) = 0$, and finally

$$g(AX, Y) = 0$$

for any $X, Y \in \mathfrak{D}$ orthogonal to $\phi\xi_1, \phi\xi_2$ and $\phi\xi_3$.

From the assumption we know that

$$g\big((A\phi - \phi A)X, \xi_i\big) = 0$$

for any $X \in T_0$ and $\xi_i \in \mathfrak{D}^\perp$. Putting $X = \phi\xi_j \in T_0$, we have

$$g(A\phi\xi_i, \phi\xi_j) = g(A\xi_i, \xi_j) = \alpha_i\delta_{ij}.$$

Then we are able to consider the following subcases.

SUBCASE I.1. U is orthogonal to $\phi_1\xi, \phi_2\xi$ and $\phi_3\xi$.

Then if we take an orthonormal basis $\{\xi_1, \xi_2, \xi_3, \xi, U, \phi U, \phi_1\xi, \phi_2\xi, \phi_3\xi\}$ and any vectors X in T_xM, $x \in M$ orthogonal to this basis, the shape operator of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$A = \begin{bmatrix} B & & & 0 \\ & C & & \\ & & B & \\ 0 & & & 0 \end{bmatrix},$$

where the matrices B and C are given in such a way that

$$B = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now for any $X, Y, Z \in T_0$ the formula (3.6) gives that

$$\begin{aligned} g(AX,Y)g(Z,V) &+ g(AY,Z)g(X,V) + g(AZ,X)g(Y,V) \\ &= \sum_\nu \eta_\nu(Y)g(\phi_\nu X,Z) + \sum_\nu \eta_\nu(Z)g(\phi_\nu X,Y) \\ &\quad + 2\sum_\nu \eta_\nu(\phi X)g(\phi_\nu Y,\phi Z) \\ &\quad + \sum_\nu \eta_\nu(\phi Z)g(\phi_\nu X,\phi Y) + \sum_\nu g(\phi_\nu X,\phi Z)\eta_\nu(\phi Y). \end{aligned} \tag{3.9}$$

Now from this, we are going to prove that $\alpha_i = 0$, $i = 1, 2, 3$. That is, the matrix B should be zero.

In fact, by substituting $X = \xi_1, Y = \xi_1$ and $Z = V$ we have

$$g(A\xi_1, \xi_1)g(V,V) = \alpha_1 g(V,V) = 0.$$

Next we are able to show that the function $\gamma = g(AU,U) = 0$ on the open set $\mathfrak{U}$ in M.

In fact, by putting $X = Y = U$ and $Z = V$ in (3.6) we have

$$g(AU,U)g(V,V) = \gamma\beta = 0.$$

Otherwise by putting $X = U$ and $Y \in \mathfrak{D}$ orthogonal to $\xi, U, \phi U, \phi_1\xi, \phi_2\xi, \phi_3\xi$ and $Z = V$ in (3.6) we know that

$$g(AU,Y) = 0$$

for any $Y \in \mathfrak{D}$ orthogonal to $\{\xi, U, \phi U, \phi_1\xi, \phi_2\xi, \phi_3\xi\}$. Moreover, we know that

$$\begin{aligned} g(AU,\phi U) &= -g(\phi AU, U) \\ &= g(U, A\phi U) = g(U, \phi AU) = g(\phi AU, U), \end{aligned}$$

where we have used the formula (*) in the third equality. From this we know $g(AU, \phi U) = 0$, which gives the matrix C. Summing up this situation, the shape operator A is given by

$$A = \begin{bmatrix} \alpha & \beta & & & & \\ \beta & 0 & & & \mathbf{0} & \\ & & 0 & & & \\ & & & 0 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

for the basis $\{\xi, U, \phi U, \xi_1, \xi_2, \xi_3, \phi_1\xi, \phi_2\xi, \phi_3\xi\}$ of M in $G_2(\mathbb{C}^{m+2})$. Then on a distribution $T_0(x) = \{X \in T_xM \mid X \perp \xi\}$ the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ satisfying the formula (*) and (**) is given by

$$\begin{cases} A\xi &= \alpha\xi + \beta U, \\ AU &= \beta\xi, \\ AX &= 0 \end{cases}$$

for any X orthogonal to ξ and U. From such an expression for the shape operator we know that the distribution $T_0(x)$ is integrable.

