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# MEASURE, COMPACTIFICATION AND REPRESENTATION

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**Introduction.** The theory of measure on topological spaces has in recent years found its most natural setting in the study of pavings and measures on such pavings (see e.g. [**1–3**; **5**; **6**; **10**; **19**; **22**; **32**; **33**]. In this setting the relationship between measure and topology crystallizes since one concentrates primarily on the simpler internal lattice structure associated with sublattices of the topology rather than on the more complex topological structure.

Particularly simple in this context are the two valued lattice regular measures. When one restricts attention to these lattice regular measures, one has at one's disposal a very powerful tool for studying many topological questions as well as a natural setting for obtaining simple proofs of many difficult integral representation theorems.

In this paper we establish the basic results in the first three sections. We then apply the results to get a variety of extensions of the main representation theorem of Alexandroff [**2**] including a very quick proof of the main theorem of Kirk [**21**]. (An even quicker proof gives a modification of this appearing in [**22**].) When our results are applied to topological questions, we obtain many theorems in the theory of compactification including results and generalizations of results of Alo Shapiro [**4**], Banaschewski [**7**], Biles [**8**], Evstigneev [**13**], Fomin and Iliades [**14**], Mrowka [**26**], Varadarajan [**34**], Wallman [**35**] and others, as well as a quick and analytically simpler measure theoretic proof of the main theorem of [**30**]. Further corollaries follow easily.

In the course of development, we obtain in a simple manner a (known) characterization of a class of function algebras important in measure and topological questions which brings together the work of several authors. The equivalence and relationships between several different approaches to the study of Wallman compactifications follows, giving a complete explanation surrounding a variety of recent methods for constructing all Hausdorff compactifications (see e.g. [**9**; **13**; **31**]).

**1. Preliminaries.** By a *paving* $(X, \mathscr{L})$ we will mean a set $X$ together with a sublattice $\mathscr{L}$ of the power set of $X$, such that $\emptyset, X \in \mathscr{L}$. If $\mathscr{L}$ is closed under countable intersections then $(X, \mathscr{L})$ is called a *delta paving*. We say that the paving $(X, \mathscr{L})$ is

(a) *separating* if whenever $x, y \in X$ and $x \neq y$, there is an $A \in \mathscr{L}$ such that $x \in A$ and $y \notin A$;

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(b) *disjunctive* if whenever $x, y \in X$ and $x \notin A \in \mathscr{L}$ there is a $B \in \mathscr{L}$ such that $x \in B$ and $B \cap A = \emptyset$;

(c) *normal* if whenever $A, B \in \mathscr{L}$ and $A \cap B = \emptyset$, there exist $C, D \in \mathscr{L}$ with $A \subset C', B \subset D'$ and $C' \cap D' = \emptyset$. (Here the prime denotes complement.) If $(X, \mathscr{L})$ is separating, disjunctive and normal then $(X, \mathscr{L})$ is called a *strongly normal paving* and in this case $\mathscr{L}$ is called a *Wallman base*. We may always form the space $W(\mathscr{L})$ of all lattice ultrafilters as in [**35**] and topologize $W(\mathscr{L})$ with the Wallman topology having as a base for the closed sets, sets of the form $W(A) = \{\mathscr{F} \in W(\mathscr{L}): A \in \mathscr{F}\}$ .With this topology $W(\mathscr{L})$ becomes a compact $T_1$ space. If $(X, \mathscr{L})$ is a separating and disjunctive lattice $W(\mathscr{L})$ is actually a compactification of $X$ where $X$ carries the topology having $\mathscr{L}$ as a base for its closed sets. If in addition $(X, \mathscr{L})$ is normal then $W(\mathscr{L})$ is a Hausdorff compactification of $X$. (It should be noted that an outstanding problem in topology is whether every Hausdorff compactification is $W(\mathscr{L})$ for some Wallman base. Considerable effort has been expended in trying to answer this question. For more about this and the basic properties of $W(\mathscr{L})$ the reader may consult [**29**].) Convergence of nets in $W(\mathscr{L})$ will be denoted as follows: $x_\alpha \rightarrow x(W)$. More generally if $X$ is a topological space with topology $\tau$, convergence of nets will be denoted by $x_\alpha \rightarrow x(\tau)$.

A bounded real valued function defined on $X$ is called $\mathscr{L}$-*continuous* (or just *continuous* in the sense of Alexandroff) if $f^{-1}(C) \in \mathscr{L}$ for every closed set $C \subset R$ (the real line). $C_b(\mathscr{L})$ will denote the collection of all bounded $\mathscr{L}$-continuous functions.

THEOREM 1.1. *If $(X, \mathscr{L})$ is a strongly normal delta paving then $C_b(\mathscr{L})$ consists precisely of the restrictions of the continuous functions of $W(\mathscr{L})$ to $X$. The unique continuous extension of $f$ to $W(\mathscr{L})$ is given by $f(\mathscr{F}) = \lim f(\mathscr{F})$.*

*Proof.* See [**30**].

According to Frink [**15**], $C_b(\mathscr{L})$ in this case consists of all the "$\mathscr{L}$-uniformly continuous functions" and, as is easy to see, these are nothing more than the collection of uniformly continuous functions when $X$ carries the weak uniformity generated $C_b(\mathscr{L})$.

