SOME GENERALIZATIONS OF NORMAL SERIES IN INFINITE GROUPS

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1. Introduction

In this paper, we are concerned with certain generalizations of subnormal and ascendent (transfinitely subnormal) subgroups of a group. A subgroup A of a group G is called f-ascendent in G if there is a well ordered ascending complete series of subgroups of G,

$$A = G_0 \subset G_1 \subset \dots \subset G_\alpha \subset \dots \subset G_\lambda = G$$

where for all $\alpha < \lambda$, either $G_\alpha \lhd G_{\alpha+1}$ or $[G_{\alpha+1}:G_\alpha] < \infty$. If such a series has finite length, A is called F-subnormal in G.

Several authors ([2, p. 414], [3], [8, p. 10] and [10, p. 256]) have proved theorems of the following type:

Let Σ be a radical class of groups (the word radical may mean one of several things: see Definition 2.1) and A be a subnormal (ascendent) Σ-subgroup of G. Then A^G is also a Σ-group.

We show here (Theorem 3.1) that by modifying somewhat the class Σ the same type of theorem holds for f-subnormal and f-ascendent subgroups. As a consequence of Theorem 3.1 we have

COROLLARY 3.2. *If $A \in L(\mathfrak{N}\mathfrak{F})$, $B \in L(\mathfrak{N}\mathfrak{F})$ and both A and B are f-ascendent* in G, then $\langle A, B \rangle \in L(\mathfrak{N}\mathfrak{F})$.

Also studied in this paper are corresponding generalizations of the normalizer condition (see Definitions 4.1 and 4.2). In particular, we determine the relationships between these classes and the class of FC-hypercentral groups.

2. Definitions and notation

A class Σ of groups is a collection of groups containing the unit group E and closed under isomorphic images. Let Σ be a class of groups:

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(i) $s(\Sigma)$ is the class of all groups which are subgroups of Σ groups.

(ii) $q(\Sigma)$ is the class of all groups which are quotients of Σ groups.

(iii) $L(\Sigma)$ is the class of all groups G in which every finitely generated subgroup is a Σ group.

We use the following abbreviations for certain classes of groups:

$\mathfrak{F}$ is the class of finite groups,
$\mathfrak{N}$ is the class of nilpotent groups,
$\mathfrak{A}$ is the class of abelian groups.

If G is a group, $G_{\mathfrak{F}}$ is the class of all $\mathfrak{F}$ groups which are subgroups of some factor of G. As usual, if Σ and $\mathfrak{X}$ are classes of groups, $\Sigma\mathfrak{X}$ denotes the class of all extensions of Σ groups by $\mathfrak{X}$ groups.

DEFINITION 2.1. Let Σ be a class of groups. Σ is a *weak radical class* if $s\Sigma = \Sigma$ and every group has a normal Σ subgroup that contains all normal Σ-subgroups of G. Σ is a *strong radical class* if Σ is a weak radical class and in addition $L(\Sigma) = \Sigma$.

If Σ is a radical class (weak or strong) and G is a group, $\Sigma(G)$ denotes the unique maximal normal Σ subgroup of G and is called the Σ-*radical* of G. $\Sigma(G)$ is a characteristic subgroup of G.

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A proof of the following lemma may be found in [8, p. 9].

LEMMA 3.1. *Let Σ be a weak radical class. If $A \lhd G$, then $\Sigma(A) \lhd \Sigma(G)$ and $\Sigma(A) = A \cap \Sigma(G)$.*

LEMMA 3.2. *Let G be a group and Σ be a weak radical class such that $\Sigma G_{\mathfrak{F}} = \Sigma$. If $A \subset G$ and $[G:A] < \infty$, then $\Sigma(A) = \Sigma(G) \cap A$, and consequently $[\Sigma(G):\Sigma(A)] < \infty$.*

PROOF. Let $B = \mathrm{Core}_G(A)$. Then $B \lhd G$ and G/B is finite. By Lemma 3.1, $\Sigma(B) = B \cap \Sigma(A)$. Thus

$$B\Sigma(A)/B \simeq \Sigma(A)/(\Sigma(A) \cap B) = \Sigma(A)/\Sigma(B).$$

So, $\Sigma(A)/\Sigma(B)$ is finite.

