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STABILITY OF GELFAND–KIRILLOV DIMENSION FOR RINGS WITH THE STRONG SECOND LAYER CONDITION

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We study the influence of the link structure of the prime spectrum of a Noetherian ring on the representation theory of the ring in the case that the ring satisfies the strong second layer condition and has exact integer Gelfand–Kirillov dimension. In particular, we show that Jategaonkar's density condition is satisfied and that the growth of an injective module is controlled by the growth of its first layer.

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Recently, there have been a number of studies examining the influence on the representation theory of a Noetherian ring of the prime ideal structure of the ring, see e.g. [**1, 3, 7**, Chapter 9 and **9**]. This note is a further contribution to the theme. In [**9**] Gelfand–Kirillov dimension was used to analyse extension of modules; however, one disadvantage of this approach is that for a finitely generated module M the analysis used $\mathrm{GK\,dim}(R/\mathrm{ann}(M))$ rather than $\mathrm{GK\,dim}(M)$, and there can be considerable disparity between these numbers. In Section 6 of [**9**] a start was made on the analysis of $\mathrm{GK\,dim}(M)$ under extensions and we continue this work here. In particular, we show that rings with finite Gelfand–Kirillov dimension and the strong second layer condition satisfy the density condition of Jategaonkar and we also show that for such rings with exact Gelfand–Kirillov dimension the growth of any injective module is controlled by the growth of its "first layer". Throughout the note we will assume that R is a Noetherian k-algebra, where k is a field, and in addition assume that R satisfies the strong second layer condition and, for certain results, that R has finite exact Gelfand–Kirillov dimension. This includes rings such as enveloping algebras of finite dimensional solvable Lie algebras [**7**, A.3.5, **8**, Chapters 6, 7], and Noetherian PI algebras with finite Gelfand–Kirillov dimension [**7**, 8.1.1, **8**, Chapter 10].

1. The strong second layer condition

The best source for the definition of the strong second layer condition from our point of view is [**6**] and so we will give the necessary background from there. Recall that an *affiliated series* of an R-module M is a sequence $0=M_0\subseteq M_1\subseteq\cdots\subseteq M_n=M$ of submodules together with an ordered set of prime ideals $\{P_1,\ldots,P_n\}$ called *affiliated* *primes* such that $M_i \neq M_{i-1}$, each P_i is a maximal annihilator prime of M/M_{i-1} and $M_i/M_{i-1} = \mathrm{ann}_{M/M_{i-1}}(P_i)$.

If P is a prime ideal of R and M a right R-module then M is a *P-prime module* if $\mathrm{ann}_R(M') = P$ for each nonzero submodule M' of M.

One can easily check that $M_i = \mathrm{ann}_M(P_i P_{i-1}, \ldots, P_1)$ and that M_i/M_{i-1} is a P_i-prime module. Jategaonkar's Main Lemma analyses the behaviour of affiliated series of length two. The version we present is that of [**6**, Theorem 11.1]. We will use freely the language of links and cliques; suitable references are [**7**] and [**6**].

Theorem 1.1. *Let R be a Noetherian ring and let M be a right R-module with an affiliated series $0 \subseteq U \subseteq M$ and affiliated primes $\{P, Q\}$, such that U is essential in M. Let M' be a submodule of M, properly containing U, such that the ideal $A = \mathrm{ann}_R(M')$ is maximal among annihilators of submodules of M properly containing U. Then exactly one of the following two alternatives occurs:*

(i) *$Q \subsetneqq P$ and $M'Q = 0$. In this case, M' and M'/U are faithful torsion R/Q-modules.*

(ii) *$Q \rightsquigarrow P$ and $B = Q \cap P/A$ is a linking bimodule between Q and P. In this case, if U is torsion free as a right R/P-module, then M'/U is torsion free as a right R/Q-module.*

The Noetherian ring R is said to have the *(right) strong second layer condition* if, given the hypotheses of the above theorem, the conclusion (i) never occurs. The ring R is said to have *exact Gelfand–Kirillov dimension* if the following condition is satisfied: if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then

$$\mathrm{GK}\dim(B) = \max\{\mathrm{GK}\dim(A), \mathrm{GK}\dim(C)\}.$$

The two invariance properties that we need to analyse the dimension of module extensions are given in the following two lemmas.