If $\mathscr{A}(\mathscr{L})$ is the algebra of sets generated by $\mathscr{L}$, then by an $\mathscr{L}$-*regular measure* we will mean a bounded, real valued, finitely additive set function $\mu$ defined on $\mathscr{A}(\mathscr{L})$, such that $\mu(\emptyset) = 0$, and which satisfies the following regularity condition:

$$\mu(E) = \sup\{\mu(L): L \subset E, L \in \mathscr{L}\} = \inf\{\mu(L'); L' \supset E, L \in \mathscr{L}\}.$$

The collection of these measures will be denoted by $MR(\mathscr{L})$. Properties of $MR(\mathscr{L})$ and $C_b(\mathscr{L})$ were studied extensively in [**1–3**] and one may consult those papers for a more complete discussion.

THEOREM 1.2. *If $(X, \mathscr{L})$ is a normal delta paving then the dual $C_b(\mathscr{L})^*$, of $C_b(\mathscr{L})$, is $MR(\mathscr{L})$. More precisely to each bounded linear functional $\Phi$ defined on $C_b(\mathscr{L})$, there is a unique measure $\mu \in MR(\mathscr{L})$ such that $\Phi(f) = \int f\, d\mu$ for* *each* $f \in C_b(\mathscr{L})$. *Moreover for each* $A \in \mathscr{L}$, $\mu(A) = \inf\{\Phi(f): f \in C_b(\mathscr{L}), K_A \leqq f \leqq 1\}$. (*Here* $K_A$ *denotes the characteristic function of* $A$.) *Furthermore,* $\|\Phi\| = |\mu|$ (*the total variation of* $\mu$) *and if* $\Phi$ *is nonnegative* (i.e. $f \geqq 0 \Rightarrow \Phi(f) \geqq 0$), *then the measure* $\mu$ *is nonnegative.*

The above theorem is the main representation theorem of Alexandroff [**2**]. Several generalizations of this will be given later.

A measure $\mu \in MR(\mathscr{L})$ is called *two valued* if $\mu$ takes on only the values 0 and 1. The collection of the two valued (regular) measures will be denoted by $IR(\mathscr{L})$.

If $(X, \mathscr{L}_1)$ and $(X, \mathscr{L}_2)$ are two pavings we say that $\mathscr{L}_1$ *separates* $\mathscr{L}_2$ if whenever $A, B \in \mathscr{L}_2$ and $A \cap B = \emptyset$, there exist $C, D \in \mathscr{L}_1$ with $A \subset C$, $B \subset D$ and $B \cap D = \emptyset$.

When we use the letter $F$ throughout this paper, it will denote a Banach algebra (under the sup norm) of bounded real valued functions on a set $X$. We will always assume that $F$ contains the constant function 1 (and hence all constant functions). $H_F$ will denote the structure space of $X$, that is, the collection of nonzero real homomorphisms of $F$, and $M_F$ will denote the collection of maximal ideals of $F$. Customarily, $H_F$ is topologized with the weak star ($w^*$) topology characterized by the following convergence of nets: $h_\alpha \rightarrow h(w^*)$ if and only if $h_\alpha(f) \rightarrow h(f)$ for all $f \in F$. It is well known that $H_F$ is a compact Hausdorff space and that if $F$ separates points of $X$ then $H_F$ is a compactification of $X$. Generally, $M_F$ is given the hull kernel topology having as a base for the closed sets, sets of the form $C(f) = \{M \in M_F: f \in M\}$. If $F$ separates points of $X$ then $H_F$ with the weak star topology and $M_F$ with the hull kernel topology are homeomorphic, as is well known. The structure space of $C_b(\mathscr{L})$ will be denoted by $\mathscr{H}$. If $R$ is any ring of real valued functions, we will denote by $\mathscr{Z}(R)$ the collection of zero sets of $R$, i.e. sets of the form $f^{-1}\{0\}$, $f \in F$. In the special case $C_b(\mathscr{L})$, $\mathscr{Z}(C_b(\mathscr{L}))$ will be denoted by $\mathscr{Z}$. If $X$ is a topological space, then $C(X)$ will denote the Banach algebra of bounded real valued functions defined on $X$ with the sup norm.

**2.** In this section we establish the basic correspondence between $IR(\mathscr{L})$ and $W(\mathscr{L})$ and some basic facts about $IR(\mathscr{L})$.

THEOREM 2.1. *There is a* 1-1 *correspondence between points of* $IR(\mathscr{L})$ *and points of* $W(\mathscr{L})$. *This correspondence is achieved by associating with each* $\mu \in IR(\mathscr{L})$ *the lattice ultrafilter* $\mathscr{F}(\mu) = \{A \in \mathscr{L}: \mu(A) = 1\}$.

*Proof.* Suppose $\mu \in IR(\mathscr{L})$. Clearly $\mathscr{F}(\mu)$ is a filter. To show that $\mathscr{F}(\mu)$ is maximal suppose there is an $\mathscr{H} \in W(\mathscr{L})$ such that $\mathscr{H} \supsetneq \mathscr{F}(\mu)$. Then there is an $L \in \mathscr{L}$ such that $L \in \mathscr{H} - \mathscr{F}(\mu)$ and thus $\mu(L) = 0$. By the regularity of $\mu$ there is an $L_0 \in \mathscr{L}$ such that $L \subset L_0'$ and $\mu(L_0') = 0$. It follows that $\mu(L_0) = 1$ and $L_0 \in \mathscr{F}(\mu) \subset \mathscr{H}$, contradicting $L \cap L_0 = \emptyset$. Thus $\mathscr{H} = \mathscr{F}(\mu)$ and $\mathscr{F}(\mu)$ is in $W(\mathscr{L})$.