Let σ be the natural homomorphism from G to $G/\Sigma(B)$. Then $(\Sigma(A))\sigma$ is finite and

$$(\Sigma(A))\sigma \lhd A\sigma.$$

Further, $A\sigma$ has finite index in $G\sigma$. So, if $x \in (\Sigma(A))\sigma$, x has finite order and has only a finite number of conjugates in $G\sigma$. By a theorem of Deitzmann (for a proof see [5, p. 154]),

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$$((\Sigma(A))\sigma)^{G\sigma}$$

is a finite group. Thus, $(\Sigma(A))_G/\Sigma(B)$ is finite, and consequently is in $G_{\mathfrak{F}}$. Since $\Sigma G_{\mathfrak{F}} = \Sigma$, $(\Sigma(A))^G \in \Sigma$. Thus

$$(\Sigma(A))^G \subset \Sigma(G),$$

and therefore $\Sigma(A) \subset \Sigma(G) \cap A$. The inclusion $\Sigma(G) \cap A \subset \Sigma(A)$ is clear, and it follows that $\Sigma(A) = \Sigma(G) \cap A$.

THEOREM 3.1. *Let G be a group and Σ be a class of groups such that* $\Sigma G_{\mathfrak{F}} = \Sigma$.

i) *if Σ is a weak radical class and A is f-subnormal in G, then $\Sigma(A)$ is an f-subnormal subgroup of $\Sigma(G)$.*

ii) *if Σ is a strong radical class and A if f-ascendent in G, then $\Sigma(A)$ is an f-ascendent subgroup of $\Sigma(G)$.*

This theorem can be proved using Lemmas 3.1 and 3.2 and minor modifications of the argument of Plotkin [8, pp. 9–11].

COROLLARY 3.1. *If $A \in L(\mathfrak{F})$, $B \in L(\mathfrak{F})$ and both A and B are f-ascendent in G, then $\langle A, B\rangle \in L(\mathfrak{F})$.*

The techniques of [9, p. 96] can be used to show that the class $L(\mathfrak{N}\mathfrak{F})$ satisfies all the conditions of Theorem 3.1 (ii). Accordingly, we have

COROLLARY 3.2. *If $A \in L(\mathfrak{N}\mathfrak{F})$, $B \in L(\mathfrak{N}\mathfrak{F})$ and both A and B are f-ascendent in G, then $\langle A, B\rangle \in L(\mathfrak{N}\mathfrak{F})$.*

We note that Corollary 3.2 holds if $\mathfrak{N}\mathfrak{F}$ is replaced by $\mathfrak{M}$, the class of Noetherian groups.

Let Σ be a class of groups with $\{s, q\}\Sigma = \Sigma$. Following R. Baer [2, p. 241], we define the class sub-Σ to be the class of all groups G such that every non-E homomorphic image of G has a non-E subnormal Σ subgroup. The method of [7, p. 350] shows that sub-Σ is a weak radical class. As a final consequence of Theorem 3.1 (ii), we have

COROLLARY 3.3. *Let Σ be a class with $\{s,q\}\Sigma = \Sigma$. Let $\mathfrak{X}$ be a class of groups such that $\{s, q\}\mathfrak{X} = \mathfrak{X}$, and every finite $\mathfrak{X}$ group is a Σ-group. If $A \in \Sigma$ and A is f-subnormal in $G \in \mathfrak{X}$, then $A^G \in$ sub-Σ.*

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We consider the following two generalizations of the normalizer condition.

DEFINITION 4.1. A group G is an *f–N group* if for every proper subgroup A of G, there is a $B \subset G$ such that A is a proper subgroup of B and either $A \triangleleft B$ or $[B: A] < \infty$.

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DEFINITION 4.2. A group G is an $f-N_1$ *group* if for every proper subgroup A of G, there is a $B \subset G$ such that A is a proper subgroup of B and $A/\mathrm{Core}_B(A)$ is finite.

Evidently, $f-N \subset f-N_1$. It is shown later that this inclusion is strict. It is also clear that $G \in f-N$ if and only if every proper subgroup of G is f-ascendent in G. Similarly, $G \in f-N_1$ if and only if for every proper subgroup A of G there is a complete well ordered ascending series

$$A = A_0 \subset A_1 \subset \ldots \subset A_\gamma = G$$

such that for all $\alpha < \lambda$, $A_\alpha/\mathrm{Core}_{A_{\alpha+1}}(A_\alpha) \in \mathfrak{F}$. From these characterizations it is easy to deduce that both of these classes are s and q closed.