Lemma 1.2. *Let R and S be k-algebras. Suppose that M is a right R-module and that ${}_RB_S$ is a bimodule such that ${}_RB$ is finitely generated. Then*

$$\mathrm{GK}\dim_S(M \otimes B) \leqq \mathrm{GK}\dim_R(M).$$

Proof. cf. [**8**, Proposition 5.6]. □

If M is a right R-module and ${}_SB_R$ is a bimodule then $\mathrm{Hom}(B_R, M)$ has a natural structure as a right S-module via $(\theta s)(b) := \theta(sb)$, for $\theta \in \mathrm{Hom}(B_R, M)$, $s \in S$ and $b \in B$.

Lemma 1.3. *Let R and S be k-algebras. Suppose that M is a right R-module and that ${}_SB_R$ is a bimodule such that B_R is finitely generated. Then*

$$\mathrm{GK}\dim_S(\mathrm{Hom}(B_R, M)) \leqq \mathrm{GK}\dim(M_R).$$

Proof. [9, Lemma 6.1 (ii)]. □

Let R be a Noetherian ring containing prime ideals P and Q and suppose that U and V are uniform modules that are P-prime and Q-prime respectively. Suppose that there is a short exact sequence of R-modules

$$0 \to U \to M \to V \to 0$$

with M uniform and such that $U = \operatorname{ann}_M(P)$. In this setting, Theorem 1.1 gives information about the connection between the prime ideals P and Q; in order to make further progress it is necessary to establish connections between U and V. The groundwork for this has been done in [3, 2.5, 2.7] where the next result is established. However the exact result that we need is a little difficult to extract from [3]; so we give a complete proof of the result, following a suggestion of the referee.

Proposition 1.4. *Let R be a Noetherian ring containing prime ideals P and Q and suppose that U and V are uniform modules that are P-prime and Q-prime respectively. If there is a short exact sequence of R-modules*

$$0 \to U \to M \to V \to 0$$

with M uniform and such that $U = \operatorname{ann}_M(P)$ then either Q is strictly contained in P or

(i) *there is an R-monomorphism from V into $\operatorname{Hom}(B, U)$, where B is the non-zero bimodule $P \cap Q/\operatorname{ann}_R(M)$, and*

(ii) *there is a nonzero homomorphism from $V \otimes_R B$ to U.*

Proof. Suppose that Q is not strictly contained in P. If $N := \operatorname{ann}_M(P \cap Q)$ strictly contains U then consider the short exact sequence

$$0 \to U \to N \to N/U \to 0.$$

If A is the annihilator of an arbitrary submodule of N that properly contains U then $P \cap Q \subseteq A$ since $N(P \cap Q) = 0$ and $A \subseteq (P \cap Q)$ since A annihilates U and a non-zero submodule of the Q-prime module N/U; thus $A = P \cap Q$. Now Theorem 1.1 applied to N yields that Q is linked to P via the bimodule $B = (P \cap Q)/(P \cap Q)$, which is absurd since $B = 0$. Thus $U = \operatorname{ann}_M(P \cap Q)$. Now, using the notation of [3], apply [3, 2.3] to M with $I = P \cap Q$ and $J = \operatorname{ann}_R(M)$, to obtain a monomorphism from $V = M/U$ into $\operatorname{Hom}(B, M)$. However, $\operatorname{Hom}(B, M) \cong \operatorname{Hom}(B, U)$, since $BP = 0$ and $U = \operatorname{ann}_M(P)$, so (i) is established.

(ii) follows from the nonzero homomorphism in (i) and the natural isomorphism $\operatorname{Hom}(V \otimes B, U) \cong \operatorname{Hom}(V, \operatorname{Hom}(B, U))$. □

The prototype of the kind of result that we are aiming at is provided by the following result.