On the other hand, Chen and Nagano [5] showed that the maximal totally geodesic submanifolds of $G_2(\mathbb{C}^{m+2})$ are

$$G_2(\mathbb{C}^{m+1}),\ CP^m,\ CP^k \times CP^{m-k}\ (1 \le k \le [m/2]),\ G_2(R^{m+2})$$

and $\mathbb{H}P^n$ (if $m = 2n$). Among them the totally geodesic submanifold in $G_2(\mathbb{C}^{m+2})$ with maximal dimension $4(m-1)$ is $G_2(\mathbb{C}^{m+1})$. Then the integral submanifold is a complex hypersurface with the distribution T_0 given by

$$T_0(x) = T_x(G_2(\mathbb{C}^{m+1})) \oplus U \oplus \phi U,$$

where

$$\dim G_2(\mathbb{C}^{m+1}) = 4(m-1) = \dim G_2(\mathbb{C}^m) - \dim\{N, \xi, U, \phi U\}$$

and N denotes the unit normal to M in $G_2(\mathbb{C}^{m+2})$.

SUBCASE I.2. $U = \phi\xi_1$ is orthogonal to $\phi_2\xi$ and $\phi_3\xi$.

In this case we may put

$$A\xi = \alpha\xi + \beta\phi_1\xi.$$

By using a similar method to that given in Subcase I.1 we are going to prove that

$$g(AX, Y) = 0$$

for any $X, Y \perp \xi, U = \phi_1\xi$. Then for an orthonomal basis $\{\xi_1, \xi_2, \xi_3, \xi, \phi_1\xi, \phi_2\xi, \phi_3\xi\}$ and any vectors X in T_xM, $x \in M$ orthogonal to this basis, the shape operator A of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$A = \begin{bmatrix} D & & & 0 \\ & E & & \\ & & 0 & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix},$$

https://doi.org/10.1017/S0004972700038934 Published online by Cambridge University Press

where the matrices D and E are given in such a way that

$$D = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

and

$$E = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{bmatrix}.$$

Now if we put $X = \xi_1, Y = \xi_2$ and $Z = V$ in (3.6), we have

$$g(A\xi_1, \xi_1)g(V, V) = \alpha_1 g(V, V) = 0,$$

and similarly by putting $X = \xi_2, Y = \xi_2$ (respectively $X = \xi_2, Y = \xi_2$) and $Z = V$ in (3.6) we know the following respectively

$$g(A\xi_2, \xi_2)g(V, V) = g(A\xi_3, \xi_3)g(V, V) = 0,$$

which means $\alpha_2 = \alpha_3 = 0$ in this Subcase. In such a case, the integral submanifold is foliated by a complex hypersurface with the distribution

$$T_0(x) = T_x\big(G_2(\mathbb{C}^{m+1})\big) \oplus \phi\xi_1 \oplus \xi_1.$$

CASE II. $\xi \in \mathfrak{D}$ and ξ is principal.

Then in this case by Theorem A due to Berndt and Suh [**2**] we assert that M is locally congruent to a tube over totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ or a tube over a totally real totally geodesic $\mathbb{H}P^n$, $m = 2n$ in $G_2(\mathbb{C}^{m+2})$. If M is locally congruent to a tube over $G_2(\mathbb{C}^{m+1})$, then its shape operator A commutes with the structure tensor ϕ (see Berndt and the second author [**3**]). From such a view point we know that this type of hypersurface satisfies all the assumptions in our main theorem.

But when M is congruent to a tube over a totally real totally geodesic $\mathbb{H}P^n$, $m = 2n$ in $G_2(\mathbb{C}^{m+2})$, the shape operator A satisfies the following:

For any $X \in T_{\cot r}$ we know that $A\phi X = \tan r\phi X$, where $T_{\cot r}$ denotes the eigen space of M with eigenvalue $\cot r$. Then if this type satisfies the assumption (*), we have

$$g\big((A\phi - \phi A)X, Y\big) = (\tan r - \cot r)g(\phi X, Y) = 0,$$

which gives a contradiction. So this type of real hypersurface cannot occur.

CASE III. $\xi \in \mathfrak{D}^\perp$ and ξ is not principal.

Since we have assumed that ξ is not principal, we may put

$$A\xi = \alpha\xi + \beta U.$$

From this, together with the A-invariance of the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$, we have $U \in \mathfrak{D}^\perp$. Moreover, $\phi A\xi = \beta\phi U \in \mathfrak{D}^\perp$ and $\{\xi_1, \xi_2, \xi_3, \phi_1\xi, \phi_2\xi, \phi_3\xi\} \in \mathfrak{D}^\perp$.