A difficulty that arises with $f-N_1$ groups is that this class contains every group G with the property that every proper subgroup of G is finite. There are unsolved questions concerning groups of this type. Some of these difficulties can be avoided if we consider only locally finite $f-N_1$ groups.

THEOREM 4.1. *If G is a locally finite $f-N_1$ group, then every maximal subgroup of G has finite index in G.*

PROOF. Let G be a locally finite f-N_1 group and H be a maximal subgroup of G. Then $H/\mathrm{Core}_G(H)$ is finite, and $H/\mathrm{Core}_G(H)$ is a maximal subgroup of the locally finite group $G/\mathrm{Core}_G(H)$. Since an infinite locally finite group can not have a finite maximal subgroup, we must have $G/\mathrm{Core}_G(H)$ finite. Thus $[G:H] < \infty$.

This theorem shows that there are locally finite groups that are not $f-N_1$ groups; e.g. $A\,\mathrm{lt}(\aleph_0)$.

It is well known that every ZA group (hypercentral group) satisfies the normalizer condition. Our next result is that an FC-hypercentral group is a $f-N_1$ group. We recall the definition of an FC-hypercentral group:

Let G be a group. Define $F_0(G) = E$,

$$F_1(G) = \{x \in G \mid [G:C(x)] < \infty\}.$$

$F_1(G)$ is a characteristic subgroup of G and is called the FC-center of G [6]. If α is a nonlimit ordinal, $F_\alpha(G)$ is defined by

$$F_\alpha(G)/F_{\alpha-1}(G) = F_1(G/F_{\alpha-1}(G)).$$

If α is a limit ordinal define

$$F_\alpha(G) \text{ by } F_\alpha(G) = \cup\{F_\beta(G) \mid \beta < \alpha\}.$$

$\cup\,\{F_\alpha(G) \mid \alpha \text{ an ordinal}\}$ is the FC-hypercenter of G. G is FC-hypercentral if G coincides with its FC-hypercenter.

THEOREM 4.2. (i) *Every periodic FC-hypercentral group is an $f-N$ group.*
(ii) *Every FC-hypercentral group is an $f-N_1$ group.*

PROOF. (i) Let G be a periodic FC-hypercentral group and A be a proper subgroup of G. There is a first ordinal α such that $F_\alpha(G) \not\subset A$. α is not a limit ordinal and $F_{\alpha-1}(G) \subset A$. The group $F_\alpha(G)/F_{\alpha-1}(G)$ is a periodic FC group and is therefore locally normal [11, p. 143]. Thus, if $x \in F_\alpha(G)\backslash A$,

$$x^G F_{\alpha-1}(G)/F_{\alpha-1}(G)$$

is a finitely generated locally normal group and is hence finite.

Now

$$[x^G A : A] = [x^G : x^G \cap A] \leqq [x^G : x^G \cap F_{\alpha-1}(G)] = [x^G F_{\alpha-1}(G) : F_{\alpha-1}(G)] < \infty$$

and part (i) is proved.

(ii) Let A be a proper subgroup of the FC-hypercentral group G. As above, there is a non-limit ordinal α such that $F_\alpha(G) \not\subset G$ while $F_{\alpha-1}(G) \subset A$. Let

$$K/F_{\alpha-1}(G) = Z(F_\alpha(G)/F_{\alpha-1}(G)).$$

Case 1. $K \subset A$. Now $F_\alpha(G)/K$ is locally normal [11, p. 444]. If $x \in F_\alpha(G)\backslash A$, the argument given in (i) shows that $[x^G A : A] < \infty$.

Case 2. $K \not\subset A$. Let $x \in K \backslash A$. Then

$$x^G F_{\alpha-1}(G)/F_{\alpha-1}(G)$$

is a finitely generated abelian group. Let $R = x^G$ and $B = RA$. We will show the existence of a homomorphism σ on B such that $A\sigma$ is finite and $\operatorname{Ker}(\sigma) \subset A$. It will then follow that $A/\operatorname{Core}_B(A)$ is finite.

Let α be the natural homomorphism from B to $B/B \cap F_{\alpha-1}(V)$. $\operatorname{Ker}(\alpha) \subset A$ and $B\alpha = (R\alpha)(A\alpha)$. Now $R\alpha \subset F_1(B\alpha)$ and $R\alpha$ is a finitely generated abelian group. Further, $R\alpha \triangleleft B\alpha$.