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Corollary 1.5. *Let R be a Noetherian k-algebra satisfying the strong second layer condition. Let P and Q be prime ideals of R and suppose that U and V are uniform modules that are P-prime and Q-prime respectively. If there is a short exact sequence of R-modules*

$$0 \to U \to M \to V \to 0$$

with M uniform and such that $U = \operatorname{ann}_M(P)$ then

(i) $\operatorname{GK}\dim(V) \leqq \operatorname{GK}\dim(U)$,

(ii) *if U is GK-homogeneous then* $\operatorname{GK}\dim(U) = \operatorname{GK}\dim(V)$.

Proof. (i) $\operatorname{GK}\dim(V) \leqq \operatorname{GK}\dim(\operatorname{Hom}(B, U))$, since part (i) of the previous result provides a monomorphism from V into $\operatorname{Hom}(B, U)$; so the result follows from Lemma 1.3.

(ii) Let $0 \neq U' \subseteq U$ be the image of a nonzero homomorphism from $V \otimes B$ to U, then, since U is GK-homogeneous,

$$\operatorname{GK}\dim(U) = \operatorname{GK}\dim(U') \leqq \operatorname{GK}\dim(V \otimes B) \leqq \operatorname{GK}\dim(V),$$

by Lemma 1.2. □

In the setting of the above result, if U is a torsion free R/P-module, then part (ii) of Theorem 1.1 guarantees that V is torsion free as an R/Q-module. The question as to when the converse holds is not very well understood and is related to the density condition of Jategaonkar [**7**, p. 176], see e.g. [**2**, Section 3]. However, we see below that the converse holds for the large class of rings that we are discussing here: the reason that we are able to see this is that torsion modules over Noetherian prime rings can be detected using Gelfand–Kirillov dimension. This is shown in the following well-known result.

Lemma 1.6. *Let R be a Noetherian prime k-algebra with finite Gelfand–Kirillov dimension. Then:*

(i) *If I is right ideal of R then* $\operatorname{GK}\dim(R/I) < \operatorname{GK}\dim(R)$ *if and only if I is an essential right ideal of R.*

(ii) *An R-module M is a torsion module if and only if* $\operatorname{GK}\dim(M) < \operatorname{GK}\dim(R)$.

Proof. Most of this is in [**10**, 8.3.6]. Suppose that I is not an essential right ideal of R but $\operatorname{GK}\dim(R/I) < \operatorname{GK}\dim(R)$. Then there is a uniform right ideal U of R such that $U \cap I = 0$. Hence U embeds in R/I and so $\operatorname{GK}\dim(U) < \operatorname{GK}\dim(R)$. However, since R is prime every right ideal of R contains an isomorphic copy of U and so some essential right ideal E of R satisfies $\operatorname{GK}\dim(E) = \operatorname{GK}\dim(U) < \operatorname{GK}\dim(R)$. Now R embeds in E so that $\operatorname{GK}\dim(R) \leqq \operatorname{GK}\dim(E)$, a contradiction. □

Theorem 1.7. *Let R be a Noetherian k-algebra satisfying the strong second layer condition and with finite Gelfand–Kirillov dimension. Let P and Q be prime ideals of R and suppose that U and V are uniform modules that are P-prime and Q-prime respectively. If there is a short exact sequence of R-modules*

$$0 \to U \to M \to V \to 0$$

with M uniform and such that $U = \operatorname{ann}_M(P)$ then U is a torsion free R/P-module if and only if V is a torsion free R/Q-module.

Proof. Suppose that U is a torsion R/P-module. The previous result shows that $\operatorname{GK}\dim(U) < \operatorname{GK}\dim(R/P)$. Note that $\operatorname{GK}\dim(R/P) = \operatorname{GK}\dim(R/Q)$, since $Q \rightsquigarrow P$. Now $\operatorname{GK}\dim(V) \leqq \operatorname{GK}\dim(U)$, by part (i) of Corollary 1.5. Thus $\operatorname{GK}\dim(V) < \operatorname{GK}\dim(R/Q)$; and so V is a torsion R/Q-module. On the other hand, if U is a torsion free R/P-module then V is a torsion free R/Q-module, by Theorem 1.1. □

Let R and S be prime Noetherian rings and let B be an R-S-bimodule, finitely generated and torsion free on each side. Then B satisfies the *density condition* if the following property and its left-handed version hold: Let E be an essential right submodule of B; then there exists a regular element d of R such that $dB \subseteq E$. A Noetherian ring R is said to satisfy the *density condition* if each of the bimodules that link prime ideals of R satisfies the density condition.