Now if we put $V = Z = \phi A\xi$ into (3.6) and use the above properties, we have for any $X, Y \in \mathfrak{D}$

$$g(AX,Y)g(V,V) = \sum_\nu \eta_\nu(V)g(\phi_\nu X,Y) + \sum_\nu \eta_\nu(\phi V)g(\phi_\nu X, \phi Y).$$

Then by taking skew-symmetric part we have

$$\eta_1(V)g(\phi_1 X,Y) + \eta_2(V)g(\phi_2 X,Y) + \eta_3(V)g(\phi_3 X,Y) = 0,$$

where we have used the formula (2.6) and the symmetric property

$$\begin{aligned} g(\phi_\nu X, \phi Y) &= -g(\phi\phi_\nu X, Y) \\ &= -g(\phi_\nu\phi X, Y) \\ &= g(\phi X, \phi_\nu Y). \end{aligned} \tag{3.10}$$

Then by putting $Y = \phi_i X \in \mathfrak{D}$, $i = 1,2,3$, respectively, we have $\eta_i(V) = 0$, $i = 1,2,3$. This means that $\eta_i(V) = \beta g(\xi_i, \phi U) = 0$. Since the function $\beta \neq 0$ on an open set $\mathfrak{U} = \{p \in M \mid \beta(p) \neq 0\}$, the vector $\phi U \in \mathfrak{D}$. But we already know that $\phi A\xi = \beta U \in \mathfrak{D}^\perp$. This implies $\phi U = 0$, that is, the vector U should be zero, which gives a contradiction. Accordingly, we conclude that this case cannot occur.

CASE IV. $\xi \in \mathfrak{D}^\perp$ and ξ is principal.

Then in such a case we may put $\xi = \xi_1 \in \mathfrak{D}^\perp$. Moreover, by virtue of Theorem A due to Berndt and the present author [3] a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Moreover this type of hypersurface satisfies both formulas (*) and (**).

Then summing up all of Cases I, II, III and IV mentioned above, we give a complete proof of our main theorem in the introduction. □

REFERENCES

[1] J. Berndt, 'Real hypersurfaces in quaternionic space forms', *J. Reine Angew. Math.* **419** (1991), 9–26.

[2] J. Berndt and Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians', *Monatsh. Math.* **127** (1999), 1–14.

[3] J. Berndt and Y.J. Suh, 'Isometric flows on real hypersurfaces in complex two-plane Grassmannians', *Monatsh. Math.* **137** (2002), 87–98.

[4] T.E. Cecil and P.J. Ryan, 'Focal sets and real hypersurfaces in complex projective space', *Trans. Amer. Math. Soc.* **269** (1982), 481–499.

[5] B.Y. Chen and T. Nagano, 'Totally geodesic submanifolds of symmetric spaces II', *Duke Math. J.* **45** (1978), 405–425.

[6] M. Kimura, 'Real hypersurfaces and complex submanifolds in complex projective space', *Trans. Amer. Math. Soc.* **296** (1986), 137–149.

[7] A. Martinez and J.D. Pérez, 'Real hypersurfaces in quaternionic projective space', *Ann. Math. Pura Appl.* **145** (1986), 355–384.

[8] S. Montiel and A. Romero, 'On some real hypersurfaces of a complex hyperbolic space', *Geom. Dedicata* **20** (1986), 245–261.

[9] M. Okumura, 'On some real hypersurfaces of a complex projective space', *Trans. Amer. Math. Soc.* **212** (1975), 355–364.

[10] Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator', *Bull. Austral. Math. Soc.* **68** (2003), 493–502.

[11] Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator', *Bull. Austral Math. Soc.* **69** (2003), 379–393.

[12] Y.J. Suh, 'Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives', *Canad. Math. Bull.* **49** (2006), 134–143.

[13] Y.J. Suh, 'Real hypersurfaces of type B in complex two-plane Grassmannians', *Monatsh. Math.* **147** (2006), 337–355.

[14] K. Yano and M. Kon, *CR-submanifolds of Kaehlerian and Sasakian manifolds* (Birkhäuser, Boston, Basel, Strutgart, 1983).

Department of Mathematics
Kyungpook National University
Taegu 702-701
Korea
e-mail: yjsuh@mail.knu.ac.kr

https://doi.org/10.1017/S0004972700038934 Published online by Cambridge University Press