Conversely, if $\mathscr{F} \in W(\mathscr{L})$ one can define the set function $\mu_1$ on $\mathscr{L}$ as follows: $\mu_1(\mathscr{L}) = 1$ if and only if $L \in \mathscr{F}$ and $\mu_1(L) = 0$ otherwise. $\mu_1$ is a finitely additive set function defined on $\mathscr{L}$ taking the empty set into 0. If $\mathscr{E}(\mathscr{F})$ represents the collection of those subsets $E$ of $X$ such that $E \supset L$ for some $L \in \mathscr{F}$ or $E \subset L'$ for some $L \in \mathscr{F}$, then $\mathscr{E}(\mathscr{F})$ is an algebra and contains $\mathscr{L}$ since $\mathscr{F}$ is an $\mathscr{L}$-ultrafilter. Hence $\mathscr{E}(\mathscr{F})$ contains $\mathscr{A}(\mathscr{L})$. It is now easy to see that the extension $\mu_{\mathscr{F}}$ of $\mu_1$ to $\mathscr{A}(\mathscr{L})$ given by $\mu_{\mathscr{F}}(E) = 1$ if $E \supset L \in \mathscr{F}$ and 0 otherwise is the required measure.

From the construction above it follows that if $\mu \in IR(\mathscr{L})$, $\mu = \mu_{\mathscr{F}(\mu)}$ and thus that the correspondence $\mu \rightarrow \mathscr{F}(\mu)$ is 1-1. Thus we have the following:

COROLLARY 2.2. *If $(X, \mathscr{L})$ is a paving and $IR(\mathscr{L})$ is given the Wallman topology, that is the topology having as a base for the closed sets, sets of the form $W(A) = \{\mu \in IR(\mathscr{L}): \mu(A) = 1\}$ where $A \in \mathscr{L}$ then $W(\mathscr{L})$ and $IR(\mathscr{L})$ are homeomorphic.*

*Proof.* The topology on $IR(\mathscr{L})$ is obviously the topology of transference.

*Remark* 2.3. It is easy to see that if $(X, \mathscr{L})$ is separating and disjunctive, then the correspondence in Theorem 2.1 takes a two valued measure concentrated at a point $p$ into the lattice ultrafilter of supersets of $p$. In general, however, we will not require that $(X, \mathscr{L})$ be separating and disjunctive and thus the homeomorphism of Corollary 2.2 is between compact spaces, not necessarily compactifications of $X$.

A very simple corollary of Theorem 2.1 is the following generalization of a theorem of Evstigneev [**13**] (see also in this connection [**4**, p. 46]).

COROLLARY 2.4. *A space is compact if and only if there is a separating disjunctive base $\mathscr{L}$ for the closed sets of $X$ such that every $\mu \in IR(\mathscr{L})$ is fixed*

In [**8**] Biles defines a Wallman ring to be a ring of real valued functions $A$ whose zero sets form a Wallman base. He singles out those ideals having the property that the zero sets of functions in them forms a $\mathscr{Z}(A)$ ultrafilter. He denotes this collection by $\mathscr{F}(A)$ and notes that $\mathscr{F}(A)$ is in 1-1 correspondence with $W(\mathscr{Z}(A))$. He topologizes $\mathscr{F}(A)$ with a base for the closed sets, sets of the form $C(f) = \{I \in \mathscr{F}(A): f \in I\}$. If one denotes for each $I \in \mathscr{F}(A)$, the two valued $\mathscr{Z}(A)$ regular measure associated with the zero sets of $I$ as in Theorem 2.1, by $\mu_I$, we see that $\mu_I(Z(f)) = 1$ if and only if $f \in I$. Thus it follows that $\mathscr{F}(A)$ carries the topology of transference from $IR(\mathscr{L})$ with the Wallman topology, under the 1-1 map $\mu_I \rightarrow I$. An immediate consequence of this observation is that $\mathscr{F}[A]$ is compact. But one has more, namely one has the following generalization of Theorem 2.2 [**8**].

COROLLARY 2.6. *If $A$ is a ring of real valued functions with identity and $\mathscr{F}(A)$ denotes the collection of ideals of $A$ whose zero sets are zero set ultrafilters, then $\mathscr{F}(A)$ with the topology having as a base for its closed sets, sets of the form $C(f) = \{I \in \mathscr{F}(A): f \in I\}$ is homeomorphic to $W(\mathscr{Z}(A))$.*

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One can go on with general applications of the above type but this will not be done here.

Many different uses of the correspondence of Theorem 2.1 occur in the literature. For example in [**34**], Varadarajan uses the correspondence with $\mathscr{L}$ being the lattice of zero sets of a Tychonoff space to study the Stone-Cech compactification. It is used in [**21**] to show that a certain "dual map" is injective. Certain subcollections of these two valued measures with $\mathscr{L}$ the lattice of closed sets were used in [**16**] to characterize the $\alpha$-complete spaces of Dykes [**11**], and were used in general very effectively in [**6**] to get a variety of old and new theorems concerning realcompactness and the general theory of repleteness. One can find similar correspondences elsewhere.

The following lemma will be useful later on and characterizes convergence in the Wallman topology on $IR(\mathscr{L})$.