Let $\{x_1, \dots, x_n\}$ be a generating set of $R\alpha$. Then

$$C_{B\alpha}(R\alpha) = \bigcap_{i=1}^{n} C_{B\alpha}(x_i).$$

Since for each i $[B\alpha : C_{B\alpha}(x_i)] < \infty$, we have

$$[B\alpha : C_{B\alpha}(R\alpha)] < \infty.$$

Now $C_{B\alpha}(R\alpha) \triangleleft B\alpha$ and since $R\alpha$ is abelian, $R\alpha \subset C_{B\alpha}(R\alpha)$. Thus

$$B\alpha = (C_{B\alpha}(R\alpha)) \cdot (A\alpha).$$

Further, $C_{B\alpha}(R\alpha) \cap A\alpha \triangleleft (R\alpha)(A\alpha) = B\alpha$.

Let ρ be the natural homomorphism from $B\alpha$ to $B\alpha/(C_{B\alpha}(R\alpha) \cap A\alpha)$. Then $\operatorname{Ker}(\rho) \subset A\alpha$ and

$$B\alpha\rho = ((C_{B\alpha}(R\alpha))\rho)(A\alpha\rho).$$

Further, $(C_{B\alpha}(R\alpha))\rho \cap A\alpha\rho = E$ and $(C_{B\alpha}(R\alpha))\rho$ has finite index in $B\alpha\rho$. It follows that $A\alpha\rho$ is finite. Let $\sigma = \alpha\rho$. Then $A\sigma$ is finite and $\operatorname{Ker}\sigma \subset A$.

Let G be the dihedral group of the infinite cyclic group Z. Then $G = \langle a\rangle\langle b\rangle$, where $o(a) = \infty$, $o(b) = 2$ and $a^b = a^{-1}$. Now $G \in \mathfrak{A}\mathfrak{F}$ so that $F_2(G) = G$. By Theorem 4.2 (ii), $G \in f\text{-}N_2$. On the other hand, it is easily shown that

$$N_G(\langle b\rangle) = \langle b\rangle$$

and that any subgroup of G properly containing $\langle b\rangle$ is infinite. Thus $G \notin f\text{-}N$, and it follows that $f\text{-}N$ is a proper subclass of $f\text{-}N_1$.

We conclude this section with some remarks about the local properties of the classes $f\text{-}N$ and $f\text{-}N_1$. Since the locally finite group $A\mathrm{lt}(\aleph_0)$ is not an $f\text{-}N_1$ group, neither of these classes satisfy the local theorem. However, both of these classes are countably recognizable in the sense of Baer [1]; i.e., if every countable subgroup of a group G is $f\text{-}N$ $(f\text{-}N_1)$, then G is also an $f\text{-}N$ $(f\text{-}N_1)$ group. These facts can be proved by modification of the techniques of Baer [1, p. 353].

5. Direct sums

THEOREM 5.1. *Let A and B be isomorphic groups. Then $A \times B \in f\text{-}N_1$ if and only if A (and consequently B) is FC-hypercentral.*

PROOF. The 'if' part of the theorem is clear. Suppose $A \times B \in f\text{-}N_1$, and that A is not FC-hypercentral. If $F(A)$ and $F(B)$ denote the FC-hypercenters of A and B, then

$$A/F(A) \simeq B/F(B)$$

and the nontrivial group $A/F(A) \times B/F(B)$ has trivial FC-center. So, without loss of generality we may assume $A \neq E$ and $F_1(A) = E$.

Let C be a diagonal in $A \times B = G$. Then C is a proper subgroup of $A \times B$. Thus, there exist subgroups K and D of G such that

$$K \triangleleft D,\ K \subset C,\ [C:K] < \infty,$$

and C is proper in D. So, $G = C \cdot A$ and $D = C \cdot (D \cap A)$. Also

$$[A \cap D, K] \subset A \cap K = E.$$

Now, let L be the projection of K on A. Since, $[C:K] < \infty$,

$$[A:L] < \infty.$$

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Since $[A \cap D, K] = E$,

$$[A \cap D, L] = E.$$

Thus, $A \cap D$ is centralized by a subgroup of finite index in A and it follows that

$$A \cap D \subset F_1(A) = E.$$

This forces $C = D$ which is a contradiction.

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We close by noting that not all $f-N_1$ groups are FC-hypercentral. The group constructed by Heineken and Mohamad in [4] is an $f-N$ group with trivial center.

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