Corollary 1.8. *Let R be a Noetherian k-algebra satisfying the strong second layer condition and with finite Gelfand–Kirillov dimension. Then R satisfies the density condition.*

Proof. The truth of the conclusion of the above result is equivalent to the truth of the conclusion of this corollary, by [7, Theorem 6.3.11]. □

Remark. Strictly, [7, 6.3.11] only gives that the strongest link bimodules have the density condition when the conclusion of the above theorem holds. However, every link is a bimodule factor of a strongest link and the density condition passes to factor bimodules, as is obvious from the first definition of the density condition given on p. 176 of [7].

The density condition does not hold in general, see [5, pp. 235–236]. However, the example constructed by Goodearl and Schofield is rather esoteric and the above corollary justifies the feeling that the density condition should hold in reasonable rings.

2. Injective modules

Our next aim is to say something about the Gelfand–Kirillov dimension of injective modules. Without loss of generality, we may assume that we are dealing with an indecomposable injective module E. Recall that the set of *associated prime ideals* of a module M, $\operatorname{ass}(M)$, consists of those prime ideals P of R such that M contains a P-prime submodule. Since E is a uniform module there is a unique prime ideal P of R with $\operatorname{ass}(E)=\{P\}$.

Theorem 2.1. *Let R be a Noetherian ring with the strong second layer condition and finite exact Gelfand–Kirillov dimension. Let E be an indecomposable injective R-module with $\operatorname{ass}(E)=\{P\}$ for some prime ideal P of R. Then*

$$\operatorname{GK}\dim(E)=\operatorname{GK}\dim(\operatorname{ann}_E(P))=\operatorname{GK}\dim(E_{R/P}(U)),$$

where U is any P-prime submodule of E.

Proof. The second equality occurs since $\operatorname{ann}_E(P)\cong E_{R/P}(U)$. Let M be any finitely generated submodule of E and set $U=M\cap\operatorname{ann}_E(P)$ and note that U is P-prime. It is enough to show that $\operatorname{GK}\dim(M)\leqq\operatorname{GK}\dim(U)$, since certainly $\operatorname{GK}\dim(U)\leqq\operatorname{GK}\dim(M)$. Now M is annihilated by a product of prime ideals, each in the right clique of P, by [**6**, Theorem 11.4], so let n be the least integer such that there are semiprime ideals S_i, $i=1,\dots,n$, with $MS_n\dots S_1=0$ such that each S_i is a finite intersection of prime ideals from the right clique of P. First we show that one may take $S_1=P$. For, if not, set $N=MS_n\dots S_2$ so that $NP\neq 0$ while $NS_1=0$. If $S_1\nsubseteq P$ set $X=S_1$ while if $S_1\subseteq P$ set X to be the semiprime ideal with $S_1\subsetneqq X$ and $S_1=X\cap P$. In both cases $X\nsubseteq P$. Now $U'=NP\cap U\neq 0$ while $U'X\subseteq NPX\subseteq NS_1=0$; so $X\subseteq P$, since U is P-prime, a contradiction.

We prove the result by induction on n. If $n=1$ the result is trivial, since $M=U$. Set $I=S_n\dots S_2\cap S_{n-1}\dots S_1$ and $J=S_n\dots S_1$, so that $J\subseteq I$. Set $N=\operatorname{ann}_M(I)$. Note that $N(S_n\cap S_{n-1})\dots(S_2\cap S_1)\subseteq NI=0$; so $N\subsetneqq M$ by the choice of n and $\operatorname{GK}\dim(N)=\operatorname{GK}\dim(N\cap U)\leqq\operatorname{GK}\dim(U)$, by induction.