LEMMA 2.7. *$\mu_\alpha \to \mu(W)$ if and only if $\mu_\alpha(A) \to \mu(A)$ whenever $\mu(A) = 0$.*

*Proof.* $\mu_\alpha \to \mu(W)$ if and only if $\mu_\alpha$ is eventually in every basic neighborhood $(W(A))'$ of $\mu$ where $A \in \mathscr{L}$. By the definition of $W(A)$, $\mu \in (W(A))'$ if and only if $\mu(A) = 0$. Thus $\mu_\alpha \to \mu(W)$ if and only if whenever $\mu(A) = 0$, $\mu_\alpha(A) \to \mu(A)$.

The next few lemmas are of critical importance for what follows but are of independent interest.

LEMMA 2.8. *If $(X, \mathscr{L}_1)$ and $(X, \mathscr{L}_2)$ are pavings and $\mathscr{L}_1 \subset \mathscr{L}_2$ then every $\mu \in IR(\mathscr{L}_1)$ extends to a $\nu \in IR(\mathscr{L}_2)$. If $\mathscr{L}_1$ separates $\mathscr{L}_2$ and a $\mu \in MR(\mathscr{L}_1)$ extends to a $\nu \in MR(\mathscr{L}_2)$ then the extension is unique.*

*Proof.* Suppose $\mu \in IR(\mathscr{L}_1)$, then by Zorn's lemma the $\mathscr{L}_1$-filter $F(\mu) = \{A \in \mathscr{L}_1: \mu(A) = 1\}$ extends to an $\mathscr{L}_2$ ultrafilter $\mathscr{H}$. If $\nu$ is the measure corresponding to $\mathscr{H}$ via Theorem 2.1 then by construction of $\nu$, $\nu$ extends $\mu$.

For the second part, suppose $\mu \in MR(\mathscr{L}_1)$ has two distinct extensions $\nu_1$ and $\nu_2$ to $MR(\mathscr{L}_2)$. Then there is an $A \in \mathscr{L}_2$ such that $\nu_1(A) \neq \nu_2(A)$. We may suppose without loss of generality that $\nu_2(A) - \nu_1(A) = \delta > 0$, and that both measures $\nu_1$ and $\nu_2$ are nonnegative. Choose $\epsilon < \delta$. By $\mathscr{L}_2$ regularity there is a $B \in \mathscr{L}_2$ such that $B' \supset A$ and such that $\nu_1(B') - \nu_1(A) < \epsilon$. By separation there exist $C, D \in \mathscr{L}_1$ such that $A \subset C$, $B \subset D$ and $C \cap D = \emptyset$. It follows that

$$\delta = \nu_2(A) - \nu_1(A) \leqq \nu_2(C) - \nu_1(A) \leqq \nu_1(D') - \nu_1(A) \leqq \nu_1(B') - \nu_1(A) < \epsilon,$$

a contradiction. Thus $\nu_1 = \nu_2$ and the proof is completed.

LEMMA 2.9. *If $(X, \mathscr{L}_1)$ and $(X, \mathscr{L}_2)$ are pavings such that $\mathscr{L}_1 \subset \mathscr{L}_2$ and $\mathscr{L}_1$ separates $\mathscr{L}_2$ then the restriction of $\nu \in MR(\mathscr{L}_2)$ to $\mathscr{A}(\mathscr{L}_1)$ is in $MR(\mathscr{L}_1)$.*

*Proof.* Again we may assume $\nu \geqq 0$ since a measure can be decomposed into its positive and negative parts. Call the restriction of $\nu$ to $\mathscr{A}(\mathscr{L}_1)$, $\mu$. If $E \in$

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$\mathscr{A}(\mathscr{L}_1)$; then as is well known, $E$ can be written as $E = \bigcup_1^n A_i - B_i$ where the union is disjoint, $A_i, B_i \in \mathscr{L}$, and $A_i \supset B_i$. Since $\mu$ is $\mathscr{L}_2$ regular there exist $B_i^* \in \mathscr{L}_2$, $i = 1, 2, 3, \ldots, n$, such that $B_i^* \subset A_i - B_i$ and $\mu(A_i - B_i) < \mu(B_i^*) + \epsilon/2^i$. By separation there are $C_i \in \mathscr{L}_1$ such that $B_i^* \subset C_i$ and $C_i \cap B_i = \emptyset$. Clearly $B_i^* \subset A_i \cap C_i \subset A_i - B_i$. It follows that

$$\mu(\bigcup A_i \cap C_i) = \sum \mu(A_i \cap C_i) \leqq \mu(E) = \sum \mu(A_i - B_i)$$
$$\leqq (\sum \mu(B_i^*)) + \epsilon \leqq \sum \mu(A_i \cap C_i) + \epsilon.$$

Thus $\mu$ is $\mathscr{L}_1$ regular.

LEMMA 2.11. *If $(X, \mathscr{L}_1)$ and $(X, \mathscr{L}_2)$ are pavings and $\mathscr{L}_1 \subset \mathscr{L}_2$, then the restriction map $\phi$: $IR(\mathscr{L}_2) \rightarrow IR(\mathscr{L}_1)$ given by $\phi(\nu)$ is the restriction of $\nu$ to $\mathscr{A}(\mathscr{L}_1)$, is well defined and* 1-1 *if $\mathscr{L}_1$ separates $\mathscr{L}_2$. If, in addition, $IR(\mathscr{L}_1)$ and $IR(\mathscr{L}_2)$ carry the Wallman topologies and $\mathscr{L}_1$ is normal, then the restriction map $\phi$ is a homeomorphism.*

*Proof.* The fact that $\phi$ is a bijection is immediate from the previous two lemmas. The continuity of $\phi$ is immediate when one looks at the preimage of a basic closed set. Since $IR(\mathscr{L}_2)$ is a compact space and $IR(\mathscr{L}_1)$ is a Hausdorff space, the map $\phi$ is a homeomorphism.

*Remark* 2.12. Lemma 2.11 generalizes many theorems in the literature. For example, if one takes $\mathscr{L}_1$ to be the zero sets of a Tychonoff space and $\mathscr{L}_2$ to be the lattice of closed sets, then $IR(\mathscr{L}_1)$ is the Stone-Cech compactification (see [**34**]) and $IR(\mathscr{L}_2)$ is the ordinary Wallman compactification [**35**]. If $X$ is a normal space, then Lemma 2.11 says that the Stone-Cech and the Wallman compactification coincide, since in this case clearly $\mathscr{L}_1$ separates $\mathscr{L}_2$ and $\mathscr{L}_1$ is a normal lattice as is well known. Again, if $\mathscr{L}_1$ is the lattice of clopen (= open and closed) sets and $\mathscr{L}_2$ the lattice of zero sets in a strongly zero dimensional Hausdorff space, then one has, since $\mathscr{L}_1$ separates $\mathscr{L}_2$ [**27**], that $IR(\mathscr{L}_1)$ is homeomorphic to $IR(\mathscr{L}_2)$. $IR(\mathscr{L}_1)$ in this case is the Banaschewski compactification [**7**] and again $IR(\mathscr{L}_2)$ is the Stone-Cech compactification. In a similar manner if one takes $\mathscr{L}_1 = \{\text{clopen sets}\}$ and $\mathscr{L}_2 = \{\text{open sets}\}$ in an extremally disconnected space (i.e. the closure of every open set is open), then one has that $IR(\mathscr{L}_1)$, the Banaschewski compactification, is homeomorphic to $IR(\mathscr{L}_2)$ (a compactification considered by Iliades and Fomin [**14**]).

*Remark* 2.13. One does not have to assume that $\mathscr{L}_1$ be normal in Lemma 2.11. The proof, however, proceeds along different lines. For a proof and further applications of these other results for two valued measures in an abstract lattice setting, see [**23**]. Further applications of Lemmas 2.8 and 2.9 will be given later.

**3. $\mathscr{H}$ and $IR(\mathscr{L})$.** In what follows $(X, \mathscr{L})$ will be a normal delta paving. According to the general theory, if $C_b(\mathscr{L})$ separates points of $X$, then $\mathscr{H}$ with the weak star topology, is the uniquely determined compactification of $X$ such that $C_b(\mathscr{L})$ consists precisely of the restrictions of continuous functions on $\mathscr{H}$.

In view of Theorem 1.1 then, we have that when $(X, \mathscr{L})$ is strongly normal, $W(\mathscr{L})$ and hence $IR(\mathscr{L})$ is homeomorphic to $\mathscr{H}$. In this section we exhibit a precise correspondence between $\mathscr{H}$ and $IR(\mathscr{L})$ in general, and show that when they carry the weak * and Wallman topologies this correspondence is a homeomorphism. We do not assume separation of points, and thus $\mathscr{H}$ and $W(\mathscr{L})$ are homeomorphic as compact spaces, not as compactifications of $X$. We also show that for proper subalgebras $F$ of $C_b(\mathscr{L})$, $H_F$ and $W(\mathscr{L})$ need not be homeomorphic even when both are Hausdorff compactifications of $X$. (Of course without normality of $\mathscr{L}$, $\mathscr{H}$ and $W(\mathscr{L})$ are definitely not homeomorphic since $W(\mathscr{L})$ is not even Hausdorff.)

LEMMA 3.1. *Every $h \in \mathscr{H}$ can be written uniquely as an integral with respect to a $\mu \in IR(\mathscr{L})$.*

*Proof.* It is well known that every $h \in \mathscr{H}$ is bounded and of norm 1 (see e.g. [**12**, p. 39]). Clearly $h$ is nonnegative since $C_b(\mathscr{L})$ contains square roots. Therefore, by Theorem 1.2, $h(f) = \int f\, d\mu$ for some unique nonnegative $\mu \in MR(\mathscr{L})$. It suffices to show that $\mu$ is two valued. However this follows immediately from the relationships

$$\mu(A) = \inf \{h(f) : K_A \leqq f \leqq 1\} = \inf \{h(f^2) : K_A \leqq f^2 \leqq 1\}$$
$$= (\inf \{h(g) : K_A \leqq g \leqq 1\})^2 = (\mu(A))^2.$$

LEMMA 3.2. *If $\mu \in IR(\mathscr{L})$ then the linear functional $h$ defined on $C_b(\mathscr{L})$ by $h(f) = \int f\, d\mu$ is a nonzero homomorphism.*

*Proof.* We may suppose that $f \geqq 0$ since for any $f \in C_b(\mathscr{L})$, $f = f \wedge 0 + f \vee 0$; $h(f) = \int f\, d\mu = \sup \sum_{i=1}^{i=n} (m_i \mu(E_i))$ where $m_i \leqq f(x)$, and the $E_i$ partition $X$. Since $\mu$ is two valued, exactly one $E_i$ has measure 1 and the rest have measure 0. Thus $h(f) = \sup m_i \mu(E_i)$. Also $(h(f))^2 = \sup m_i^2 (\mu(E_i))^2 = \sup (\sum m_i \mu(E_i))^2 = \int f^2\, d\mu = h(f^2)$. If $f$ and $g$ are $\geqq 0$, then $h(fg) = h(\frac{1}{4}(f+g)^2 - (f-g)^2) = h(f)h(g)$ and thus $h \in \mathscr{H}$.