Now $M/N=\operatorname{ann}_M(J)/\operatorname{ann}_M(I)$ embeds in $\operatorname{Hom}(I/J,M)$, by [**3**, 2.3], and, since I/J is a right R/P-module any image of I/J in M must in fact be in U so that M/N embeds in $\operatorname{Hom}(I/J,U)$. Thus $\operatorname{GK}\dim(M/N)\leqq\operatorname{GK}\dim(U)$, by Lemma 1.3, and so $\operatorname{GK}\dim(M)\leqq\operatorname{GK}\dim(U)$, by exactness of Gelfand–Kirillov dimension. □

The above result says that the Gelfand–Kirillov dimension of an injective module is controlled by the Gelfand–Kirillov dimension of the "first layer". Unfortunately, it does not say that if M is an essential extension of a P-prime module U then $\operatorname{GK}\dim(M)=\operatorname{GK}\dim(U)$. The snag is that we have not shown that $\operatorname{GK}\dim(M)=\operatorname{GK}\dim(U)$ where both M and U are P-prime modules, and, indeed, the present methods will not touch this problem since $MP=0$ and the question is one concerning the prime ring R/P and not one involving the structure of the clique of P in R. However, in the case of a torsion module we can make an explicit statement.

Corollary 2.2. *Let R be a Noetherian ring with the strong second layer condition and finite exact Gelfand–Kirillov dimension. Let P be a prime ideal of R and suppose that U is a finitely generated torsion R/P-module. If $E=E(U)$ is the R-injective envelope of U then*

$$\operatorname{GK}\dim(E) < \operatorname{GK}\dim(R/P).$$

Proof. Without loss of generality, we may assume that U is uniform; and so E is also uniform. Let M be any submodule of E such that $U \subseteq M$ and $MP=0$. Then U is essential in M and so M is also a torsion R/P-module; so $\operatorname{GK}\dim(M) < \operatorname{GK}\dim(R/P)$ by Lemma 1.6 (ii). The result now follows from the above theorem. □

3. Examples

In an attempt to clarify the general problem, one might ask the following question. Let R be the second Weyl algebra; so that $\operatorname{GK}\dim(R)=4$. If U is a holonomic simple R-module then $\operatorname{GK}\dim(U)=2$. Stafford [**12**] has shown that there are simple nonholonomic modules. If V is such a module then $\operatorname{GK}\dim(V)=3$. Is it possible to have a nonsplit extension of a holonomic simple module U by a nonholonomic simple module V: that is, is there a uniserial module M_R of length 2 with a holonomic submodule U such that $V=M/U$ is a nonholonomic module?

We finish by giving two examples that place the previous results and discussion in perspective.

The first example shows that if one replaces Gelfand–Kirillov dimension by Krull dimension then the problem discussed above does arise.

Let $R=\mathbf{C}[x,y]$, $xy-yx=x$, the enveloping algebra of the two dimensional nonabelian solvable Lie algebra. The ring R has the strong second layer condition [**7,** A.3.9] and exact integer Gelfand–Kirillov dimension. Also the Krull dimension is an exact integer dimension function. Musson [**11**] has constructed a uniform module M with a simple submodule U such that M/U is 1-critical for Krull dimension. Thus $\operatorname{K}\dim(U)=0$ while $\operatorname{K}\dim(M)=1$. This example is discussed in detail in [**4,** Ex. 7.15] and from that discussion one can see that both U and M/U are 1-critical modules for Gelfand–Kirillov dimension.

The second example illustrates what can happen when the strong second layer condition is dropped. Let $S=\mathbf{C}[x,y]$, $xy-yx=1$, be the first Weyl algebra and set $I=xS$, a maximal right ideal. Then the idealizer ring $R=\mathbb{I}(I)=\mathbf{C}+xS$ is a Noetherian domain in which I is the unique proper ideal. When considered as an R-module, $M=S/I$ is uniserial of length two, [**10,** 5.5.5.], the only nontrivial submodule being $U=R/I$. Thus $0 \subseteq U \subseteq M$ is an affiliated series of length two with affiliated prime ideals $\{I,0\}$ and so we have an extension of type (i) in Theorem 1.1. Hence R does not have the strong second layer condition. Now $U=R/I=\mathbf{C}+xS/xS$ so that $\operatorname{GK}\dim(U)=0$; while $M/U=\mathbf{C}[x,y]/x\mathbf{C}[x,y]$, so that $\operatorname{GK}\dim(M/U)=1$.

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