*Remark* 3.3. (a) We did not use the normality in Lemma 3.2; thus Lemma 3.2 is true for any Banach algebra $F$ of real valued functions containing constants and contained in $C_b(\mathscr{L})$.

(b) If $(X, \mathscr{L})$ is normal and $F \subset C_b(\mathscr{L})$, the map $T: IR(\mathscr{L}) \to H_F$ given by $T(\mu) = h$ where $h(f) = \int f\, d\mu$ is not necessarily 1-1, and thus $H_F$ and $IR(\mathscr{L})$ (which is homeomorphic to $IR(\mathscr{Z})$ by Lemma 2.11) need not be homeomorphic via the map $T$. As an example, we may take $X$ to be the real line, $Y$ to be its one point compactification and $F$ to be the collection of restrictions of continuous real valued functions from $Y$ to $X$. Then clearly $F \subset C_b(\mathscr{Z}(F))$. $H_F$ is homeomorphic to $Y$ but $W(\mathscr{Z}(F))$ is homeomorphic to the Stone-Cech compactification of $X$. All of this is related to the fact that every Hausdorff compactification $Y$ of $X$ may be realized as a quotient space of $W(\mathscr{Z})$, where $\mathscr{Z}$ is the collection of zero sets of restrictions of continuous functions from $Y$ to $X$ (see e.g. [**9**] or [**31**] for a more direct proof).

It is customary in the study of topological measure theory, to topologize $MR(\mathscr{L})$, where $\mathscr{L}$ is a delta normal lattice, with the following *vague topology* $(V)$: $\mu_\alpha \to \mu(V)$ where $\mu_\alpha$, $\mu \in MR(\mathscr{L})$ if and only if $\int f\, d\mu_\alpha \to \int f\, d\mu$ for all $f \in C_b(\mathscr{L})$. (The assumption of normality is to guarantee that the limit of a sequence in this topology is unique.) This induces on $IR(\mathscr{L})$ the relative vague topology and the following is clear in view of Lemmas 3.1 and 3.2.

PROPOSITION 3.4. *If $(X, \mathscr{L})$ is a normal delta paving, then there is a* 1-1 *correspondence between $\mathscr{H}$ and $IR(\mathscr{L})$ given by $h \to \mu$ where $h(f) = \int f\, d\mu$. If $\mathscr{H}$ carries the weak * topology and $IR(\mathscr{L})$ the relative vague topology, then the correspondence is a homeomorphism.*

It remains to establish the coincidence of the Wallman topology on $IR(\mathscr{L})$ and the relative vague topology on $IR(\mathscr{L})$ when $(X, \mathscr{L})$ is a normal delta paving. For this we need the following well known theorem.

THEOREM 3.5 (Portmanteau). *The following are equivalent for a normal delta paving $(X, \mathscr{L})$.*

(1) $\mu_\alpha \to \mu(V)$, *where* $\mu_\alpha$, $\mu \in MR(\mathscr{L})$.

(2) $\mu_\alpha(X) \to \mu(X)$ *and* lim sup $\mu_\alpha(A) \leqq \mu(A)$, *for all* $A \in \mathscr{L}$.

(3) $\mu_\alpha(X) \to \mu(X)$ *and* lim inf $\mu_\alpha(A') \geqq \mu(A')$, *for all* $A \in \mathscr{L}$.

For a proof see, for example, [**3**, p. 180].

PROPOSITION 3.6. *If $(X, \mathscr{L})$ is a normal delta paving $\mu_\alpha \to \mu(V)$ where $\mu_\alpha$, $\mu \in IR(\mathscr{L})$ if and only if whenever $\mu(A) = 0$ where $A \in \mathscr{L}$, $\mu_\alpha(A) \to \mu(A)$.*

*Proof.* Suppose $\mu(A) = 0$ and $\mu_\alpha \to \mu(V)$. Then by Theorem 3.5 lim sup $\mu_\alpha(A) \leqq \mu(A) = 0$ which implies that $\mu_\alpha(A) = 0$ after some point and thus that $\mu_\alpha(A) \to \mu(A)$.

To prove the converse, note trivially, that lim sup $\mu_\alpha(A) \leqq \mu(A)$ when $\mu(A) = 1$ and by hypothesis lim sup $\mu_\alpha(A) = 0$ when $\mu(A) = 0$. Thus lim sup $\mu_\alpha(A) \leqq \mu(A)$ for all $A \in L$ and $\mu_\alpha \to \mu(V)$ by Theorem 3.5.

THEOREM 3.7. *If $(X, \mathscr{L})$ is a delta normal paving, then the relative vague topologies on $IR(\mathscr{L})$ and the Wallman topologies on $IR(\mathscr{L})$ coincide.*

*Proof.* This follows from the previous proposition and Lemma 2.7.

**4. Some applications.** Having established the basic correspondences, we give some immediate applications of the preceding theorems. The first theorem is an alternate proof of Theorem 1.1. It is, of course, clear from several points of view because of the preceding propositions but provides us with some useful observations.

THEOREM 4.1. *If $(X, \mathscr{L})$ is a strongly normal delta paving then $C_b(\mathscr{L})$ consists precisely of the restrictions of continuous functions on $IR(\mathscr{L})$.*

*Proof.* By Proposition 3.4 and Theorem 3.7, $IR(\mathscr{L})$ is homeomorphic to $\mathscr{H}$ via a homeomorphism which carries $\mu_x$ to $x^*$, the evaluation functional at $x$.

As is well known there is a basic isomorphism between $C_b(\mathscr{L})$ and $C(\mathscr{H})$. This isomorphism is given by $f \rightarrow f^*$, where $f^*(h) = h(f)$. Thus we have the series of isomorphisms $C_b(\mathscr{L}) \rightarrow C(\mathscr{H}) \rightarrow C(IR(\mathscr{L}))$. If $f \rightarrow f^* \rightarrow f^{**}$ via this series, we have $f^{**}(\mu_x) = f^*(x^*) = f(x)$. Thus $f^{**}$ "extends" $f$ and this is the desired result.

It follows from the above theorem using Theorem 1.1 that if $\mu \in IR(\mathscr{L})$ and $\mathscr{F}$ is the $\mathscr{L}$-ultrafilter associated with $\mu$ via Theorem 2.1, then $f^*(h) = h(f) = \int f \, d\mu = \lim f(F)$.

Many diverse corollaries follow from Theorem 4.1. One is the following (see [**30**]).

COROLLARY 4.2. *If $K$ is any Hausdorff compactification of a set $X$, and if the ring $F$ of the restrictions of continuous functions to $X$ have the property that*

(*) *whenever $Z_1 \cap Z_2 = \emptyset$, where $Z_1, Z_2 \in \mathscr{Z}(F)$, there is an $f \in F$ such that $f(Z_1) = 0$ and $f(Z_2) = 1$*

*then $K$ is homeomorphic to $W(Z(F))$. (A particular consequence; every compactification of a pseudocompact space is a Wallman compactification.)*

In [**21**] a Banach algebra $F$ (under the sup norm) of bounded, real valued functions defined on $X$ is said to be *$Z$-separating* if it has a unit element, separates points of $X$ and has property (*). It follows then from Theorem 1.1 and Corollary 4.2 (since $F$ consists of the restrictions of continuous functions from $H_F$ to $X$), that the $Z$-separating algebras of Kirk are nothing more than all the bounded $\mathscr{Z}(F)$ continuous functions. In fact, the following theorem is true.

THEOREM 4.3. *If $F$ is a Banach algebra of bounded real valued functions defined on a set $X$ containing constants and separating points of $X$, then the following are equivalent.*

(1) *$F$ consists of all the bounded $\mathscr{Z}(F)$-continuous functions.*

(2) *$F$ consists of all the bounded functions in an algebra $A$ which is closed under uniform convergence and closed under inversion of functions with empty zero sets.*

(3) *Whenever $f, g \in F$, $g \neq 0$ and $f/g$ is bounded then $f/g \in F$.*

(4) *$F$ is $Z$-separating.*

*Proof.* (1) implies (2). Take $A$ to be all the $Z(A)$ continuous functions.

(2) implies (3). Clear.

(3) implies (4). If $f^{-1}\{0\} \cap g^{-1}\{0\} = \emptyset$ where $f, g \in F$, then $h = f^2/(f^2 + g^2)$ is the separating function.

(4) implies (1). This follows from the remarks preceding this theorem.

This last theorem is known and is scattered throughout the literature. Much of it is implicit in Alexandroff's papers. One should also see in this connection the papers of Mrowka [**24–26**], Hager [**17–18**] and Isbell [**20**]. Another particular consequence of Theorem 4.3 is the following: $F$ of Theorem 4.3 separates

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the zero sets of all the continuous functions on a Tychonoff space $X$ if and only if $F$ is all the continuous functions on $X$. (See e.g. [**21**, Th. 3.12; **20**, p. 115]).

The next corollary is of much more interest and is the main theorem of [**21**]. The proof here is much simpler.

COROLLARY 4.4. *Let $A$ be a uniformly closed algebra of bounded real valued functions defined on a set $X$ and suppose that $A$ contains constants and separates points of $X$. Let $\mathscr{L}$ be any lattice of sets containing $\mathscr{Z}(A)$ which is a base for the closed sets of the weak topology generated by $A$. (This has as a base for the closed sets the zero sets of functions in $A$.) Then if*

(**) *whenever $G_1, G_2 \in \mathscr{L}$ and $G_1 \cap G_2 = \emptyset$ there is an $f \in A, 0 \leqq f \leqq 1$ such that $f(G_1) = 0$ and $f(G_1) = 1$,*

*then the dual of $A$ (with the* sup *norm), $A^*$, is isomorphic to $MR(\mathscr{L})$. This isomorphism is given by $\phi(f) = \int f\, d\mu$ where $\phi \in A^*$.*

*Proof.* We need only show that any $\mu \in MR(\mathscr{Z}(A))$ has an extension to a $\nu \in MR(\mathscr{L})$, since by Lemmas 2.8 and 2.9 the restriction map from $MR(\mathscr{Z}(A))$ to $MR(\mathscr{L})$ is a bijection. Suppose $\mu \in MR(\mathscr{Z}(A))$. Define $\Phi$ by $\Phi(f) = \int f\, d\mu$. Then $\Phi \in A^*$. (**) implies that $A$ is $Z$-separating, and thus by the remarks following Corollary 4.2, $A$ consists precisely of the restrictions of continuous functions on $IR(\mathscr{Z}(A))$, which is homeomorphic to $IR(\mathscr{L})$ by Lemma 2.11. Thus $\Phi$ induces on $C(IR(\mathscr{L}))$ a bounded linear functional $\Phi^*$ defined on $C(IR(\mathscr{L}))$ by $\Phi^*(f^*) = \Phi(f)$, where $f^*$ is the unique continuous extension of $f$ to $IR(\mathscr{L})$. $\Phi^*(f^*) = \int f^*\, d\mu_1$ where $\mu_1$ is a (unique) regular Borel measure. Restrict $\mu_1$ to $\mathscr{A}(\overline{\mathscr{L}})$ where $\overline{\mathscr{L}} = \{\bar{L}: L \in \mathscr{L}\}$ and the closures are taken in $IR(\mathscr{L})$. Call the restriction $\omega$. $\omega \in MR(\overline{\mathscr{L}})$ since in $IR(\mathscr{L})$, $\bar{L}$ separates the lattice of closed sets. This follows since $IR(\mathscr{L})$ is compact. Finally, define $\nu$ on $\mathscr{A}(\mathscr{L})$ as follows: $\nu(B) = \omega(C)$, where $C \in \mathscr{A}(\overline{\mathscr{L}})$ and $C \cap X = B$. We need only show that $\nu$ is well defined. (It is clearly $\mathscr{L}$-regular and as is easily seen extends $\mu$ since $\mu(Z) = \omega(\bar{Z}) = \nu(Z)$ for all $Z \in \mathscr{Z}(A)$). Thus, suppose $C_1 \cap X = C_2 \cap X$, where $C_1$ and $C_2 \in A(\overline{\mathscr{L}})$. Let $\omega_*$ be the inner measure associated with $\omega$ as in [**2**], i.e. $\omega_*(E) = \sup \{\omega(\bar{L}): \bar{L} \subset E\}$. Then since $\bar{L} \subset X'$ implies $\bar{L} \cap X = \emptyset$ and hence that $\bar{L} = \emptyset$, we have that $\omega_*(X') = 0$. Since $C_1 \Delta C_2 \subset X'$ where $\Delta$ denotes the symmetric difference, we have

$$\omega(C_1 \Delta C_2) \leqq \omega_*(X') = 0$$

and hence $\omega(C_1) = \omega(C_2)$. Thus $\nu$ is well defined.

*Remark* 4.5. In [**22**] Kirk defines a *standard representation* of the dual, $F^*$, of a Banach algebra $F$ of real valued bounded functions containing constants and separating points of a paving $(X, \mathscr{L})$. This is a linear map $I$ from $F^*$ to $MR(\mathscr{L})$ with the property that if $0 \leqq \phi \in F^*$ then $I_\phi(A) = \inf \{\phi(f), f \in F, K_A \leqq f\}$. In this case he says that $MR(\mathscr{L})$ *represents* $F^*$. He proves that if $H_F$ is $W(\mathscr{L})$ for some Wallman base $\mathscr{L}$ on $X$ then $F^*$ has a standard representation. It is clear that by modifying the above proof of Corollary 4.4 we may also obtain Kirk's more general theorem. The proof is in fact simpler, since we need not use the lattice $\mathscr{Z}[A]$ or require that $F$ have property (**) of Corollary 4.4. We need not appeal to Corollary 4.2 or Lemma 2.11, since they were only used to show that $H_F$ was homeomorphic to $W(\mathscr{L})$, a condition built into the hypothesis of his more general theorem.

*Remark* 4.6. An alternate proof of Corollary 4.4 appears in [**32**].

The following extension of Alexandroff's Theorem 1.2 is also a useful consequence of Theorem 4.1.

COROLLARY 4.7. *If* $(X, \mathscr{L}_1)$ *and* $(X, \mathscr{L}_2)$ *are strongly normal delta pavings such that* $\mathscr{L}_1$ *separates* $\mathscr{L}_2$ *and* $\mathscr{L}_2$ *separates* $\mathscr{L}_1$ *(here we do not assume that* $\mathscr{L}_1 \subset \mathscr{L}_2$*), then* $MR(\mathscr{L}_1)$ *and* $MR(\mathscr{L}_2)$ *are isometrically isomorphic.*

*Proof.* $C_b(\mathscr{L}_1)$ is isomorphic to $C(IR(\mathscr{L}_1))$ by Theorem 4.1. But $IR(\mathscr{L}_1)$ is homeomorphic to $IR(\mathscr{L}_2)$ by [**29**, Th. 7] in conjunction with Theorem 2.1. Thus $C_b(\mathscr{L}_1)$ is isomorphic to $C(IR(\mathscr{L}_2))$ and again by Theorem 4.1, $C_b(\mathscr{L}_2)$ is isomorphic to $C_b(\mathscr{L}_1)$. Thus the duals of $C_b(\mathscr{L}_1)$ and $C_b(\mathscr{L}_2)$ are congruent and by Theorem 1.2 so are $MR(\mathscr{L}_1)$ and $MR(\mathscr{L}_2)$.

*Remark* 4.8. The above correspondence also sets up a topological isomorphism when $MR(\mathscr{L}_1)$ and $MR(\mathscr{L}_2)$ carry their vague topologies, since the vague topologies are just weak * topologies.

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